Zero sum problems in abelian groups and related extremal problems

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A meeting on operator theory, topology and combinatorics: Celebrating Bhaskar Bagchi's career at ISI Bangalore.

(Based on joint work with Eshita Mazumdar and Kevin Zhao)

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Bhaskar Bagchi

Bhaskar Bagchi, COO, IndiaHomes ...
indiahomesblog.wordpress.com


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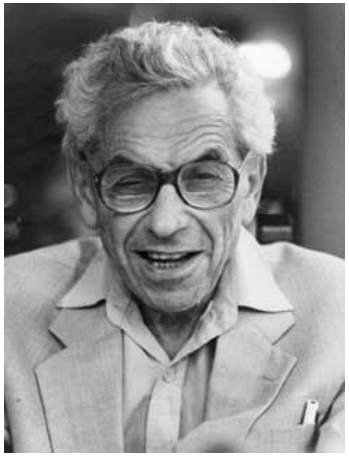
Bhaskar Bagchi, ISI Bangalore

An Erdős problem for 'Epsilons'



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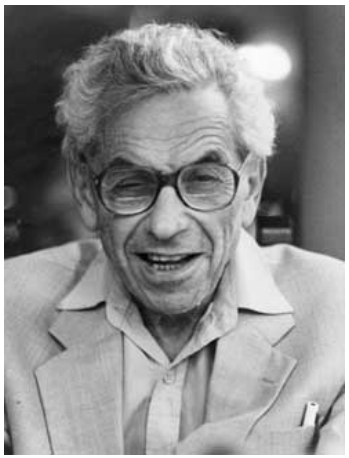

Given a sequence of n integers $(a_1, \ldots, a_n)$ there exists a nontrivial subsequence of these integers whose sum equals zero modulo n.

This is tight: $\underbrace{(1, \ldots, 1)}_{n-1 \text{ times}}$ does not have a nontrivial zero sum subsequence.

Zero sum problems in finite abelian groups

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The Davenport constant $D(G)$ of the group G is the smallest m such that every G-sequence of length m contains a non-trivial zero-sum G-subsequence.

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The Erdős problem $\Leftrightarrow D(\mathbb{Z}_n) = n$.

The Davenport constant in Algebraic Number theory

The Davenport constant is an important invariant of the ideal class group:

If R is the ring of integers of an algebraic number field and G its ideal class group, then $D(G)$ is the max no. of prime ideals occurring in the prime ideal decomposition of an irreducible in R.

Other (Combinatorial) Invariants:

- Erdős-Ginzburg-Ziv constant (EGZ), $\mathfrak{s}(G)$: $\min m \in \mathbb{N}$ such that every sequence of elements from G of length m contains a zero-sum subsequence of length $\exp(G)$.

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- Harborth constant, $\mathrm{g}(G)$: $\min\ m$ such that every *subset* of size m of G admits a zero-sum subsequence of size $\exp(G)$.

A Weighted Davenport constant

Notation: Suppose $\mathbf{x} = (x_1, \ldots, x_m)$ is a G-sequence and $\mathbf{a} = (a_1, \ldots, a_m)$ is a sequence of integers. Let $\mathbf{0}_m = \underbrace{(0, \ldots, 0)}_{m \text{ times}} \in \mathbb{Z}^m$.

$$\langle \mathbf{a}, \mathbf{x} \rangle := \sum_i a_i x_i.$$

Here $nx := \underbrace{x + \cdots + x}_{n \text{ times}}$ if $n > 0$, and for n negative, $nx := (-n)(-x)$.

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An equivalent description of the Davenport constant: Least m such that for every G-sequence $\mathbf{x} = (x_1, \ldots, x_m)$, there exists $\mathbf{e} = (\varepsilon_1, \ldots, \varepsilon_m) \in \{0, 1\}^m$ with $\mathbf{e} \neq \mathbf{0}_m$ (nontrivial $\mathbf{e}$) such that

$$\langle \mathbf{e}, \mathbf{x} \rangle = 0.$$

A Weighted Davenport constant - contd.

Theorem

(Adhikari, et al, 2006) Let $m = \lfloor \log_2 n \rfloor + 1$. *Then every* $\mathbb{Z}_n$*-sequence* $\mathbf{x} = (x_1, \ldots, x_m)$ *admits a nontrivial* $\mathbf{e} = (\varepsilon_1, \ldots, \varepsilon_m)$ *with* $\varepsilon_i \in \{-1, 0, 1\}$ *such that*

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$$D_{\pm 1}(\mathbb{Z}_n) = \lfloor \log_2 n \rfloor + 1.$$

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Definition

Suppose G is an abelian group. The Weighted Davenport constant of G w.r.t A, denoted $D_A(G)$, is the least k such that:

For any G-sequence $(x_1, \ldots, x_k)$, there exists a nontrivial $\mathbf{a} \in (A \cup \{0\})^k$ such that

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We may always assume $A \subset [1, n-1]$ where $n = \exp(G)$.

Weighted Davenport constant $D_A(G)$

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If $A = [1, p-1]$, then this is precisely the dimension n.

For arbitrary $A \subset [1, p-1]$, $D_A(G)$ measures how large a sequence of vectors in $\mathbb{F}_p^n$ can be, if the sense of 'independence' restricts the coefficients of the vectors to A.

Some known results

- $D_{\pm}(\mathbb{Z}_n) = \lfloor \log_2 n \rfloor + 1$. (Adhikari *et al*, 2006)
- $D_A(\mathbb{Z}_n) = 2$ for $A = \mathbb{Z}_n \setminus \{0\}$. (Adhikari *et al*, 2006)
- $D_A(\mathbb{Z}_n) = a + 1$ for $A = \mathbb{Z}_n^*$ where $a = \sum_{i=1}^{k} a_i$ and $n = p_1^{a_1} \cdots p_k^{a_k}$. (Griffiths, 2008)
- $D_A(\mathbb{Z}_n) = \lceil \frac{n}{r} \rceil$ if $A = \{1, \ldots, r\}$ for $1 \leq r \leq n-1$. (Adhikari, David, Urroz, 2006; Adhikari, Rath, 2008)

A related extremal problem

Suppose G is a finite abelian group with $\exp(G) = n$, and suppose $k \geq 2$ is an integer.

Definition

$$\begin{aligned} f_G^{(D)}(k) &:= \min\left\{|A| : \emptyset \neq A \subseteq [1, n-1] \text{ satisfies } D_A(G) \leq k\right\}, \\ &:= \infty \text{ if there is no such } A. \end{aligned}$$

Notation: If $G = \mathbb{Z}_n$, then denote $f_G^{(D)}(k)$ by $f^{(D)}(n, k)$.

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Natural extremal problem: Given a finite abelian group G, and $k \geq 2$,

$$\text{Determine } f_G^{(D)}(k).$$

The function $f_G^{(D)}(k)$

Proposition

If $G = \mathbb{Z}_p \times H$ is a finite abelian group with $p \nmid |H|$, then for any integer k, $f_G^{(D)}(n,k) \leq f^{(D)}(p,k)$. More generally, if $G = H_1 \times \cdots \times H_r$ where H_i is a p_i-group with $p_1 < \cdots < p_r$, then for all k,

$$f_G^{(D)}(k) \leq \min\left\{ f_{H_i}^{(D)}(k) : 1 \leq i \leq r \right\}.$$

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If $G_n = (\mathbb{Z}_p)^n$, $f_n := f_G^{(D)}(k)$, then $f_1 \leq f_2 \leq \cdots$

Theorem

Let p be a prime and $m \geq 1, k \geq 2$ be positive integers., Then for $G = \mathbb{Z}_{p^m}$,

$$p^{1/k} - 1 \leq f_G^{(D)}(k) = f^{(D)}(p, k).$$

Thus for all k,

$$f^{(D)}(p, k) = f_{\mathbb{Z}_{p^2}}^{(D)}(k) = f_{\mathbb{Z}_{p^3}}^{(D)}(k) = \cdots$$

Concrete Upper bounds for $f_G^{(D)}(k)$

Theorem

Let $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_s}$, where $1 < n_1 \mid \cdots \mid n_s$. Let $1 \leq r < (n-1)/2$, and let $A = \{\pm 1, \pm 2, \ldots, \pm r\}$. Then

$$1 + \sum_{i=1}^{s} \left\lceil \log_{r+1} n_i \right\rceil \geq D_A(G) \geq 1 + \sum_{i=1}^{s} \left\lfloor \log_{r+1} n_i \right\rfloor \text{ for } s \geq 2$$

$$D_A(\mathbb{Z}_n) = \left\lfloor \log_{r+1} n \right\rfloor + 1.$$

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Consequently, $p^{1/k} - 1 \leq f^{(D)}(p,k)$

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Consequently, $p^{1/k} - 1 \leq f^{(D)}(p,k) \leq 2(p^{1/(k-1)} - 1)$.

Almost tight upper bound for $f^{(D)}(p,k)$

Theorem

Let $k \geq 2$. There exists an integer $p_0(k)$ and an absolute constant $C = C(k) > 0$ such that for all prime $p > p_0(k)$

$$f^{(D)}(p,k) \leq C(p \log p)^{1/k}.$$

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So we have

$$p^{1/k} - 1 \leq f^{(D)}(p,k) \leq C(p \log p)^{1/k} \text{ for all sufficiently large } p.$$

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- *There exists an absolute constant* $C > 0$ *such that*

$$f^{(D)}(p, 4) \leq C p^{1/4}.$$

Something about the proofs

$f^{(D)}(p,k) \geq p^{1/k} - 1$:

Consider $\mathcal{G} = (V, E)$ with $V = \mathcal{X} \cup \mathcal{Y}$, where $\mathcal{X} = \{\mathbf{a} \in (A \cup \{0\})^k \setminus \{\mathbf{0}_k\}$ and $\mathcal{Y} = \{\mathbf{x} : x_i \neq 0, x_i \in \mathbb{Z}_p\}$ and $\mathbf{a} \leftrightarrow \mathbf{x}$ in $\mathcal{G}$ if and only if $\langle \mathbf{a}, \mathbf{x} \rangle = 0$. Let A be an optimal sized set.

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Fix $\mathbf{a} \in \mathcal{X}$, and wlog let $a_1 \neq 0$. For any $x_2, \ldots, x_k \in \mathbb{Z}_p^*$, the equation $a_1 x_1 = -(a_2 x_2 + \cdots + a_k x_k)$ admits a unique solution for $x_1 \in \mathbb{Z}_p$, so $\mathbf{a} \in \mathcal{X}$ has degree at most $(p-1)^{k-1}$. Hence $|\mathcal{X}|(p-1)^{k-1} \geq |E| \geq (p-1)^k$. Now use

$$|\mathcal{X}| = (|A| + 1)^k - 1.$$

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$f^{(D)}(p,k) \leq O((p \log p)^{1/k})$:

Theorem

Suppose $p \gg 0$ is a prime and A is a θ-random subset of $[1, p-1]$. Let $\omega(p), \omega'(p)$ be arbitrary functions satisfying $\omega(p), \omega'(p) \to \infty$ as $p \to \infty$.

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1. *If $\theta > \sqrt{\frac{2 \log p + \omega(p)}{p}}$, then whp $D_A(\mathbb{Z}_p) = 2$.*
2. *If $k \geq 3$ is an integer and θ satisfies*

$$\frac{(3kp(\log p + \omega(p)))^{1/k}}{p} < \theta < \frac{p^{1/(k-1)}}{p\ \omega'(p)},$$

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Janson's Inequality.

About the proofs: Why is the upper bound harder?

For sets $A, B \in \mathbb{Z}_p$ with $0 \notin B$, $\frac{A}{B} := \{\frac{a}{b} : a \in A, b \in B\}$.

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A **Difference Base** in a finite abelian group G is a set $B \subseteq G \setminus 0$ such that for every $g \neq 0$, $g = b - b'$ for some $b, b' \in B$. An upper bound for $f^{(D)}(p, 2)$ comes from a 'small' difference base for $\mathbb{Z}_p$.

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A difference base D is said to be **Perfect** if for every $g \neq 0$, there exists a unique pair $(b, b') \in D \times D$ s.t. $g = b - b'$.

Upper bound for $f^{(D)}(p, 2)$

Singer's theorem: There exists a Perfect Difference Set for $\mathbb{Z}_{p^2+p+1}$ of order $p+1$ for p prime.

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The best known upper bound for a size of a difference base in $\mathbb{Z}_n$ is $\frac{2}{\sqrt{3}}\sqrt{n} \leq 1.15471\sqrt{n}$ for $n \gg 0$. (Banakh-Gavrylkiv, 2017)

Upper bound for $f^{(D)}(p, 2k)$

For $A, B \subseteq \mathbb{Z}_p$, and $\alpha \in \mathbb{Z}_p^*$, $\alpha A := \{\alpha a : a \in A\}$, and $A + B := \{a + b : a \in A, b \in B\}$.

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Towards an optimal upper bound for $f^{(D)}(p, 2k)$: A set $A \subseteq \mathbb{Z}_p^*$ of 'optimal' size such that for any $\alpha_1, \ldots, \alpha_{k-1}, \beta_1, \ldots, \beta_{k-1} \in \mathbb{Z}_p^*$,

$$\mathbb{Z}_p^* \subseteq \frac{A + \alpha_1 A + \cdots + \alpha_{k-1} A}{(A + \beta_1 A + \cdots + \beta_{k-1} A)^*}.$$

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Theorem

There exists a set $A \subseteq \mathbb{Z}_p^*$ with $|A| \leq Cp^{1/4}$ for some absolute constant $C > 0$ s.t. for all $\alpha, \beta \in \mathbb{Z}_p^*$

$$\mathbb{Z}_p^* \subseteq \frac{A + \alpha A}{(A + \beta A)^*}.$$

Another related extremal problem

Suppose G is a finite abelian group with $\exp(G) = n$, and suppose $k \geq 2$ is an integer. Determine

$$\max \{D_A(G) : |A| = k, A \subset [1, n-1]\}.$$

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We know $D_A(\mathbb{Z}_n) = \lceil n/k \rceil$ if $A = \{1, \ldots, k\}$ for $1 \leq k \leq n-1$, so this maximum is at least $\lceil n/k \rceil$.

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Combinatorial Nullstellensatz. Also works for a 'list version' of this result.

The Harborth Constant

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This is not always well defined: If $G = \mathbb{Z}_{2n}$, $\exp(G) = 2n$. But

$$\sum_{x \in \mathbb{Z}_{2n}} x = n.$$

In these cases, we adopt the convention: $\mathsf{g}(G) = |G| + 1$.

Some known results on the Harborth constant

Theorem

(Marchan, Ordaz, Ramos, Schmid, 2013) Let $n \in \mathbb{N}$. We have

- $$\mathsf{g}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) = \begin{cases} 2n+2 & \text{if } 2 \mid n, \\ 2n+3 & \text{otherwise.} \end{cases}$$

- $$\mathsf{g}_{\pm 1}(\mathbb{Z}_n) = \begin{cases} n+1 & \text{if } n \equiv 2 \pmod 4, \\ n & \text{otherwise.} \end{cases}$$

- *If $n \geq 3$, then $\mathsf{g}_{\pm 1}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) = 2n+2$.*

Harborth constant for the dihedral groups

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Theorem

(B., Mazumdar, Zhao, 2018)
For any integer $n \geq 3$ and $G = D_{2n}$,

$$\mathrm{g}(G) = \begin{cases} n+2 & \text{if } 2 \mid n, \\ 2n+1 & \text{otherwise.} \end{cases}$$

Main ingredient lemmas

Lemma

Suppose n *is even and let* $s \geq 2$. *Let* $S = \{xy^{\alpha_1}, \ldots, xy^{\alpha_{2s}}\}$ *with* $\alpha_i \neq \alpha_j$. *Then*

$$\left|\prod\nolimits_{2s}(S)\right| \geq s.$$

If equality holds, then $2s$ *divides* n *and* $\{\alpha_1, \ldots, \alpha_{2s}\}$ *is a coset of the subgroup of* $\mathbb{Z}_n$ *of order* $2s$.

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Lemma

Suppose n is even and let $S = \{xy^{\alpha_1}, \ldots, xy^{\alpha_{2s+1}}\}$ with $\alpha_i \neq \alpha_j$. Then

$$\left|\prod_{2s+1}(S)\right| \geq s + 1.$$

If equality holds then $2s+2$ divides n and there is a coset K of the subgroup H of $\mathbb{Z}_n$ of order $2s+2$ such that $\{\alpha_1, \ldots, \alpha_{2s+1}\} \subset K$.

Open Questions

- Our conjecture:

Conjecture

$f^{(D)}(p,k) = \Theta(p^{1/k})$ *for all sufficiently large* p.

Stronger conjecture:

$$f^{(D)}(p,k) \leq (1+o(1))p^{1/k}.$$

- We believe

$$\max\{D_A(G) : |A| = k, A \subset [1, \exp(G) - 1]\} = \left\lceil \frac{|G|}{k} \right\rceil$$

holds for $G = \mathbb{Z}_n$, and also for $G = (\mathbb{Z}_n)^\ell$ for all n and all ℓ.

THANK YOU