The uniqueness of the generalized octagon of order $(2, 4)$ containing a suboctagon of order $(2, 1)$

Bart De Bruyn

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Generalized polygons

Let $\mathcal{S}$ be a point-line geometry with point set $\mathcal{P} \neq \emptyset$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$.

$\mathcal{S}$ is called a generalized k-gon with $k \geq 3$ if:

(GP1) Every two distinct points are incident with at most one line.

(GP2) $\mathcal{S}$ has no subgeometries that are ordinary l-gons with $3 \leq l < k$.

(GP3) $\mathcal{S}$ has sufficiently many subgeometries that are ordinary k-gons: any two elements of $\mathcal{S}$ are contained in one such a subgeometry.

- The generalized 3-gons are precisely the projective planes.

Generalized polygons

A generalized polygon $\mathcal{S}$ is said to have order (s, t) if every line is incident with precisely $s + 1$ points and if every point is incident with precisely $t + 1$ lines. If $s = t$, then $\mathcal{S}$ is also said to have order s.

$\mathcal{S}$ is called thick if every point is incident with at least three lines and if every line is incident with at least three points. If $\mathcal{S}$ is thick, then $\mathcal{S}$ also has an order.

Type of problem under consideration

Problem. Classify all finite generalized k-gons of order (s, t) for certain specific values of k, s and t.

Every non-thick generalized polygon is either an ordinary polygon or can be constructed from another smaller generalized polygon.

So, we may assume that the generalized k-gon is thick, i.e. that $s, t \geq 2$.

Restrictions on k, s and t

Bruck-Ryser Theorem (1949): If there exists a projective plane whose order n is 1 or 2 mod 4, then n should be the sum of two squares.

Feit-Higman theorem (1964): Finite thick generalized k-gons only exist for $k \in \{3, 4, 6, 8\}$ + some other conditions.

Higman's inequality (1975): If there exists a thick generalized quadrangle or octagon of order (s, t), then $\sqrt{s} \leq t \leq s^2$.

Inequality of Haemers-Roos (1981): If there exists a thick generalized hexagon of order (s, t), then $\sqrt[3]{s} \leq t \leq s^3$.

Known classification results

- Complete classification of all projective planes of order at most 10.

- Complete classification of all finite GQ's of order $(2, t)$, $(3, t)$ and $(4, 4)$. (Dixmier - Zara, 1975; Cameron; Payne 1977)

- Complete classification of all finite GH's of order $(2, t)$. (Cohen - Tits, 1985)

- Unique GH of order 3, assuming there is at least one subhexagon of order $(3, 1)$. (De Medts - Van Maldeghem, 2009)

What about generalized octagons

Till very recently nothing was known, not even for the smallest possible case $(s,t) = (2,4)$. A GO(2,4) contains 1755 points and 2925 lines.

Theorem

There exists, up to isomorphism, a unique generalized octagon of order $(2,4)$ *that contains at least one suboctagon of order (2,1).*

This generalized octagon belongs to an infinite family of generalized octagons constructed by Tits (1961), using the Ree groups of type ${}^2F_4$.

Open Problem. Does there exist other finite thick generalized octagons besides these Ree-Tits octagons and their point-line duals?

Other characterization result

Theorem (Cohen, O'Brien and Shpectorov)

Suppose $\mathcal{S}$ is a generalized octagon of order $(2, 4)$, and G is a group of automorphisms of $\mathcal{S}$ stabilizing all lines through a distinguished point x and acting transitively on the set of points opposite to x. Then $\mathcal{S}$ is isomorphic to the Ree-Tits generalized octagon of order $(2, 4)$.

How will we proceed?

• Any GO(2,4) contains 1755 points and 2925 lines, while a GO(2,1) contains 45 points and 30 lines. There is up to isomorphism a unique $GO(2,1)$.

• We have a known configuration of 45 points and 30 lines. We prove that there is at most one way in which this configuration can be extended to a $GO(2,4)$.

• We use techniques/tricks which are modifications of techniques/tricks from the theory of near polygons that were developed by P. Vandecasteele and myself (around 2003) for the purpose of classifying dense near polygons.

• We also make use of a computer (GAP) to do several computations.

Valuations

Let $\mathcal{S}$ be a generalized $2d$-gon, $d \geq 2$, with point set $\mathcal{P}$. A map $f : \mathcal{P} \to \mathbb{N}$ is called a valuation if the following three conditions are satisfied:

1. There is a point with value 0.
2. Every line L contains a unique point x_L such that $f(x) = f(x_L) + 1, \forall x \in L \setminus \{x_L\}$.
3. If x is a point with non-maximal value, then there is at most one line through x containing a point with value $f(x) - 1$.

The set of points with non-maximal value is a hyperplane H_f of $\mathcal{S}$.

Valuations

A valuation f can be completely reconstructed from its associated hyperplane H_f.

Theorem

Let f be a valuation of a generalized $2d$-gon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$. Let M denote the maximal distance from a point of $\mathcal{S}$ to $\mathcal{P} \setminus H_f$. Then $f(x) = M - d(x, \mathcal{P} \setminus H_f)$ for every point x of $\mathcal{S}$.

Examples of valuations

Let $\mathcal{S}$ be a generalized $2d$-gon, $d \in \mathbb{N} \setminus \{0, 1\}$.

Let x be a fixed point and put $f(y) := d(x, y)$ for every point y. Then f is a so-called classical valuation.

Let O be an ovoid. Put $f(x) = 0$ if $x \in O$ and $f(x) = 1$ if $x \notin O$. Then f is a so-called ovoidal valuation.

Suppose $x \in \mathcal{P}$ and $O \subseteq \Gamma_d(x)$ such that every line of $\mathcal{S}$ at distance $d - 1$ from x has a unique point in common with O.

- $f(y) := d(x, y)$ if $d(x, y) \leq d - 1$,
- $f(y) := d - 2$ if $d(x, y) = d$ and $y \in O$,
- $f(y) := d - 1$ if $d(x, y) = d$ and $y \notin O$.

Then f is a so-called semi-classical valuation of $\mathcal{S}$.

Distance-j-ovoids

A distance-j-ovoid of $\mathcal{S}$ is a set X of points satisfying the following properties:

- $|X| \geq 2$ and the minimal distance between two distinct points of X is equal to j;
- for every point a of $\mathcal{S}$, there exists a point $x \in X$ such that $d(a, x) \leq \frac{j}{2}$;
- for every line L of $\mathcal{S}$, there exists a point $x \in X$ such that $d(x, L) \leq \frac{j-1}{2}$.

Distance-2-ovoids = ovoids.

Distance-j-ovoidal valuations

Let X be a *distance-j-ovoid* with j even. The map $\mathcal{P} \to \mathbb{N}; x \mapsto d(x, X)$ is a so-called distance-j-ovoidal valuation of $\mathcal{S}$.

The distance-2-ovoidal valuations are precisely the ovoidal valuations.

A few definitions

Let f be a valuation of a generalized $2d$-gon $\mathcal{S}$.

- $\mathcal{O}_f$: the set of points with f-value 0.
- $\mathcal{A}_f$: the set of all points x of $\mathcal{S}$ that are not collinear with a point having f-value $f(x) - 1$.
- M_f: the maximal value attained by f.

Clearly, $\mathcal{O}_f \subseteq \mathcal{A}_f$ and $M_f \in \{1, 2, \ldots, d\}$.

Characterizations of valuations

Theorem

Suppose f is a valuation of $\mathcal{S}$. Then:

- $M_f = 1$ *if and only if f is ovoidal;*
- $M_f = d - 1$ *if and only if f is semi-classical;*
- $M_f = d$ *if and only if f is classical;*
- $\mathcal{O}_f = \mathcal{A}_f$ *if and only if f is either classical or distance-j-ovoidal for some even j.*

Why are valuations useful?

Theorem

Let $\mathcal{S}$ be embedded as a full sub-$2d$-gon in a larger generalized $2d$-gon $\mathcal{S}'$. Let y be a point of $\mathcal{S}'$ at distance m from $\mathcal{S}$. For every point x of $\mathcal{S}$, we define

$$f_y(x) := d(x, y) - m.$$

Then f_y is a valuation of $\mathcal{S}$.

We call f_y the valuation of $\mathcal{S}$ induced by y.

The valuations of $GO(2,1)$

- There exists an easy way to determine all hyperplanes of a given point-line geometry with three points per line.
- $GO(2,1)$ has 92 hyperplanes.
- 12 of these hyperplanes are associated with valuations.

There are 3429 valuations which fall into 12 isomorphism classes: A, B1, B2, C1, C2, C3, C4, C5, C6, C7, D1, D2.

The valuations of $GO(2,1)$

Type	#	M_f	$\|\mathcal{O}_f\|$	$\|\mathcal{A}_f\|$	$\|H_f\|$	Type
A	45	4	1	1	29	classical
B1	90	3	1	9	21	semi-classical
B2	90	3	1	9	21	semi-classical
C1	720	2	1	13	17	–
C2	720	2	2	11	19	–
C3	720	2	2	11	19	–
C4	360	2	3	9	21	–
C5	180	2	1	13	17	–
C6	180	2	1	13	17	–
C7	36	2	5	5	25	distance-4-ovoidal
D1	144	1	15	15	15	ovoidal
D2	144	1	15	15	15	ovoidal

L-sets of valuations

Let $\mathcal{S}$ be a generalized $2d$-gon.

Let f_i, $i \in I$, be a collection of mutually distinct valuations of $\mathcal{S}$.

The set $\{f_i \mid i \in I\}$ is called an *L-set* of valuations if for every point x of $\mathcal{S}$, there exists a (necessarily unique) $i \in I$ such that $f_j(x) - M_{f_j} = f_i(x) - M_{f_i} + 1$ for every $j \in I \setminus \{i\}$.

Admissible L-sets of valuations

The set $\{f_i \mid i \in I\}$ is called an admissible set of valuations if the following holds for all $i_1, i_2 \in I$ with $i_1 \neq i_2$, for every $x \in \mathcal{A}_{f_{i_1}}$ and every $y \in \mathcal{A}_{f_{i_2}}$.

- If $M_{f_{i_1}} = M_{f_{i_2}} = d$, then $d(x, y) = 1$.
- If $x = y$, then $(f_{i_1}(x) - M_{f_{i_1}}) - (f_{i_2}(x) - M_{f_{i_2}}) \in \{-1, 0, 1\}$.
- If $x \neq y$ and at least one of $M_{f_{i_1}}$, $M_{f_{i_2}}$ is distinct from d, then $d(x, y) + f_{i_1}(x) + f_{i_2}(y) - M_{f_{i_1}} - M_{f_{i_2}} \geq -1$.

Importance of admissible L-sets

Theorem

Let $\mathcal{S}$ be embedded as a full sub-$2d$-gon in a larger generalized $2d$-gon $\mathcal{S}'$. For every point y of $\mathcal{S}'$, let f_y be the valuation of $\mathcal{S}$ induced by y.

- *If y_1 and y_2 are two distinct collinear points of $\mathcal{S}$, then $f_{y_1} \neq f_{y_2}$.*
- *For every line L of $\mathcal{S}$, the set $\{f_y \mid y \in L\}$ is an admissible L-set of valuations of $\mathcal{S}$.*

Admissible L-sets of $GO(2,1)$

$GO(2,1)$ has 45966 admissible L-sets which fall into 58 isomorphism classes. According to the types of the valuations they contain, we can distinguish 52 types of admissible L-sets.

[A,A,A], [A,B1,B1], [A,B2,B2], [B1,C1,C4], [B1,C2,C2], [B1,C3,C3], [B1,C4,C4], [B1,C5,C5], [B1,C5,C7], [B1,C6,C6], [B2,C1,C3], [B2,C2,C2], [B2,C4,C5], [B2,C6,C6], [C1,C1,C1], [C1,C1,C2], [C1,C1,C3], [C1,C1,C4], [C1,C1,C5], [C1,C1,C6], [C1,C1,D1], [C1,C1,D2], [C1,C2,C3], [C1,C2,C5], [C1,C2,C6], [C1,C2,D1], [C1,C2,D2], [C1,C3,C6], [C1,C3,D1], [C1,C4,D2], [C1,C5,C6], [C1,C5,D1], [C1,D1,D2], [C2,C2,C2], [C2,C2,C6], [C2,C2,D2], [C2,C3,D1], [C2,C4,D2], [C2,C5,D1], [C2,D1,D2], [C3,C3,C5], [C3,C5,D2], [C3,C6,D1], [C3,D2,D2], [C4,C6,D2], [C4,D1,D1], [C5,C5,C5], [C5,D2,D2], [C6,C6,C6], [C6,D1,D2], [C7,D2,D2], [D1,D1,D2]

The valuation geometry of $GO(2,1)$

Let $\mathcal{V}$ denote the following point-line geometry.

- The points of $\mathcal{V}$ are the valuations of $GO(2,1)$.
- The lines of $\mathcal{V}$ are the admissible L-sets of valuations of $GO(2,1)$.

The partial linear space $\mathcal{V}$ is called the valuation geometry of $GO(2,1)$.

Conclusion

In the sequel, $\mathcal{S}$ denotes an arbitrary generalized octagon of order $(2,4)$ that contains a suboctagon $GO(2,1)$ of order $(2,1)$.

Theorem

Let θ be the map which associates with each point x of $\mathcal{S}$ the associated valuation f_x of $GO(2,1)$. Then θ is a map between the point sets of $\mathcal{S}$ and $\mathcal{V}$ mapping lines of $\mathcal{S}$ to lines of $\mathcal{V}$.

For every line L of $\mathcal{S}$, we call $\theta(L)$ the line of $\mathcal{V}$ induced by L.

Two basic definitions

A point x of $\mathcal{S}$ is said to be of Type T if the valuation $\theta(x)$ has Type T.

A line L of $\mathcal{S}$ is said to be of Type $[T_1, T_2, T_3]$ if the admissible L-set $\theta(L)$ is of Type $[T_1, T_2, T_3]$.

The points of $\mathcal{S}$

Type T	$\mathcal{S}$	$\mathcal{V}$	N
A	45	45	1
B1	270	90	3
C2	720	720	1
C5	180	180	1
C6	360	180	2
C7	36	36	1
D1	144	144	1
Total	1755	–	–

N = The number of times a valuation of Type T is induced by a point of $\mathcal{S}$.

The lines of $\mathcal{S}$

Type $[T_1, T_2, T_3]$	$\mathcal{S}$	$\mathcal{V}$	N
[A,A,A]	30	30	1
[A,B1,B1]	135	45	3
[B1,C2,C2]	720	720	1
[B1,C5,C7]	180	180	1
[B1,C6,C6]	180	90	2
[C2,C2,C6]	720	720	1
[C2,C5,D1]	720	720	1
[C6,C6,C6]	240	240	1
Total	2925	–	–

N = the number of times an admissible L-set of Type $[T_1, T_2, T_3]$ is induced by a line of $\mathcal{S}$.

Uniqueness of the GO

The subgraph of the collinearity graph of $\mathcal{S}$ induced on the points of Type C2, C5, C7 and D1 is isomorphic to the subgraph of the collinearity graph of $\mathcal{V}$ induced on the points of Type C2, C5, C7 and D1.

The fact combined with the nonexistence of subgeometries that ordinary l-gons with $l < 8$ can be used to show that the whole of $\mathcal{S}$ is uniquely determined.

Other applications

Theorem (De Medts and Van Maldeghem, 2009)

There exists up to isomorphism a unique generalized hexagon of order 3 *containing a subhexagon of order* $(3, 1)$*, namely the split Cayley hexagon* $H(3)$*.*

Question: Can we also use the above techniques to prove that result?

Answer: Yes.

First of all, we need to construct the valuation geometry.

The valuation geometry $\mathcal{V}$ of $GH(3, 1)$

There are 8 types of points in $\mathcal{V}$: A, B1, B2, C1, C2, C3, C4 and C5.

Type	Max. value	Number
A	3	52
B1	2	312
B2	2	1872
C1	1	144
C2	1	432
C3	1	468
C4	1	936
C5	1	1872
Total	–	6088

The valuation geometry $\mathcal{V}$ of $GH(3,1)$

There are 309700 lines in $\mathcal{V}$ of 44 distinct types:

[A,A,A,A], [A,B1,B1,B1], [A,B1,B2,B2], [B1,B1,B1,B1], [B1,B2,B2,C5], [B1,B2,C3,C5], [B1,B2,C5,C5], [B1,C1,C1,C1], [B1,C1,C5,C5], [B1,C2,C2,C2], [B1,C2,C4,C4], [B1,C3,C3,C3], [B1,C3,C4,C5], [B1,C4,C5,C5], [B1,C5,C5,C5], [B2,B2,B2,C3], [B2,B2,C2,C3], [B2,B2,C3,C5], [B2,B2,C5,C5], [B2,C1,C3,C4], [B2,C1,C3,C5], [B2,C2,C3,C4], [B2,C2,C4,C5], [B2,C2,C5,C5], [B2,C3,C3,C3], [B2,C3,C4,C4], [B2,C3,C4,C5], [B2,C3,C5,C5], [B2,C4,C4,C5], [B2,C4,C5,C5], [B2,C5,C5,C5], [C1,C2,C2,C2], [C1,C3,C4,C4], [C1,C5,C5,C5], [C2,C2,C4,C5], [C2,C2,C5,C5], [C2,C3,C5,C5], [C2,C4,C5,C5], [C3,C4,C4,C4], [C3,C4,C5,C5], [C3,C5,C5,C5], [C4,C4,C5,C5], [C4,C5,C5,C5], [C5,C5,C5,C5].

The valuation geometry can be used to study generalized hexagons of order $(3, t)$ containing a subhexagon of order $(3, 1)$.

A generalized hexagon of order $(3, 3)$ containing a subhexagon of order $(3, 1)$ can only contain points of Type A and B.

Let $\mathcal{V}_1$ be the subgeometry of $\mathcal{V}$ consisting of all points of Type A and B, and all lines containing only points of Type A and B.

GH(3,3) containing a GH(3,1)

There are three types of points in $\mathcal{V}_1$: A, B1 and B2.

There are four types of lines in $\mathcal{V}_1$: [A,A,A,A], [A,B1,B1,B1], [A,B1,B2,B2], [B1,B1,B1,B1].

A generalized hexagon of order $(3,3)$ containing a subhexagon of order $(3,1)$ cannot contain points of Type B2 (such a point can only be incident with one line).

Let $\mathcal{V}_2$ be the subgeometry of $\mathcal{V}$ consisting of all points of Type A and B1, and all lines containing only points of Type A and B1.

The geometry $\mathcal{V}_2$ contains 364 points, the same number of points of a GH(3,3), and the set of B1 points is connected.

From this it can be derived that any $GH(3,3)$ containing a subhexagon of order $(3,1)$ must be isomorphic to $\mathcal{V}_2$ (and hence to $H(3)$).

Semi-finite generalized polygons

A generalized polygon of order (s, t) is called semi-finite if s is finite and t is infinite.

The question whether thick semi-finite generalized polygons exist is one of the most famous open problems in the theory of the generalized polygons.

A number of results have been obtained for generalized quadrangles (Cameron, Brouwer, Kantor, Cherlin).

Semi-finite generalized polygons

Theorem (BDB and Vanhove, 2013)

Let $s \in \mathbb{N} \setminus \{0,1\}$. Then there are no semi-finite generalized hexagons that contain a subhexagon of order (s, s^3).

Every $GH(2,2)$ is isomorphic to either $H(2)$ or its dual $H^D(2)$. Using valuations, the following can be proved:

Theorem (BDB)

There are no semi-finite $GH(2,t)$ which contain a subhexagon of order $(2,2)$.

Dense near polygons

A partial linear space is called a near polygon if for every point x and every line L there exists a unique point on L nearest to x. A near polygon is called dense if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours.

If x and y are two points of a dense near polygon at distance δ from each other, then x and y are contained in a unique convex subspace of diameter δ (Shult and Yanushka; Brouwer and Wilbrink).

The point-line geometry induced on every convex subspace is again a near polygon.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near polygon. A valuation of $\mathcal{S}$ is a map $f : \mathcal{P} \to \mathbb{N}$ satisfying the following properties:

(V1) There exists at least one point x with $f(x) = 0$.

(V2) Every line L contains a unique point x_L such that $f(x) = f(x_L) + 1, \forall x \in L \setminus \{x_L\}$.

(V3) Every point x of $\mathcal{S}$ is contained in a convex subspace F_x that satisfies the following properties:

- $f(y) \leq f(x)$ for every point y of F_x;
- every point z of $\mathcal{S}$ which is collinear with a point y of F_x and which satisfies $f(z) = f(y) - 1$ also belongs to F_x.

Importance of valuations

Theorem (BDB and Vandecasteele, 2005)

Suppose $\mathcal{S}$ is a dense near polygon which is fully and isometrically embedded as a subgeometry in another dense near polygon $\mathcal{S}'$. For every point x of $\mathcal{S}'$ and every point y of $\mathcal{S}$, we put $f_x(y) = \mathrm{d}(x, y) - m$ where $m := \mathrm{d}(x, \mathcal{S})$. Then f_x is a valuation of $\mathcal{S}$.

So, the valuations of $\mathcal{S}$ provide some information on how $\mathcal{S}$ can be fully and isometrically embedded as a subgeometry in another dense near polygon.

Applications of valuations

Theorem (BDB and Vandecasteele, 2007)

Up to isomorphism, there are 26 dense near octagons with three points per line.

Theorem (BDB, 2011; BDB and Vanhove, 201?)

There are 28 known dense near octagons with four points per line. Every other such near octagon must be the direct product of an unknown dense near hexagon with a line of size 4.

Other applications: Constructions of hyperplanes; classification of hyperplanes; study of isometric embeddings.