A note on Demazure character formula for negative dominant characters

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Introduction

We first set up some notation.
Let G be a simple simply connected algebraic group of rank l over the field $\mathbb{C}$ of complex numbers.
Let T be a maximal torus of G.
Let $N_G(T)$ denote the normaliser of T in G. Let $W = N_G(T)/T$ denote the Weyl group of G with respect to T.
We denote by $\mathfrak{g}$ the Lie algebra of G.
We denote by $\mathfrak{h} \subseteq \mathfrak{g}$ the Lie algebra of T.
Let R denote the roots of G with respect to T.

Notations

Let R^+ denote the set of positive roots. Let B^+ be the Borel sub group of G containing T with respect to R^+. Let $S = \{\alpha_1, \ldots, \alpha_l\}$ denote the set of simple roots in R^+, where l is the rank of G. Let B be the Borel subgroup of G containing T with respect to the set of negative roots $R^- = -R^+$.
For $\beta \in R^+$ we also use the notation $\beta > 0$. The simple reflection in the Weyl group corresponding to α_i is denoted by s_{α_i}.

Notations

We have $X(T) \bigotimes \mathbb{R} = Hom_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$, the dual of the real form of $\mathfrak{h}$. The positive definite W-invariant form on $Hom_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$ induced by the Killing form of the Lie algebra $\mathfrak{g}$ of G is denoted by $(\ ,\ )$. We use the notation $\langle\ ,\ \rangle$ to denote $\langle \nu, \alpha \rangle = \frac{2(\nu,\alpha)}{(\alpha,\alpha)}$.

Notations

Let $\leq$ denote the partial order on $X(T)$ given by $\mu \leq \lambda$ if $\lambda - \mu$ is a non negative integral linear combination of simple roots. We also say that $\mu < \lambda$ if $\mu \leq \lambda$, and $\mu \neq \lambda$.

We denote by $X(T)^+$ the set of dominant characters of T with respect to B^+.

For any simple root α, we denote the fundamental weight corresponding to α by ω_α.

Let ρ denote the half sum of all positive roots of G with respect to T and B^+.

For every simple root α, s_α permutes all positive roots other than α. So, $\rho = \sum_{\alpha \in S} \omega_\alpha =$ sum of all fundamental weights.

Notations

We denote by $X(T)^+_{reg}$ the set of all regular dominant characters of T.
For $w \in W$, let $l(w)$ denote the length of w. We define the dot action by $w \cdot \lambda = w(\lambda + \rho) - \rho$.
Let w_0 denote the longest element of the Weyl group W.

Flag Variety and Schubert Variety

Recall,

Flag Variety and Schubert Variety

We denote by G/B, the flag variety of all Borel subgroups of G.
For any $w \in W$, we denote by $X(w) = \overline{BwB/B} \subset G/B$ the Schubert Variety corresponding to w.

Notations

Let $\leq$ denote the Bruhat order on W.
i.e $w \leq \tau$ if and only if $X(w) \subseteq X(\tau)$.
We also say that $w < \tau$ if $w \leq \tau$, and $w \neq \tau$.
For a subset $J \subseteq S$ denote $W^J = \{w \in W | w(\alpha) > 0, \alpha \in J\}$.

Minimal parabolic subgroup and its Levi factor

Unipotent radical of B

We denote by U (resp. U^+) the unipotent radical of B (resp B^+). We denote by P_α the minimal parabolic subgroup of G containing B and s_α. Let L_α denote the Levi subgroup of P_α containing T. We denote by B_α the intersection of L_α and B. Then L_α is the product of T and a homomorphic image G_α of $SL(2)$ via a homomorphism $\psi : SL(2) \longrightarrow L_\alpha$. (cf. [7,II.1.1.4]).

One dimensional reprsentation of B

We refer to [6] for notation and preliminaries on semisimple Lie algebras and their root systems.
For a fixed $w \in W$, the set of all positive roots α which are made negative by w^{-1} is denoted by $R^+(w^{-1})$.
For any character λ of B, we denote by $\mathbb{C}_\lambda$ the one dimensional representation of B corresponding to λ.

Cohomology of line bundles on $\mathbb{P}^1$

We make use of following points in computing cohomologies.
Since G is simply connected, the morphism $\psi : SL(2) \longrightarrow G_\alpha$ is an isomorphism, and hence $\psi : SL(2) \longrightarrow L_\alpha$ is injective. We denote this copy of $SL(2)$ in L_α by $SL(2, \alpha)$ We denote by B'_α the intersection of B_α and $SL(2, \alpha)$ in L_α.
We also note that the morphism $SL(2, \alpha)/B'_\alpha \hookrightarrow L_\alpha/B_\alpha$ induced by ψ is an isomorphism.
Since $L_\alpha/B_\alpha \hookrightarrow P_\alpha/B$ is an isomorphism, to compute the cohomology $H^i(P_\alpha/B, \mathcal{L}(V))$ for any B- module V, we treat V as a B_α- module and we compute $H^i(L_\alpha/B_\alpha, \mathcal{L}(V))$

Bott-Samelson- Demazure-Hansen sheme

We recall some basic facts and results about Schubert varieties. A good reference for all this is the book by Jantzen (cf [7, II, Chapter 14]).
Let $w = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \dots s_{\alpha_{i_n}}$ be a reduced expression for $w \in W$. Define

$$Z(w) = \frac{P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \dots \times P_{\alpha_{i_n}}}{B \times \dots \times B},$$

where the action of $B \times \dots \times B$ on $P_{\alpha_{i_1}} \times P_{\alpha_{i_2}} \times \dots \times P_{\alpha_{i_n}}$ is given by $(p_1, \dots, p_n)(b_1, \dots, b_n) = (p_1 \cdot b_1, b_1^{-1} \cdot p_2 \cdot b_2, \dots, b_{n-1}^{-1} \cdot p_n \cdot b_n)$, $p_j \in P_{\alpha_{i_j}}$, $b_j \in B$. We denote by ϕ_w the birational surjective morphism $\phi_w : Z(w) \longrightarrow X(w)$.
We note that for each reduced expression for w, $Z(w)$ is smooth, however, $Z(w)$ may not be independent of a reduced expression.

$\mathbb{P}^1$-fibration

Let $f_n : Z(w) \longrightarrow Z(ws_{\alpha_n})$ denote the map induced by the projection $P_{\alpha_1} \times P_{\alpha_2} \times \ldots \times P_{\alpha_n} \longrightarrow P_{\alpha_1} \times P_{\alpha_2} \times \ldots \times P_{\alpha_{n-1}}$. Then we observe that f_n is a $P_{\alpha_n}/B \simeq \mathbf{P}^1$-fibration.
Let V be a B-module. Let $\mathcal{L}_w(V)$ denote the pull back to $X(w)$ of the homogeneous vector bundle on G/B associated to V. *By abuse of notation* we denote the pull back of $\mathcal{L}_w(V)$ to $Z(w)$ also by $\mathcal{L}_w(V)$, when there is no cause for confusion.

$\mathbb{P}^1$-fibration

$$R^i f_{n*}\mathcal{L}_w(V) = \mathcal{L}_{ws_{\alpha_n}}(H^i(P_{\alpha_n}/B, \mathcal{L}_w(V)).$$

This together with easy applications of Leray spectral sequences is the constantly used tool in what follows. We term this the *descending 1-step construction*.

$\mathbb{P}^1$-base

We also have the *ascending 1-step construction* which too is used extensively in what follows sometimes in conjunction with the descending construction. We recall this for the convenience of the reader.
Let the notations be as above and write $\tau = s_\gamma w$, with $l(\tau) = l(w) + 1$, for some simple root γ. Then we have an induced morphism

$$g_1 : Z(\tau) \longrightarrow P_\gamma/B \simeq \mathbf{P}^1,$$

with fibres given by $Z(w)$. Again, by an application of the Leray spectral sequences together with the fact that the base is a $\mathbf{P}^1$, we obtain for every B-module V the following exact sequence of P_γ-modules:

Short exact sequence of B-modules

$$(0) \longrightarrow H^1(P_\gamma/B, R^{i-1}g_{1*}\mathcal{L}_w(V)) \longrightarrow H^i(Z(\tau), \mathcal{L}_\tau(V)) \longrightarrow H^0(P_\gamma/B, R^i g_{1*}\mathcal{L}_w(V)) \longrightarrow (0).$$

This short exact sequence of B-modules will be used frequently. So, we denote this short exact sequence by *SES* when ever this is being used.

Vanishing of higher direct images

We also recall the following well-known isomorphisms:

- $\phi_{w*}\mathcal{O}_{Z(w)} = \mathcal{O}_{X(w)}$.
- $R^q\phi_{w*}\mathcal{O}_{Z(w)} = 0$ for $q > 0$.

This together with [7, II. 14.6] implies that we may use the Bott-Samelson schemes $Z(w)$ for the computation and study of all the cohomology modules $H^i(X(w), \mathcal{L}_w(V))$. Henceforth we shall use the Bott-Samelson schemes and their cohomology modules in all the computations.

The Notation $H^i(w, \lambda)$

Simplicity of Notation If V is a B-module and $\mathcal{L}_w(V)$ is the induced vector bundle on $Z(w)$ we denote the cohomology modules $H^i(Z(w), \mathcal{L}_w(V))$ by $H^i(w, V)$.
In particular if λ is a character of B we denote the cohomology modules $H^i(Z(w), \mathcal{L}_\lambda)$ by $H^i(w, \lambda)$.

Some constructions from Demazure's paper

We recall briefly two exact sequences from [4] that Demazure used in his short proof of the Borel-Weil-Bott theorem (cf. [3]). We use the same notation as in [4].

Let α be a simple root and let $\lambda \in X(T)$ be such that $\langle \lambda, \alpha \rangle \geq 0$. For such a λ, we denote by $V_{\lambda,\alpha}$ the module $H^0(P_\alpha/B, \mathcal{L}_\lambda)$. Let $\mathbb{C}_\lambda$ denote the one dimensional B- module.

Lemma due to Demazure

Lemma (Demazure)

$$(0) \longrightarrow K \longrightarrow V_{\lambda,\alpha} \longrightarrow \mathbb{C}_{\lambda} \longrightarrow (0).$$

$$(0) \longrightarrow \mathbb{C}_{s_{\alpha}(\lambda)} \longrightarrow K \longrightarrow V_{\lambda-\alpha,\alpha} \longrightarrow (0).$$

A consequence of the Lemma

Lemma (Demazure)

1. *Let* $\tau = ws_\alpha$, $l(\tau) = l(w) + 1$. *If* $\langle \lambda, \alpha \rangle \geq 0$ *then* $H^j(\tau, \lambda) = H^j(w, V_{\lambda,\alpha})$ *for all* $j \geq 0$.
2. *Let* $\tau = ws_\alpha$, $l(\tau) = l(w) + 1$. *If* $\langle \lambda, \alpha \rangle \geq 0$, *then* $H^i(\tau, \lambda) = H^{i+1}(\tau, s_\alpha \cdot \lambda)$. *Further, if* $\langle \lambda, \alpha \rangle \leq -2$, *then* $H^i(\tau, \lambda) = H^{i-1}(\tau, s_\alpha \cdot \lambda)$.
3. *If* $\langle \lambda, \alpha \rangle = -1$, *then* $H^i(\tau, \lambda)$ *vanishes for all* $i \geq 0$ *(cf. Prop 5.2(b), [7]).*

Demazure character formula

Let D_α denote the Demazure operator on $\mathbb{Z}[X(T)]$ corresponding to a simple root α.
We recall from Jantzen [7, II, 14.17] that

$$D_\alpha(e^\lambda) = \frac{e^\lambda - e^{s_\alpha(\lambda)-\alpha}}{1 - e^{-\alpha}}.$$

Let $w = s_{\alpha_1} s_{\alpha_2} \cdots .s_{\alpha_n}$ be a reduced expression of w.
Then define $D_w = D_{\alpha_1} \circ D_{\alpha_2} \circ \cdots \circ D_{\alpha_n}$.
(It is a well known lemma that D_w is independent of reduced expression of w.)

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Demazure character formula:

$$D_w(e^\lambda) = \sum_{i=0}^{l(w)} (-1)^i char(H^i(w, \lambda))$$

Character of $(H^{l(w)}(w, -\lambda))^*$

Let λ be a regular dominant character of T. That is λ satisfies $\langle \lambda, \alpha \rangle \geq 1$ for each simple root α.
We recall notation from section 2: For $w \in W$, we denote by $H^{l(w)}(w, -\lambda)$ the top cohomology of the line bundle $\mathcal{L}_{-\lambda}$ on $X(w)$ associated to $-\lambda$.
Let e^{ρ} denote the element of the representation ring $\mathbb{Z}[X(T)]$ of T corresponding to ρ. Here, we use exponential notation e^{ρ} for using multiplication in the ring $\mathbb{Z}[X(T)]$.

Character of $(H^{l(w)}(w, -\lambda))^*$

Let $h^0(\tau, \mu)$ denote the character of the T- module $H^0(\tau, \mu)$.
Let $h^{l(w)}(w, -\lambda)$ denote the character of the T- module $H^{l(w)}(w, -\lambda)$.
Let $h^{l(w)}(w, -\lambda)^*$ denote the character of the dual $(H^{l(w)}(w, -\lambda))^*$ of the T- module $H^{l(w)}(w, -\lambda)$.
We have the following theorem:

Theorem (.....)

For any $\tau \in W$, we have $\sum_{w \leq \tau} h^{l(w)}(w, -\lambda)^* = e^{\rho} \cdot h^0(\tau, \lambda - \rho)$.

Twist Character in Serre duality

Let $w \in W$ be such that $X(w)$ is Gorenstein. Let χ'_w be the character of T such that $\mathcal{L}_{-2\rho+\chi'_w}$ is the canonical line bundle on $X(w)$.
When G is of type A_n, χ'_w is described in a nice combinatorial way by Woo and Yong in [11].
Then, by Serre duality, there is a character ψ_w of T such that the B-modules $H^{l(w)}(w, -\lambda)^*$ and $H^0(w, \lambda - 2\rho + \chi'_w) \bigotimes \mathbb{C}_{\psi_w}$ are isomorphic.
The character ψ_w is described by O.Mathieu in [10].

Twist Character in Serre duality

With notation as above, the following corollary relates the two characters χ'_w and ψ_w of T.

Corollary (.....)

Then, we have $\psi_w = \rho + w(\rho) - w(\chi'_w)$.

Kernels of Demazure operators

Let N_α denote the kernel of D_α.
Let N denote the intersection $\bigcap_{\alpha \in S} N_\alpha$ of the kernels of all D_α's, α running over all simple roots.
Then, we have

Corollary (.....)

$\{\sum_{w \in W} h^{l(w)}(w, -\lambda) : \lambda \in X(T)^+_{reg}\}$ *forms a* $\mathbb{Z}$ *basis for* N.

Character of $H^{l(\tau)}(\tau, -\lambda)$ for $\lambda \in X(T)^+_{reg}$

The following theorem gives a formula for the character of $H^{l(\tau)}(\tau, -\lambda)$ for any $\tau \in W$, and for any regular dominant character λ of T.

Theorem (......)

Let $\tau \in W$. Let $\lambda \in X(T)^+_{reg}$. Then, the character of $H^{l(\tau)}(\tau, -\lambda)$ is equal to the sum $e^{-\rho}(\sum_{w \leq \tau}(-1)^{l(\tau)-l(w)} h^0(w, \lambda - \rho)^*)$

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