Annual Foundational School Part II
May 7 - June 2, 2007
Indian Statistical Institute
8th Mile Mysore Road
Bangalore 560 059.

Lectures on complex analysis

Gadadhar Misra

Schwarz Lemma: Let $f : B(0,1) \to B[0,1]$ be holomorphic with $f(0) = 0$, where $B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$ and $B[0,1] = \{z \in \mathbb{C} : |z| \le 1\}$. Then

(α) $|f(z)| \le |z|$ for all $z \in B(0,1)$ and $|f'(0)| \le 1$;

(β) if there exists $z_0 \in B(0,1)\backslash\{0\}$ such that $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$, then f has the form $f(z) = cz$ for all $z \in B(0,1)$, where $|c| = 1$.

Proof: The function

$$g(z) := \begin{cases} f(z)/z, & z \in B(0,1)\backslash\{0\} \\ f'(0), & z = 0 \end{cases}$$

has a removable singularity at 0. Hence g is holomorphic on $B(0,1)$. For $|z| < r < 1$, we infer from the weak maximum principle:

$$|g(z)| \le \sup_{|w|=r} \{|g(w)|\} = \sup_{|w|=r} \left\{ \frac{|f(w)|}{r} \right\} \le \frac{1}{r},$$

and letting $r \to 1_-$, we get $|g(z)| \le 1$. Clearly, $|f'(0)| \le 1$.

In each of the two cases in (β), $|g|$ assumes its supremum in $B(0,1)$. So, by the maximum modulus theorem, g is a constant. $\square$

Theorem: For a fixed $\alpha \in B(0,1)$, the fractional linear transformation $\varphi_\alpha(z) := \frac{z-\alpha}{1-\bar{\alpha}z}$ is a one-one map, which maps the unit circle $\mathbb{T}$ onto $\mathbb{T}$, and $B(0,1)$ onto $B(0,1)$ and α to 0. The inverse of φ_α is $\varphi_{-\alpha}$. We also have

$$\varphi'_\alpha(0) = 1 - |\alpha|^2, \ \varphi'_\alpha(\alpha) = (1 - |\alpha|^2)^{-1}.$$

Proof: The map φ_α is holomorphic in the whole plane except for $z = 1/\bar{\alpha}$ which is outside $B[0,1]$. We see that $\varphi_{-\alpha}(\varphi_\alpha(z)) = z$. Thus φ_α is one-one and $\varphi_{-\alpha}$ is its inverse. Since for real t,

$$\left| \frac{e^{it} - \alpha}{1 - \bar{\alpha}e^{it}} \right| = \left| \frac{e^{it} - \alpha}{e^{-it} - \bar{\alpha}} \right| = 1;$$

φ_α maps $\mathbb{T}$ onto $\mathbb{T}$. The same is true of $\varphi_{-\alpha}$ hence $\varphi_\alpha(\mathbb{T}) = \mathbb{T}$. Now, maximum modulus principle shows that $\varphi_\alpha(B(0,1)) \subseteq B(0,1)$. So is $\varphi_{-\alpha}(B(0,1)) \subseteq B(0,1)$.

Suppose $\alpha \in B(0,1)$. How large can $|f'(\alpha)|$ be if $f : B(0,1) \to B(0,1)$ is holomorphic?

Put $g = \varphi_\beta \circ f \circ \varphi_\alpha$, where $f(\alpha) = \beta$. Since φ_β and φ_α map $B(0,1)$ onto itself, we see that g is a holomorphic map from $B(0,1)$ into $B(0,1)$. Also, $g(0) = 0$. Thus $|g'(0)| \leq 1$. But, using chain rule, we have

$$g'(0) = \varphi'_\beta(\beta)\ f'(\alpha)\ \varphi'_{-\alpha}(0)$$

which then gives the inequality

$$|f'(\alpha)| \leq \frac{1 - |f(\alpha)|^2}{1 - |\alpha|^2}.$$

Equality occurs in this inequality if and only if $g(z) = cz$, for some $c : |c| = 1$.

A remarkable feature: Thus the extremal solution $F_{\alpha,\beta}$ to the problem

$$\sup\{|f'(\alpha)|\ |f : B(0,1) \to B(0,1) \text{ is holomorphic and } f(\alpha) = \beta\},$$

$F_{\alpha,\beta}(z) = \varphi_{-\beta}(c\varphi_\alpha(z))$, is a rational function, although, no continuity assumption was made on f near the boundary.

Let $\mathrm{Hol}(B(0,1))$ denote the space of all holomorphic functions defined on the open unit disc $B(0,1)$.

Theorem: Suppose f is in $\mathrm{Hol}(B(0,1))$, it is one to one, and onto, $f(\alpha) = 0$ for some $\alpha \in B(0,1)$. Then there exists a constant $c : |c| = 1$ such that

$$f(z) = c\ \varphi_\alpha(z),\ z \in B(0,1).$$

Proof: Let g be the inverse of f, defined by $g(f(z)) = z,\ z \in B(0,1)$. Since f is one-one, f' has no zero in $B(0,1)$, so $g \in$ Hol $(B(0,1))$. By the chain rule, $g'(0)\ f'(\alpha) = 1$. We have $|f'(\alpha)| \leq \frac{1}{1-|\alpha|^2}$, and therefore $|g'(0)| \leq 1 - |\alpha|^2$. Since $g'(0)\ f'(\alpha) = 1$, it follows that equality must hold in these inequalities. As in the previous problem (with $\beta = 0$) this forces f to be of the form $c\ \varphi_\alpha$.

Lemma: Let $f : B(0,1) \to B(0,1)$ be holomorphic. Then for any $a, b \in B(0,1)$.

$$\left|\frac{f(a) - f(b)}{1 - \overline{f(a)}f(b)}\right| \leq \left|\frac{a - b}{1 - \bar{a}b}\right|.$$

In particular, $\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}$ for all $z \in B(0,1)$.

Proof: The function,

$$g(z) := \left(\frac{f(z)-f(b)}{1-\overline{f(b)}f(z)}\right) / \left(\frac{z-b}{1-\bar{b}z}\right)$$

is defined on $B(0,1)\backslash\{b\}$ is holomorphic at b as well. As in the usual Schwarz Lemma, $|g| \leq 1$ on $B(0,1)$ by the maximum principle, since $\left|\frac{z-b}{1-\bar{b}z}\right| \to 1$ as $|z| \to 1$, while $\left|\frac{f(z)-f(b)}{1-\overline{f(b)}z}\right| \leq 1$ for all $z \in B(0,1)$.

Riemannian Metric: Put on D, the Riemannian metric $ds = \varphi(z)|dz|$, $\varphi > 0$ and twice continuously differentiable. If $f : B(0,1) \to B(0,1)$ is holomorphic, then define the pull-back of $ds = \varphi(z)|dz|$ under f to be the metric

$$f^*(\varphi(z)|dz|) = |f'(z)|(\varphi \circ f)(z)|dz|$$

Then the Schwarz Lemma amounts to saying: A holomorphic map $f : B(0,1) \to B(0,1)$ is distance decreasing, that is, $f^*(ds) \leq ds$, where $ds = \frac{|dz|}{1-|z|^2}$.

Ahlfors Lemma: Let us start with a computation. For a holomorphic function f on an open set $\Omega \subseteq \mathbb{C}$ with $|f(z)| \leq M$,

$$\begin{aligned}
\Delta \log(M^2-|f|^2)^{-1} &= -4\frac{\partial^2}{\partial\bar{z}\partial z}\text{`}\log(M^2-|f|^2) \\
&= 4\frac{\partial}{\partial\bar{z}}\left(\frac{\bar{f}f'}{M^2-|f|^2}\right) \\
&= 4f'\left(\frac{\bar{f}'}{M^2-|f|^2} + \frac{\bar{f}\ f\bar{f}'}{(M^2-|f|^2)^2}\right) \\
&= 4|f'|^2\left(\frac{M^2-|f|^2+|f|^2}{(M^2-|f|^2)^2}\right) \\
&= 4\left(\frac{M|f'|}{M^2-|f|^2}\right)^2.
\end{aligned}$$

Near points where $f' \neq 0$, $\log|f'|$ is harmonic, that is, $\Delta\log|f'| = 0$. Hence

$$\Delta \log \frac{M|f'|}{M^2-|f|^2} = 4\left(\frac{M|f'|}{M^2-|f|^2}\right)^2$$

on the open set $\{z : f'(z) \neq 0\}$.

In particular,

(i) $\Delta \log \frac{r}{r^2-|z|^2} = 4\left(\frac{r}{r^2-|z|^2}\right)^2$ on $B(0,r)$;
(ii) $\Delta \log \frac{|f'|}{1-|f|^2} = 4\left(\frac{|f'|}{(1-|f|^2)}\right)^2$ on $B(0,1)\backslash\{z : f'(z) \neq 0\}$ for any holomorphic function $f : B(0,1) \to B(0,1)$.

Definition: The Gaussian curvature of a metric $ds = \varphi|dz|$, with $\varphi > 0$ and twice continuously differentiable on an open set $\Omega \subseteq \mathbb{R}^2 \cong \mathbb{C}$ is defined to be

$$K(z,\varphi) = -\varphi(z)^{-2}\ \Delta \log \varphi(z).$$

Thus any holomorphic function $f : \Omega \to B(0,M) = \{z : |z| < M\}$ provides a metric of constant negative curvature on $\Omega\backslash\{z : f^(z)\prime = 0\}$, namely

$$\frac{M|f'(z)|}{M^2 - |f(z)|^2}|dz|.$$

The basic observation of Ahlfors is the following.

Lemma: Let $\varphi \geq 0$ be a continuous function on $B(0,1) = \{z : |z| < 1\}$ which is twice differentiable on the open set $D_\varphi := \{z : \varphi(z) > 0\}$. Suppose $\Delta \log \varphi \geq 4\varphi^2$ on D_φ. Then

$$\varphi(z) \leq \frac{1}{1-|z|^2}, \text{ for all } z \in B(0,1).$$

Proof: Fix $\alpha \in B(0,1)$, and let $r \in (|\alpha|, 1)$. Put $p_r(z) = \frac{r}{r^2-|z|^2}$ on $B(0,r) = \{z : |z| < r\}$. Since $p_r(z) \to \infty$ as $|z| \to r$ and φ is continuous on $\{|z| < r\}$, it is clear that $\psi := \frac{\varphi}{p_r}$ attain its maximum on $B(0,r)$ at some $q \in B(0,1)_r$. If $\varphi(q) = 0$, then $\varphi \equiv 0$. Hence we may assume that $q \in D_\varphi$. Then q is also a local maximum of $\log \psi$, hence $\Delta \log \psi \leq 0$ at q. Now, at q:

$$\begin{aligned} 0 \geq \Delta \log \psi &= \Delta \log \varphi - \Delta \log p_r \\ &\geq 4(\varphi^2 - p_r^2), \end{aligned}$$

that is, $\psi(q) \leq 1$. Thus $\varphi \leq p_r$ on D_r. Letting $r \uparrow 1$, we conclude that $\varphi(\alpha) \leq \frac{1}{1-|\alpha|^2}$ as observed.

To get some immediate corollaries of the Ahlfors' Lemma, it is convenient to have the following Definition.

Definition: Let $\Omega \subseteq \mathbb{C}$ be an open set. Then $NC(\Omega)$ denotes the set of continuous functions $\varphi \geq 0$ on Ω such that φ is twice continuously differentiable on $D_\varphi := \{z : \varphi(z) > 0\}$ and $\Delta \log \varphi \geq 4\varphi^2$ on D_φ. Thus $NC(\Omega)$ consists of $\varphi \geq 0$ such that $\varphi|dz|$ is a metric whose Gaussian curvature is bounded by -4.

Remark: Let $f : \Omega \to \Omega'$ be a holomorphic maps of open sets in $\mathbb{C}$. Then $\varphi \in NC(\Omega')$ implies $|f'|\ \varphi(f) \in NC(\Omega)$. Ahlfors' Lemma has the following Corollaries.

Corollary: $NC(\mathbb{C}) = \{0\}$.

Proof: Let $\varphi \in NC(\mathbb{C})$. Fix $a \in \mathbb{C}$. Then for any $r > |a|$, Ahlfors' Lemma yields $\varphi(a) \leq \frac{r}{r^2 - |a|^2}$. Letting $r \to \infty$, we get $\varphi(a) = 0$ as asserted.

Corollary: Let $NC(\Omega) \neq \{0\}$. Then every holomorphic map $f : \mathbb{C} \to \Omega$ is constant.

Taking $\Omega = B(0,1)$, we get Liouville's theorem. Thus to prove Picard's theorem, for example, we need only show $NC(\Omega) \neq \{0\}$ for $\Omega = \mathbb{C} \backslash \{0,1\}$. This is rather easy to do. Consider

$$\varphi(z) = |z|^{\beta/2-1}|1 - z|^{\beta/2-1}(1 + |z|^\beta)(1 + |z - 1|)^\beta$$

for $\beta > 0$.

A straight-forward computation using the fact that

$$\Delta \log(1 + |f|^2) = \frac{4|f'|^2}{(1 + |f|^2)^2}$$

whenever f is a holomorphic function shows that $\varphi \in NC(\mathbb{C} \backslash \{0,1\})$ for $0 < \beta < \frac{2}{7}$. Thus we have proved Picard's theorem by elementary calculus!!

Here is another version of Ahlfors' Lemma which is easy to prove.

Ahlfors' Lemma Let $f : B(0,1) \to \Omega$ be a holomorphic function. Suppose ϱ is a metric on Ω with $K(z, \varrho) \leq -4$. Then

$$\varrho(f(z))|f'(z)| = f^*(\varrho)(z) \leq \frac{1}{1 - |z|^2} \text{ for } z \in B(0,1).$$

The following Proposition guarantees the existence of a metric in $\varphi \in NC(\mathbb{C} \backslash \{0,1\})$ with some additional properties which are essential to prove the Big Picard Theorem.

Proposition: There exists a metric ϱ on $\mathbb{C}\backslash\{0,1\}$ with the following properties:
(i) ϱ has curvature at most - 1.
(ii) $\varrho \geq c\sigma$ for some constant $c > 0$, where $\sigma(z) = (1+|z|^2)^{-2}$.

Proof:

$$\begin{aligned}
\Delta \log(1+|z|^\alpha) &= 4\frac{\partial^2}{\partial z \partial \bar{z}} \log(1+|z|^\alpha),\ z \in \mathbb{C}\backslash\{0\} \\
&= 4\frac{\partial}{\partial \bar{z}} \left(\frac{\frac{\alpha}{2} z^{\alpha/2-1} z^{-\alpha/2}}{1+|z|^\alpha} \right),\ z \in \mathbb{C}\backslash\{0\} \\
&= \alpha^2 \frac{|z|^{\alpha-2}}{(1+|z|^\alpha)^2},\ z \in \mathbb{C}\backslash\{0\}.
\end{aligned}$$

Since $\Delta \log |z|^\beta = 0,\ z \in \mathbb{C}$, we have

$$\Delta \log \frac{1+|z|^\alpha}{|z|^\beta} = \alpha^2 \frac{|z|^{\alpha-2}}{(1+|z|^\alpha)^2},\ z \in \mathbb{C}\backslash\{0\}.$$

Similarly,

$$\Delta \log\ \frac{1+|z-1|^\alpha}{|z-1|^\beta} = \alpha^2 \frac{|z-1|^{\alpha-2}}{(1+|z-1|^\alpha)^2}$$

for $z \in \mathbb{C}\backslash\{1\}$.

We consider the metric

$$\tau_{\alpha\beta}(z) = \frac{1+|z|^\alpha}{|z|^\alpha}\ \frac{1+|z-1|^\alpha}{|z-1|^\beta} \text{ for } z \in \mathbb{C}\backslash\{0,1\}.$$

Then the curvature $K(z, \tau_{\alpha\beta})$ is

$$\begin{aligned}
K(z, \tau_{\alpha\beta}) &= -\alpha^2 \frac{|z|^{\alpha+2\beta-2}}{(1+|z|^\alpha)^4}\ \frac{|z-1|^{2\beta}}{(1+|z-1|^\alpha)^2} \\
&\quad -\alpha^2 \frac{|z-1|^{\alpha+2\beta-2}}{(1+|z-1|^\alpha)^4}\ \frac{|z|^{2\beta}}{(1+|z|^\alpha)^2}.
\end{aligned}$$

For $\alpha = 1/3$ and $\beta = 5/6$, we see that
(i) $K(z,\tau) < 0$ for all $z \in \mathbb{C}\backslash\{0,1\}$
(ii) $\lim\limits_{z\to 0} K(z,\tau) = -1/36$
(iii) $\lim\limits_{z\to 1}\ K(z,\tau) = -1/36$

(iv) $\lim_{z\to\infty} K(z,\tau) = -\infty$.
Also,
(1) $\lim_{|z|\to\infty} \tau(z)/\sigma(z) \to \infty$.

Combining, (ii), (iii) and (iv) with the fact that $K(z,\tau) < 0$, which is (i), we find that there is a constant $k > 0$ such that

$$K(z,\tau) \le -k \text{ for } z \in \mathbb{C}\backslash\{0,1\}.$$

Thus if we define $\varrho(z) = \sqrt{k}\tau$ then $K(\varrho, z) \le -1$. From (1), we see that $\lim_{z\to\infty} \varrho(z)/\sigma(z) = \infty$, which is the second property.

Chain Rule:

$$\begin{aligned}
\frac{\partial}{\partial z}(f\circ g)(z) &= \frac{\partial f}{\partial z}(g(z))\,\frac{\partial g}{\partial z}(z) + \frac{\partial f}{\partial \bar{z}}(g(z))\,\frac{\partial \bar{g}}{\partial z}(z)\\
\frac{\partial}{\partial \bar{z}}(f\circ g)(z) &= \frac{\partial f}{\partial z}(g(z))\,\frac{\partial g}{\partial \bar{z}}(z) + \frac{\partial f}{\partial \bar{z}}(g(z))\,\frac{\partial \bar{g}}{\partial \bar{z}}(z).
\end{aligned}$$

Suppose f is holomorphic and non-zero.

$$\begin{aligned}
\Delta \log|f|^2 &= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\ \log|f|^2\\
&= 4\frac{\partial}{\partial z}\ \frac{\partial}{\partial \bar{z}}\ \log f + 4\frac{\partial}{\partial \bar{z}}\ \frac{\partial}{\partial z}\ \log f\\
&= 0.
\end{aligned}$$

Proposition: If Ω_1 and Ω_2 are two planar domains, and $h:\Omega_1 \to \Omega_2$ is a conformal map, ($h' \neq 0$) then

$$K(z, h^*\varrho) = K(h(z), \varrho),\ \ z \in \Omega_1,$$

where ϱ is a metric on Ω_2.

Proof: We need to calculate:

$$\begin{aligned}
K(z,h^*\varrho) &= -\frac{\Delta \log(\varrho(h(z)|h'(z)|)}{(\varrho(h(z))|h'(z)|)^2}\\
&= -\frac{\Delta \log \varrho(h(z)) - \Delta \log|h'(z)|}{\varrho(h(z))^2|h'(z)|^2}\\
&= -\frac{\Delta \log \varrho(h(z))\ |h'(z)|^2}{\varrho(h(z))^2\ |h'(z)|^2}\\
&= -\frac{\Delta \log \varrho(h(z))}{\varrho(h(z))^2} = K(h(z),\varrho).
\end{aligned}$$

Schottky's theorem: To each $M > 0$ and $r \in (0,1)$, there exists $C > 0$ with the following property:

If $f : B(0,1) \to \mathbb{C}$ is holomorphic, f omits any two values, and $|f(0)| \leq M$ then

$$\sup |f(z)| \leq C \text{ for } |z| \leq r.$$

Proof: If f omits the two values a, b then consider $F(z) = (f(z) - a)/(b - a)$ and assume without loss of generality, $a = 0,\ b = 1$. Let ϱ be the metric on $\mathbb{C}\backslash\{0,1\}$ we have constructed with $K(z,\varrho) \leq -4$. By Ahlfors' Lemma:

$$C^{-1} f^* \varrho(z) \leq C^{-1} \lambda(z),\ z \in B(0,1).$$

Let $\sigma(z) = z(1 + |z|^2)^{-1}$ be a metric on $\mathbb{C}$ with $K(z,\sigma) = 1$. Now, recall that

$$\varrho(z)/\sigma(z) \to \infty \text{ as } |z| \to \infty$$

and $\varrho \geq c\sigma$ on $\mathbb{C}\backslash\{0,1\}$. So,

$$f^*(\sigma) \leq c^{-1} f^*(\varrho) \leq c^{-1}\lambda,$$

that is,

$$(1 + |f(z)|^2)^{-1} |f'(z)| \leq c^{-1}(1 - |z|^2)^{-1},\ z \in B(0,1).$$

So, $\frac{|f'(z)|}{1+|f(z)|^2} \leq C_1,\ z \in B(0,1)_r$ where $C_1 = c^{-1}(1 - r^2)^{-1}$. Since f never takes the value zero, $t \to f(tz)$ is differentiable for any fixed $z \in B(0,r)$, and

$$\left| \frac{d}{dt}(arc\tan |f(tz)|) \right| \leq \frac{|f'(tz)||z|}{1 + |f(tz)|^2} \leq c_1.$$

So,

$$|arc\tan f(z) - arc\tan f(0)| \leq \int_0^1 |\frac{d}{dt}(arc\tan f(tz)|)|dt.$$

Therefore, it follows that

$$\begin{aligned} arc\tan |f(z)| &\leq c_1 + arc\tan |f(0)| \\ &\leq c_1 + arc\tan M. \end{aligned}$$

This proves the theorem with $c = c_1 + arc \tan M$.

Picard's theorem: If a holomorphic function f has an essential singularity then the range of f omits at most one complex number.

Proof: Assume that the singularity is at 0 and f is holomorphic on $B(0, e^{2\pi})\backslash\{0\}$.

Case-1: If $|f(z)| \to \infty$ as $z \to 0$ then there is nothing to prove since 0 is a pole.

Case-II: There exists z_n such that $z_n \to 0$ and $|f(z_n)|$ is bounded as $n \to \infty$, say $|f(z_n)| \leq M$ for all n.

Passing to a subsequence, assume

$$1 > |z_1| > \ldots > |z_n| > |z_{n+1}| \ldots \to 0.$$

For a fixed n, consider $F_n : \xi \mapsto f(z_n e^{2\pi i \xi})$.

Then F_n is holomorphic on $B(0,1)$, omits the values $\{0,1\}$ and $|F_n(0)| = |f(z_n)| \leq M$. Thus there exists $c > 0$ such that

$$|F_n(\xi)| = |f(z_n e^{2\pi i \xi})| \leq C \text{ for } \xi \in B(0, 1/2).$$

In particular,

$$|F_n(t)| \leq C \text{ for all } t \in [-1/2, 1/2].$$

Thus f is bounded by M on $\{z : |z| = |z_n|\}$. It follows that f is bounded on $B(0, |z_1|)\backslash\{0\}$. This will force 0 to be a removable singularity which is a contradiction.

Bi-holomorphic equivalence: For $w \in \Omega \subseteq \mathbb{C}^m$, let us define the tangent space of Ω at w to be

$$T_w(\Omega) := \{(w, v) : w \in \Omega\, v \in \mathbb{C}^m\} \cong \mathbb{C}^m.$$

Let $\mathrm{Hol}(\Omega, B[0,1])$ be the set of holomorphic functions defined on $\Omega \subseteq \mathbb{C}^m$ which take values in the closed unit disc $B[0,1]$. Define the Caratheodory metric C_Ω for Ω as

$$C_\Omega(w, v) = \sup\{|Df(w) \cdot v| : f \in \mathrm{Hol}(\Omega, B[0,1]),\ f(w) = 0\},\ (w, v) \in T_w(\Omega).$$

For nay holomorphic function $f : \omega \to \Omega'$, define the push-forward of the tangent vector (w, v) in $T_w(\Omega)$ under the function f to be $(f(w), Df(w) \cdot v)$ in $T_{f(w)}(\Omega')$.

Proposition : Suppose $\varphi : \Omega \to \Omega'$ is holomorphic. Then

$$C_{\Omega'}(\varphi(w), \varphi_*(v)) \leq C_\Omega(w, v), \ (w, v) \ \in T_w(\Omega)$$

that is, $D\varphi(w) : T_w(\Omega) \to T_{\varphi(w)}(\Omega')$ is a distance decreasing –

$$C_{\Omega'}(\varphi(w), D\varphi(w) \cdot v) \leq C_\Omega(v).$$

Proof: By the definition of the Caratheodory metric, we have

$$C_{\Omega'}(\varphi(w), (\varphi_*(v)) = \sup\{|(Df)(\varphi(w)) \cdot \varphi_*(v)| \ : f \in \mathrm{Hol}(\Omega', B(0,1)), \ f(\varphi(w)) = 0\}.$$

Now,

$$\begin{aligned} ((Df)(\varphi(w))) \cdot (D\varphi(w) \cdot v) &= [Df(\varphi(w))D\varphi(w)] \cdot v \\ &= (D(f \circ \varphi)(w)) \cdot v \end{aligned}$$

Since

$$\begin{array}{ccc} \Omega & \xrightarrow{f\circ\varphi} & \mathbb{D} \\ {\scriptstyle\varphi}\searrow & & \nearrow{\scriptstyle f} \\ & \Omega' & \end{array},$$

it follows that

$$\{f \circ \varphi \ : \ f \ \in \ \mathrm{Hol}(\Omega', B(0,1)), \ f(\varphi(w)) = 0\} \subseteq \{g : g \ \in \ \mathrm{Hol}(\Omega, B(0,1)), \ g(w) = 0\}$$

Remark: If φ is also invertible then

$$C_\Omega(\varphi^{-1}(\varphi(w)), (\varphi^{-1})_*(D\varphi(w) \cdot v)) \leq C_{\Omega'}(\varphi(w), (D\varphi(w) \cdot v).$$

But

$$\begin{aligned} \varphi_*^{-1}(D\varphi(w) \cdot v) &= D\varphi^{-1}(\varphi(w)) \ D\varphi(w) \cdot v \\ &= v. \end{aligned}$$

Therefore,

$$C_\Omega(w, v) \leq C_{\Omega^{prime}}(\varphi(w), D\varphi(w)\dot{v}) \leq C_\Omega(w, v).$$

Thus

$$C_\Omega(w, v) = C_{\Omega'}(\varphi(w), D\varphi(w) \cdot v).$$

This is the same as saying

$$D\varphi(w) : T_w(\Omega) \cong (\mathbb{C}^m, C_\Omega(w) \to (\mathbb{C}^m, C_{\Omega'}(\varphi(w)) \cong T_{\varphi(w)}(\Omega')$$

is an isometry.

Lemma: Suppose $\Omega = \{\alpha \in \mathbb{C}^m : \|\alpha\| < 1\}$ for some choice of a norm $\|\cdot\|$ on $\mathbb{C}^m$. Then $C_\Omega(0, v) = \|v\|$.

Proof: For $\|v\| \leq 1$, let us define $g_v : B(0,1) \to \Omega$ by $g_v(z) = z \cdot v$. Then $f \circ g_v$ defines a holomorphic map from $B(0,1)$ to itself with $f \circ g_v(0) = 0$. Therefore, $|(f \circ g_v)'(0)| \leq 1$ and we have

$$(f \circ g_v)'(0) = f'(g_v(0))g_v'(0) = f'(0) \cdot v.$$

Hence

$$\sup\{|Df(0) \cdot v| \ : \ f \in \mathrm{Hol}(\Omega, B(0,1))\, , \, f(0) = 0\} \leq 1.$$

Thus the linear map $Df(0) : T_w(\Omega) \cong \mathbb{C}^m \to \mathbb{C} \cong T_0(B(0,1))$ is in the unit ball of the dual space $(\mathbb{C}^m, \|\cdot\|)^*$.

Now, pick a linear functional ℓ on $(\mathbb{C}^m, \|\cdot\|)$, which is of norm at most 1, that is, $\ell \in (\mathbb{C}^m, \|\cdot\|)_1^*$. Then $\ell : \Omega \to B(0,1)$ by definition. Therefore $D\ell(0) = \ell$. In otherwords,

$$\{Df(0) \mid f \ \in, \mathrm{Hol}(\Omega, B(0,1)), \ f(0) = 0\} = (\mathbb{C}^m, \ \|\cdot\|)_1^*.$$

Let $\mathbb{B}^m := \{z \in \mathbb{C}^m : |z_1|^2 + \cdots |z_m|^2 < 1\}$ be the Euclidean ball and $(B(0,1))^m$ be the $m-$ fold cartesian product of the open unit disc $B(0,1)$. Suppose there exists a bi-holomorphic function $\varphi : \mathbb{B}^m \to (B(0,1))^m$. Then we may assume without loss of generality that $\varphi(0) = 0$. This will force $D\varphi(0)$ to be an isometry which is a contradiction.