$K(\pi, 1)$ problem for Artin groups

Part I

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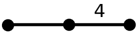

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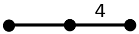

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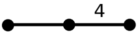

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$W = \mathfrak{S}_{n+1}$, symmetric group, where s_i is $(i, i+1)$.

Definition. If a, b are two letters and m is an integer ≥ 2, we set

$$\Pi(a, b : m) = \begin{cases} (ab)^{\frac{m}{2}} & \text{if } m \text{ is even} \\ (ab)^{\frac{m-1}{2}} a & \text{if } m \text{ is odd} \end{cases}$$

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Lemma 1. *Let Γ be a Coxeter graph.*
Then W_Γ has the following presentation.

$$W_\Gamma = \left\langle S \;\middle|\; \begin{array}{c} s^2 = 1 \text{ for all } s \in S \\ \Pi(s, t : m_{s,t}) = \Pi(t, s : m_{s,t}) \text{ for all } s, t \in S, \\ s \neq t,\; m_{s,t} \neq \infty \end{array} \right\rangle$$

Proof. It suffices to prove that the relation $(st)^m = 1$ is equivalent to the relation $\Pi(s, t : m) = \Pi(t, s : m)$ modulo the relations $s^2 = 1$ for all $s \in S$.

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Thanks to Lemma 1, the map $\Sigma \to S$, $\sigma_s \mapsto s$, induces an epimorphism $\theta : A_\Gamma \to W_\Gamma$.
The kernel of θ is the **colored Artin group** of Γ and is denoted by CA_Γ.

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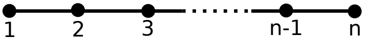

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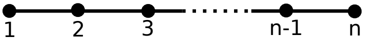

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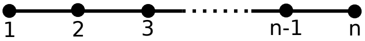

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The colored Artin group is the **pure braid group** $\mathcal{PB}_{n+1}$.

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(b) $\mathcal{A}$ is **locally finite** in I, that is, for all $x \in I$, there is an open neighborhood U_x of x in I such that the set $\{H \in \mathcal{A} \mid H \cap U_x \neq \emptyset\}$ is finite.

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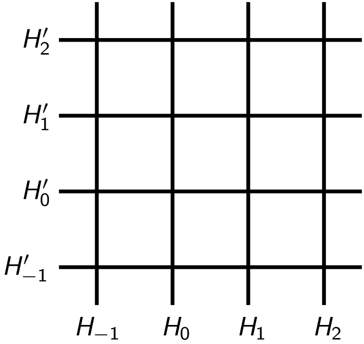

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A **wall** of $\bar{C}_0$ is the support of a (codimensional 1) face of $\bar{C}_0$, that is, a hyperplane of V generated by that face.

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The pair (W, S) is called a **Vinberg system** if $wC_0 \cap C_0 = \emptyset$ for all $w \in W \setminus \{1\}$.
In that case, the group W is called **reflection group** in Vinberg sense, S is called **canonical generating system** for W, and C_0 is called **fundamental chamber** of (W, S).

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(4) Let $x \in I$ be such that $W_x = \{w \in W \mid w(x) = x\}$ is different from $\{1\}$. Then there exists a reflection r in W such that $r(x) = x$.

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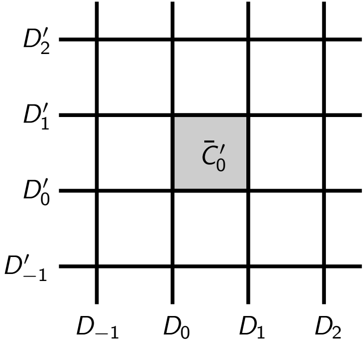

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