Annual Foundational School, part II
MAY 7 – JUNE 2, 2007

Indian Statistical Institute
8th MILE, MYSORE ROAD
BANGALORE 560 059

Lectures on Complex Analysis

Gautam Bharali

1 Lecture 1

The initial stages of this course will be devoted to defining and investigating the properties of complex-analytic functions. Let us begin with a definition:

Definition 1.1 Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ a complex-valued function. We say that f is $\mathbb{C}$-*differentiable at a point* $a \in \Omega$ if the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{1}$$

exists. If it exists, we denote the limit by $f'(a)$. We say that f *is* $\mathbb{C}$-*differentiable on* Ω.

Remark 1.2 Note that whereas the limit (1) looks no different from that of the usual ($\mathbb{R}$)-derivative of calculus, there is a major difference: h is allowed to approach 0 *unrestrictedly.* Note that if we wrote

$$\begin{aligned} z &= x + iy, \quad x, y \in \mathbb{R} \\ f(z) &= u(x,y) + iv(x,y), \quad u, v : \Omega \to \mathbb{R} \end{aligned}$$

and restricted $h \to 0$ along the x-axis, the resultant limit, if it existed, would be

$$\lim_{\mathbb{R} \ni h \to 0} \frac{f(a+h) - f(a)}{h} = \frac{\partial u}{\partial x}(a) + i \frac{\partial v}{\partial x}(a).$$

However, as $\to 0$ unrestrictedly, we can say a lot more.

One immediate impact of allowing $h \to 0$ unrestrictedly is as follows. It is possible to construct functions $f : \Omega \to \mathbb{C}$ such that $\exists a \in \Omega$ such that

* $\frac{\partial u}{\partial x}(a),\ \frac{\partial u}{\partial y}(a),\ \frac{\partial v}{\partial x}(a),\ \frac{\partial v}{\partial x}(a)$ exist; but
* f is not even continuous at $z = a$.

(**Exercise:** Construct a $f : \Omega \to \mathbb{C}$ as above.) In contrast, if the limit (1) exists, we have the following proposition.

Proposition 1.3 *If $f : \Omega \to \mathbb{C}$ is $\mathbb{C}$-differentiable at $a \in \Omega$, then f is continuous at a.*

Proof: We need to show that $\lim_{z\to a} f(z) = f(a)$. Note that

$$0 \leq \lim_{z\to a} |f(z) - f(a)| = \left(\lim_{z\to a} \frac{|f(z)-f(a)|}{|z-a|}\right) \left(\lim_{z\to a} |z-a|\right).$$

The first equality is justified by the fact that — owing to $\mathbb{C}$-differentiability at $z = a$ — each of the limits forming the product exists. Hence $\lim\limits_{z\to a}(f(z) - f(a)) = 0$, which establishes continuity. $**$

All this raises the following question. We have; via Definition 1.1, invented a new notion of differentiability; but why is this at all interesting? There could be several answers to the question, some of which are:

(a) If $f : \Omega \to \mathbb{C}$ is $\mathbb{C}$-differentiable on Ω, then f is *conformal* on $\Omega\backslash(f')^{-1}\{0\}$. Conformality is a real-variables notion, and is of considerable importance in many areas of mathematics and science. But when $\Omega \subseteq \mathbb{C}$ (or, in general, Ω is an orientable 2-manifold), conformality is closely linked to complex analysis.

(b) If the open set $\Omega \subset \mathbb{C}$ is simply connected, then any harmonic function h — i.e. a real function $h : \Omega \to \mathbb{R}$ such that h satisfies the partial differential equation $\triangle h = 0$ on Ω — is the real part of some $\mathbb{C}$-differentiable function on Ω. This leads to a very fruitful interplay between two related areas.

(c) Despite the restrictiveness of Definition 1.1, a rich theory of $\mathbb{C}$-differentiable functions can be developed which is interesting in its own right. In this lecture, we plan to show that there are many examples of $\mathbb{C}$-differentiable functions. Constructing such functions relies on the following simple lemma.

Lemma 1.4 *Let Ω be an open subset of $\mathbb{C}$ and let $f, g : \Omega \to \mathbb{C}$ be $\mathbb{C}$-differentiable on Ω. Then*

$$\begin{aligned}
(cf)' &= cf',\ c \in \mathbb{C} \\
(f+g)' &= f' + g' \\
(fg)' &= f'g + fg' \\
(f/g)'(z) &= \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}\ \forall\ z \in \Omega \textit{ where } g(z) \neq 0.
\end{aligned}$$

The proof of the above is formally the same as its analogue in real differentiation. Before presenting the $\mathbb{C}$-differentiation analogue of the Chain Rule, let us give a definition and introduce some notation:

Definition 1.5 Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$. We say that f is *complex-analytic*, or *holomorphic*, on Ω if f is $\mathbb{C}$-differentiable on Ω and f' is continuous on Ω. We will denote the set of *all* functions on Ω that are holomorphic by $\mathcal{O}(\Omega)$.

Theorem 1.6 (The Chain Rule) *Let f and g be holomorphic on Ω_1 and Ω_2 respectively, and suppose $f(\Omega_1) \subset \Omega_2$. Then:*

$$(g \circ f)'(z) = g'[f(z)]\ f'(z)\ \forall\ z \in \Omega_1.$$

In particular, $g \circ f \in \mathcal{O}(\Omega_1)$.

Proof: Let us fix a $z_0 \in \Omega_1$. We must show that:

$$\lim_{z \to z_0} \frac{g[f(z)] - g[f(z_0)]}{z - z_0} = g'[f(z_0)]\ f'(z_0).$$

The proof of this falls into two cases:

CASE 1: $f'(z_0) \neq 0$.

Exercise: Show from the definition of $f'(z_0)$ that, if $f'(z_0) \neq 0$, then $\exists\ r_0 > 0$ sufficiently small such that $|z - z_0| < r_0 \Rightarrow z \in \Omega_1$, and

$$f(z) \neq f(z_0) \quad \forall\ z : 0 < |z - z_0| < r_0.$$

In what follows, we will use the notation:

$$\begin{aligned} D(z_0; r_0) &:= \text{ the disc in } \mathbb{C} \text{ with centre } z_0 \text{ and radius } r_0, \\ D(z_0; r_0)^* &:= \text{ the punctured disc } D(z_0; r_0) \backslash \{z_0\}. \end{aligned}$$

By the above exercise, we can write

$$\begin{aligned} &\lim_{D(z_0;r_0)^* \ni z \to z_0} \frac{g[f(z)] - g[f(z_0)]}{z - z_0} \\ = &\lim_{D(z_0;r_0)^* \ni z \to z_0} \frac{g[f(z)] - g[f(z_0)]}{f(z) - f(z_0)} \cdot \frac{f(z) - f(z_0)}{z - z_0} \\ = &\left[\lim_{D(z_0;r_0)^* \ni z \to z_0} \frac{g[f(z)] - g[f(z_0)]}{f(z) - f(z_0)}\right] \left[\lim_{D(z_0;r_0)^* \ni z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}\right]. \quad (2) \end{aligned}$$

The last equality is justified because each individual limit exists. The existence of the first factor relies on Proposition 1.3. By hypothesis, f is $\mathbb{R}$-differentiable, hence continuous. Thus

$$\lim_{D(z_0;r_0)^*\ni z\to z_0} [f(z)-f(z_0)] = 0,$$

where

$$\lim_{D(z_0;r_0)^*\ni z\to z_0} \frac{g[f(z)]-g[f(z_0)]}{f(z)-f(z_0)} = g'[f(z_0)]. \tag{3}$$

Combining (2) and (3), we get

$$\lim_{D(z_0;r_0)\ni z\to z_0} \frac{g[f(z)]-g[f(z_0)]}{z-z_0} = g[f(z_0)]\ f'(z_0).$$

CASE 2: $f'(z_0)=0$.
Since the limit

$$\lim_{w\to f(z_0)} \frac{g(w)-g[(z_0)]}{w-f(z_0)}$$

exists, $\exists\ C>0$ and $\epsilon>0$ such that

$$\left|\frac{g(w)-g[f(z_0)]}{w-f(z_0)}\right| \leq C \quad \forall\ w: 0<|w-f(z_0)|<\epsilon. \tag{4}$$

By continuity of f, $\exists r_1>0$ such that $|z-z_0|<r_1 \Rightarrow |f(z)-f(z_0)<\epsilon$. Applying this to (4), we see that

$$\lim_{D(z_0;r_1)^*\ni z\to z_0} \left|\frac{g[(z)]-g[f(z_0)]}{z-z_0}\right| \leq \lim_{D(z_0;r_1)^*\ni z\to z_0} C\left|\frac{f(z)-f(z_0)}{z-z_0}\right| = 0.$$

Hence:

$$\lim_{z\to z_0} \frac{g[f(z)]-g[f(z_0)]}{z-z_0} = 0 = g'[f(z_0)]\ f'(z_0).$$

From Cases 1 and 2, and the fact that z_0 was completely arbitrary, the result follows. $**$

Example 1.7 *By the application of the binomial theorem, it is elementary to check that* $f(z) := z^m,\ m\in\mathbb{N}$, *is complex-analytic in* $\mathbb{C}$.

In general, even if one is explicitly provided a $f : \Omega \to \mathbb{C}$, it could be hard to check where (1) exists. Is there a checkable criterion for holomorphicity ? This is answered by the next result.

Theorem 1.8 (The Cauchy-Riemann Conditions) *Let Ω be an open set in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$. Express f as*

$$f(x+iy) = u(x,y) + iv(x,y),$$

where u and v are the real and imaginary parts of f, and assume u and v have continuous partial derivatives. Then, f is complex-analytic on Ω if and only if:

$$\begin{aligned} \frac{\partial u}{\partial x}(x,y) &= \frac{\partial v}{\partial y}(x,y) \\ \frac{\partial u}{\partial y}(x,y) &= -\frac{\partial v}{\partial x}(x,y) \quad \forall\ x+iy \in \Omega \end{aligned} \tag{5}$$

Remark 1.9 The equations (5) are called the Cauchy-Riemann equations.

Proof: Let us first assume that f is complex-analytic. From the calculation in Remark 1.2, we see that

$$f'(z) = \frac{\partial u}{\partial x}(z) + i\frac{\partial v}{\partial x}(z) \quad \forall\ z \in \Omega. \tag{6}$$

However, let us now let $h \to 0$ along the imaginary axis. This gives us

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(z+ih) - f(z)}{ih} \\ &= \lim_{h\to 0} \frac{u(x,y+h) - u(x,y)}{ih} + i\frac{v(x,y+h) - v(x,y)}{ih} \\ &= -i\frac{\partial u}{\partial y}(z) + \frac{\partial v}{\partial y}(z). \end{aligned} \tag{7}$$

Comparing real and imaginary parts of (6) and (7) gives the Cauchy-Riemann equations.

Let us now assume that f satisfies (5). Let us write $h = s + it$. We express

$$u(x+s, y+t) - u(x,y) = [u(x+s, y+t) - u(x, y+t)] + [u(x, y+t) - u(x,y)].$$

By applying the Mean Value Theorem to the univariate functions $u(\bullet, y+t)$ and $u(x, \bullet)$ respectively, we see that $\exists\ \sigma : |\sigma| < |s|$ and $\exists\ \tau : |\tau| < |t|$ so that

$$\begin{array}{lll} u(x+s, y+t) - u(x, y+t) & = & u_x(x+\sigma, y+t)s \\ u(x, y+t) - u(x, y) & = & u_y(x, y+\tau)t. \end{array} \tag{8}$$

If we define $\Phi(s,t) := [u(x+s, y+t) - u(x,y)] - [u_x(x,y)s + u_y(x,y)t]$, we get, by applying (8

$$\frac{\Phi(s,t)}{h} = \frac{s}{s+it}[u_x(x+\sigma, y+t) - u_x(x,y)] + \frac{t}{s+it}[u_y(x, y+\tau) - u_y(x,y)].$$

Now, as $|\sigma| < |s|$ and $|\tau| < |t|$, $h \to 0 \Rightarrow (\sigma + i\tau) \to 0$. We now use the continuity of u_x and u_y to get

$$\lim_{h\to 0} \frac{\Phi(s,t)}{h} = 0. \tag{9}$$

By a similar argument, if we define

$$\Psi(s,t) := [v(x+s, y+t) - v(x,y)] - [v_x(x,y)s + v_y(x,y)t],$$

we get:

$$\lim_{h\to 0}\ \Psi(s,t)h = 0. \tag{10}$$

From (9) and (10), we have

$$\begin{aligned} \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h\to 0} \frac{u_x(x,y)s + u_y(x,y)t + \Phi(s,t)}{h} \\ &\quad + \lim_{h\to 0}\ i\ \frac{u_x(x,y)s + v_y(x,y)t + \Psi(s,t)}{h} \\ &= \lim_{h\to 0} \frac{(s+it)u_x(x,y) + i(s+it)v_x(x,y)}{h} \\ &= u_x(x,y) + iv_x(x,y). \end{aligned}$$

The second equality above follows from the Cauchy-Riemann conditions.

We have just shown that the $\mathbb{C}$-derivative of f exists at each $z \in \Omega$. But we have also shown that

$$f'(z) = \frac{\partial u}{\partial x}(z) + i\frac{\partial v}{\partial x}\ (x,y).$$

By our hypothesis on u and v, $f' \in \mathcal{C}(\Omega)$. Hence, the function f is complex-analytic on Ω. $**$

2 Lecture 2

From the material in Lecture 1, we see that any polynomial in z, $z \in \mathbb{C}$, is holomorphic in $\mathbb{C}$. But are there other examples of holomorphic (i.e. complex analytic) functions ? For this purpose, we consider power series. A *power series* is formally a series of the form $\sum_{n=0}^{\infty} c_n(z-a)^n$, where $a \in \mathbb{C}$ and $c_n \in \mathbb{C}$, $n \in \mathbb{N}$. To determine a function

$$f : z \longmapsto \sum_{n=0}^{\infty} c_n(z-a)^n$$

on some set $\Omega \subset \mathbb{C}$, $\Omega \neq \emptyset$, we must ensure that the series at $z = z_0$ is absolutely convergent for each $z_0 \in \Omega$. To determine when this is possible, we need to recall:

THE ROOT TEST: For the series $\sum_{n=0}^{\infty} a_n$ of complex numbers, define

$$\alpha := \limsup_{n\to\infty} |\alpha_n|^{1/n}.$$

(a) If $\alpha < 1$, then the series converges absolutely.

(b) If $\alpha > 1$, then the series diverges.

(c) If $\alpha = 1$, then the test provides no information.

Remark 2.1 It is easy to see that the Root Test may be used to find a disc $D(a;R)$ with the property that, *provided* $R > 0$, the given power-series converges absolutely at each $z \in D(a;R)$, and diverges at each $z \in \mathbb{C} \setminus \overline{D(a;R)}$. Associated to each $\sum\limits_{n=0}^{\infty} c_n(z-a)^n$, there is a unique such R associated to it, which is called its *radius of convergence.*

Exercise: Define

$$R = \left\{ \limsup_{n\to\infty} |c_n|^{1/n} \right\}^{-1}.$$

Show that $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges absolutely at each $z : |z-a| < R$ and diverges at each $z : |z-a| > R$. Also show that for any $r \in (0,R)$, the series is uniformly convergent on $\overline{D(a;r)}$.

We remind the reader that if X is a compact metric space and $\mathcal{C}(\mathrm{X};\mathbb{C}) :=$ the class of all complex-valued continuous functions on X, then $\|f\|_{\mathrm{X}} := \sup_{x\in\mathrm{X}} |f(x)|$, $f \in \mathcal{C}(X;\mathbb{C})$, is a norm on $\mathcal{C}(\mathrm{X};\mathbb{C})$. We say that $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{C}(\mathrm{X};\mathbb{C})$ converges uniformly to f if $\lim_{n\to\infty} \|f_n - f\|_{\mathrm{X}} = 0$. Thus, a series

$$\sum_{n=0}^{\infty} f_n, \quad f_n \in \mathcal{C}(\mathrm{X};\mathbb{C}),$$

is said to converge uniformly if the partial sums

$$S_N := \sum_{n=0}^{N} f_n \longrightarrow \sum_{n=0}^{\infty} f_n \text{ uniformly as } N \to \infty.$$

We now have the ingredients to show that any power series having a positive radius of convergence is holomorphic on its disc of convergence. This needs the following proposition.

Proposition 2.2 *Let $f(z) = \sum_{n=0}^{\infty} C_n(z-a)^n$, $z \in D(a;R)$, where R is the radius of convergence of the R.H.S. and $R > 0$. Then, the series*

$$\sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$$

also has radius of convergence R. Furthermore, f is $\mathbb{C}$-differentiable on $D(a;R)$ and

$$f'(z) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1}.$$

Proof: For the first part of this proposition, we would be done if we showed that

$$\limsup_{n\to\infty} \{n|c_n|\}^{1/(n-1)} = 1/R.$$

We already know that $\lim\limits_{n\to\infty} n^{1/(n-1)} = 1$. Thus, it suffices to show that

$$\limsup_{n\to\infty} |c_n|^{1/(n-1)} = 1/R$$

Let $\rho :=$ The radius of convergence of

$$\sum_{n=1}^{\infty} c_n (z-a)^{n-1} = \frac{f(z) - f(a)}{z-a}.$$

It is quite evident that

(a) $\sum_{n=1}^{\infty} c_n (z-a)^{n-1}$ converges absolutely at each $z : |z-a| < R$, whence $\rho \geq R$; and

(b) $\sum_{n=1}^{\infty} c_n (z-a)^{n-1}$ diverges at each $z : |z-a| > R$, whence $\rho \leq R$.

This tells us that

$$\limsup_{n\to\infty} |c_n|^{1/(n-1)} = 1/\rho = 1/R,$$

which proves the first part of this proposition.

As for the second part, for z such that $|z-a| < R$, set

$$\begin{aligned} g(z) &:= \sum_{n=1}^{\infty} n c_n (z-a)^{n-1}, \\ S_N(z) &:= \sum_{n=0}^{N} c_n (z-a)^n, \\ f(z) &\equiv S_N(z) + R_N(z) \quad \forall\, z \in D(a;R). \end{aligned}$$

Then, if we *fix* a $w \in D(a;R)$, we can write:

$$\begin{aligned} \frac{f(z)-f(w)}{z-w} - g(w) = \left\{ \frac{S_N(z) - S_N(w)}{z-w} - S_N'(w) \right\} + \{S_N'(w) - g(w)\} \\ + \left\{ \frac{R_N(z) - R_N(w)}{z-w} \right\} \quad (11) \end{aligned}$$

Note that if we select an $r \in (0,R)$ such that $z, w \in D(a;r)$, we have

$$\left| \frac{R_N(Z) - R_N(w)}{z-w} \right| = \left| \sum_{n=N+1}^{\infty} \sum_{k=0}^{n-1} c_n (z-a)^k (w-a)^{n-1-k} \right| < \sum_{n=N+1}^{\infty} n |c_n| r^{n-1}.$$

Suppose we are given an $\epsilon > 0$. Owing to the first part of this proposition, $\exists N_1 \equiv N_1(\epsilon) \in \mathbb{N}$ such that:

$$\left| \frac{R_N(z) - R_N(w)}{z - w} \right| < \sum_{n=N+1}^{\infty} |c_n| r^{n-1} < \frac{\epsilon}{3} \quad \forall N \geq N_1. \tag{12}$$

Furthermore:

$$\begin{aligned} |S'_N(w) - g(w)| &= \left| \sum_{n=N+1}^{\infty} n c_n (w-a)^{n-1} \right| \leq \sum_{n=N+1}^{\infty} n |c_n| r^{n-1} \\ &< \frac{\epsilon}{3} \quad \forall N \geq N_1. \end{aligned} \tag{13}$$

Finally, since we know that polynomials are $\mathbb{C}$-differentiable, if we *fix* an $N_0 \geq N_1(\epsilon)$, $\exists \delta \equiv \delta(\epsilon, w)$ such that

$$0 < |z - w| < \delta \Rightarrow \left| \frac{S_{N_0}(z) - S_{N_0}(w)}{z - w} - S'_{N_0}(w) \right| < \frac{\epsilon}{3}. \tag{14}$$

Putting (11) and (12)–(13) together, with $N = N_0 \geq N_1(\epsilon)$, we have

$$\left| \frac{f(z) - f(w)}{z - w} - g(w) \right| < 3 \times \frac{\epsilon}{3} \text{ whenever } 0 < |z - w| < \delta.$$

But this is simply the restatement of

$$\lim_{z \to w} \left\{ \frac{f(z) - f(w)}{z - w} - g(w) \right\} = 0,$$

which proves the second part of this proposition. $**$

But the above proposition establishes the following important fact.

Theorem 2.3 *Let the power-series $\sum_{n=0}^{\infty} c_n (z-a)^n$ have radius of convergence $R > 0$. Then, if we define*

$$f(z) := \sum_{n=0}^{\infty} c_n (z-a)^n, \quad z \in D(a; R),$$

then $f \in \mathcal{O}(D(a; R))$.

Proof: We have already established that f is $\mathbb{C}$-differentiable in Proposition 2.2. So, we must show that f' is continuous at each $z_0 \in D(a;R)$. But since each $z_0 \in \overline{D(a:r)}$ for some $r \in (O,R)$, we must just show that $f'|_{\overline{D(a;r)}} \in \mathcal{C}(\overline{D(a;r)};\mathbb{C})$. From the first part of the previous proposition, and the exercise in Remark 2.1, we have

$$\sum_{n=1}^{N} nc_n(z-a)^{n-1} \longrightarrow f' \text{ uniformly on } \overline{D(a,r)},$$

and since $\mathcal{C}(\overline{D(a;r)};\mathbb{C})$ is complete in the uniform norm, $f'|_{\overline{D(a;r)}}$ is continuous. Thus $f \in \mathcal{O}(D(a;R))$. $**$

We remark that from this point onwards, we shall abbreviate the expression "f is complex-analytic/holomorphic on Ω" by $f \in \mathcal{O}(\Omega)$. We have already used this notation in Theorem 2.3. This theorem tells us, among other things, that the following functions

$$\begin{aligned} e^z &:= \sum_{n=0}^{\infty} \frac{z^n}{n!}, \\ \cos z &:= \frac{e^{iz}+e^{-iz}}{2}, \\ \sin z &:= \frac{e^{iz}-e^{-iz}}{2}, \end{aligned}$$

are all holomorphic on $\mathbb{C}$.

2.4 The logarithm: This section illustrates the difficulties in defining inverse functions so that they are analytic. Suppose we want to define $\log(z)$ to be the inverse of e^z, then if we write $\log(z) = u(z)+iv(z)$, u, v real-valued, then we have

$$\exp[u(z)+iv(z)] = z.$$

Suppose, we write $z = |z|e^{i\mathsf{Arg}(z)}$, where we choose the argument so that $\mathsf{Arg}(z) \in [-\pi,\pi)$. Then

$$|z| = e^{u(z)}|e^{iv(z)}| = e^{u(z)},$$

whence $u(z) = \log|z|$. But there is no unique choice for $v(z)$ since the above calculation actually reveals:

$$\{w \in \mathbb{C} : \ e^w = z\} = \{\log|z| + i(\mathsf{Arg}(z)+2\pi k) : k \in \mathbb{Z}\}.$$

Suppose we make a *choice* of $k \in \mathbb{Z}$ to define $v(z)$ and set, for instance:

$$\log(z) := \log|z| + i\mathsf{Arg}(z)\ \forall z \in \mathbb{C} \setminus \{0\}, \tag{15}$$

we are led to the question: Is $\log \in \mathcal{O}(\mathbb{C} \setminus \{0\})$? Unfortunately, log defined in (15) is not even continuous! To see this, note that whereas, by the definition of Arg,

$$\log(-1) = -i\pi,$$

$$\begin{aligned} \lim_{t\to 0^+} \log(-1+it) &= \lim_{t\to 0^+} \left\{ \log\sqrt{1+t^2} + i\cos^{-1}\left(\frac{-1}{\sqrt{1+t^2}}\right)\right\} \\ &= +i\pi \neq \log(-1). \end{aligned}$$

This problem arises because the argument is multiple-valued. The problem is solved if we restrict Arg to take values in $(-\pi, \pi)$ (note the open interval). This amounts to restricting z to $G_0 := \mathbb{C} \setminus (-\infty, 0]$. Let the logarithm restricted to G_0 be denoted by Log, i.e.

$$\mathsf{Log}(z) := \log|z| + i\mathsf{Arg}(z) \quad \forall z \in G_0 := \mathbb{C} \setminus (-\infty, 0].$$

Log is known as the *the principal analytic branch of the logarithm.* Certainly $\mathsf{Log} \in \mathcal{C}(G_0; \mathbb{C})$, but why is it analytic? Continuity, it turns out; is the crucial property, from which analyticity follows in view of the following:

Exercise: Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be open subsets. Let $f \in \mathcal{C}(\Omega_1; \mathbb{C}), g \in \mathcal{O}(\Omega_2)$, and $f(\Omega_1) \subset \Omega_2$. Suppose

$$g[f(z)] = z \quad \forall z \in \Omega_1,$$

and $g'(w) \neq 0 \forall w \in \Omega_2$. Then $f \in \mathcal{O}(\Omega_1)$ and

$$f'(z) = \frac{1}{g'[f(z)]} \quad \forall z \in \Omega_1.$$

Each choice of $v(z)$ in the definition of $\log(z)$ in (15) leads to a different inverse of e^z. These different choices of the logarithm are called the *branches* of the logarithm. We then have the following result.

Proposition 2.5 *Let $\Omega \subset \mathbb{C}$ be an open and connected set, and let $F, G \in \mathcal{O}(\Omega)$ be two analytic branches of the logarithm. Then, $\exists k_0 \in \mathbb{Z}$ such that $F(z) = G(z) + 2\pi i k_0\ \forall z \in \Omega$.*

Proof: Let us define the function

$$\nu(z) := \frac{F(z) - \mathrm{G}(z)}{2\pi i} \quad \forall z \in \Omega.$$

Since $\{w \in \mathbb{C} : e^w = z\} = \{\log|z| + i(\mathsf{Arg}(z) + 2\pi k) : k \in \mathbb{Z}\}$, we see that $\nu(z) \in \mathbb{Z}\ \forall z \in \Omega$. But ν is clearly continuous. Hence, as Ω is connected, $\exists k_0 \in \mathbb{Z}$ such that $\nu(\Omega) = \{k_0\}$. Hence

$$F(z) - G(z) = 2\pi i k_0 \quad \forall z \in \Omega. \qquad **$$

3 Lecture 3

This lecture will be dedicated to developing the concept of the line integral of a complex-valued function. To begin with, let us describe the class of paths along which we shall perform integration. For this, we need a definition.

Definition 3.1 Let $[a, b] \subset \mathbb{R}$ be a closed, bounded interval. A function $\gamma : [a, b] \to \mathbb{C}$ is called a *piecewise smooth path* if we can find points

$$a = T_0 < T_1 < \cdots < T_M = b,$$

$M \geq 1$ such that

- $\gamma|_{(T_{j-1},T_j)}$ is differentiable on $(T_{j-1}, T_j),\ j = 1, \ldots, M$, with $[\gamma|_{(T_{j-1},T_j)}]' \in \mathcal{C}((T_{j-1}, T_j); \mathbb{C})$; and
- $[\gamma|_{(T_{j-1},T_j)}]'$ extends to a continuous function on $[T_{j-1}, T_j],\ j = 1, 2, \ldots, M$.

The above is the class of paths to which we shall restrict attention. We shall often refer to the image of a path $\gamma : [a, b] \to \mathbb{C}$ as a path too. We shall denote the image, i.e. $\gamma([a, b])$, by $\langle\gamma\rangle$.

To motivate the concept of a complex line integral, let us recall the concept of the work done by a vector field. Let Ω be an open set in $\mathbb{R}^2$ and let $\mathbf{F} = (P, Q) : \Omega \to \mathbb{R}^2$ be a continuous vector field. The *work done by* $\mathbf{F}$ *along a piecewise smooth path* $\gamma : [a, b] \to \Omega$ is defined as the following limit of Riemann sums

$$\text{work} \quad = \lim_{\|\mathbb{P}\|\to 0} \sum_{j=1}^{N} \langle \mathbf{F}(\gamma(\tau_j)), \gamma(t_j) - \gamma(t_{j-1})\rangle, \tag{16}$$

provided this limit exists. In (16), $\langle , \rangle$ represents the standard inner product on $\mathbb{R}^2$ — the “dot product” — while $\mathbb{P}$ denotes any partition

$$\mathbb{P} : a = t_0 < t_1 < \ldots < t_N = b,$$

with the mesh of this partition $\|\mathbb{P}\| := \max_{j\leq N}(t_j - t_{j-1})$. Also τ_j denotes any point such that $\tau_j \in [t_{j-1}, t_j]$. Now, if we write $\gamma = (\gamma_1, \gamma_2)$, then the R.H.S. (16) simplifies to

$$\text{work} = \lim_{\|\mathbb{P}\|\to 0} \left\{ \sum_{j=1}^{N} P(\gamma(\tau_j))[\gamma_1(t_j) - \gamma_1(t_{j-1})] + \sum_{j=1}^{N} Q(\gamma_2(t_j) - \gamma_2(t_{j-1})] \right\}$$

By the theory of the Riemann integral, if P and Q are continuous, then the above limit will exist, and:

I) This limit does not depend on the choice of $\tau_j \in [t_{j-1}, t_j]$. Furthermore;

II) If the limit in (16) exists, it will be unchanged if $\mathbb{P}$ is replaced by the refinement $\mathbb{P}^*_\gamma := \mathbb{P} \cup \{T_1, \dots, T_{N-1}\}$.

The fact that the limit in (16) exists can be justified exactly as when one shows that a continuous function on $[a, b]$ is Riemann integrable. Given this fact, let us exploit (I) and (II) to compute a value for the limit in (16), which we denote as

$$\text{work } = \int_\gamma \mathbf{F} \cdot d\mathbf{r}.$$

By Mean Value Theorem (recall our second hypothesis on γ), $\exists \tau^{1j}, \tau^{2j} \in (t_{j-1}, t_j)$ such that

$$\gamma_k(t_j) - \gamma_k(t_{j-1}) = \gamma'(\tau^{kj})\Delta t_j, \quad k = 1, 2,$$

(*provided* there is no $T_\mu \in (t_{j-1}, t_j)$, which is why we will have to now replace $\mathbb{P}$ by $\mathbb{P}^*_\gamma$). By (I) and (II), therefore:

$$\int_\gamma \mathbf{F} \boldsymbol{\cdot} d\mathbf{r} = \lim_{\|\mathbb{P}^*_\gamma\| \to 0} \left\{ \sum_{j=1}^{N^*} P(\gamma(\tau^{1j}))\gamma_1'(\tau^{1j})\Delta t_j + \sum_{j=1}^{N^*} Q(\gamma(\tau^{2j}))\gamma_2'(\tau^{2j})\Delta t_j \right\}, \tag{17}$$

where we relabel $\mathbb{P}^*_\gamma$ as $\mathbb{P}^*_\gamma : a = t_0 < t_1 < \cdots < t_{N^*} = b$.

Now, by our hypothesis on γ, the functions $(P \circ \gamma) \cdot \gamma_1'$ and $(Q \circ \gamma) \cdot \gamma_2'$ are Riemann integrable on $[T_{k-1}, T_k] \subset [a, b]$. Applying this to (17), we get

$$\int_\gamma \mathbf{F} \boldsymbol{\cdot} d\mathbf{r} = \sum_{k=1}^{M} \int_{T_{k-1}}^{T_k} \{P \circ \gamma(t)\gamma_1'(t) + Q \circ \gamma(t)\gamma_2'(t)\}\, dt.$$

If we are now given $\Omega \subset \mathbb{C}$ and a complex-valued function $f \in \mathcal{C}(\Omega; \mathbb{C})$, and a piecewise smooth path $\gamma : [a, b] \to \Omega$, we define the *line integral of f along γ*, written as

$$\int_\gamma f(z)dz$$

in a very similar manner to (16). The essential difference is that, rather than the inner product on $\mathbb{R}^2$, we utilise the product on $\mathbb{C}$. Thus, given $f \in \mathcal{C}(\Omega;\mathbb{C})$, we define

$$\int_\gamma f(z)dz = \lim_{\|P\|\to 0} \sum_{j=1}^{N} f(\gamma(\tau_j))(\gamma(t_j) - \gamma(t_{j-1})), \tag{18}$$

where, as before $\mathbb{P}$ is a partition

$$\mathbb{P} : a = t_0 < t_1 < \cdots < t_N = b,$$

and $\tau_j \in [t_{j-1}, t_j]$. The proof that the limit on the right-hand side above exists is exactly the same as the existence of the limit in (16). In fact, the reader may verify that the real and imaginary parts of the right-hand side of (18) are just real integrals of the form we discussed earlier. This last fact allows us to imitate the calculations above and obtain:

$$\int_\gamma f(z)dz = \sum_{k=1}^{M} \int_{T_{k-1}}^{T_k} f(\gamma(t))\gamma'(t)dt \tag{19}$$

Notice that (19) seems to suggest that if (γ) had a different parametrisation, say $\sigma : [c,d] \to \Omega$ such that $\sigma([c,d]) = \langle\gamma\rangle$, then

$$\int_\gamma f(z)dz = \int_\sigma f(z)dz \qquad (\text{with } \gamma \neq \sigma)$$

is not necessarily true. To address the truth or the falsity of this, we need to rigourously define what we mean by a reparametrisation.

Definition 3.2 Let $\gamma : [a,b] \to \mathbb{C}$ and $\sigma : [c,d] \to \mathbb{C}$ be two piecewise smooth paths. We say that σ is a reparametrisation of γ if $\exists \phi : [c,d] \to [a,b]$ such that ϕ is strictly increasing, continuously differentiable (with $\phi'(c)$ and $\phi'(d)$ being defined via right- and left-limits respectively) on $[c,d]$ such that $\sigma = \gamma \circ \phi$.

A question that arises is why we require ϕ to be strictly increasing. If, in our chosen class of paths, we write $\gamma \sim \sigma \Leftrightarrow \sigma$ is a reparametrisation of γ, and we allow non-strictly-increasing $\phi : [c,d] \to [a,b]$, then $\sim$ is not an equivalence relation.

Exercise: Show that if we replace "ϕ is strictly increasing" in Definition 3.2 by "ϕ is non-decreasing", then $\sim$ will not be an equivalence relation.

This discussion shows that in the quantity

$$\int_\gamma f(z)dz,$$

it is not enough to merely know the point-set $\langle\gamma\rangle$. The path $\gamma : [a,b] \to \Omega$, i.e. information on how $\langle\gamma\rangle$ is traversed, is crucial. However, regarding two paths that are reparametrisations of each other, we have the following result.

Proposition 3.3 *Let Ω be an open subset in $\mathbb{C}$ and $f \in \mathcal{C}(\Omega;\mathbb{C})$. Let $\gamma : [a,b] \to \Omega$ and $\sigma : [c,d] \to \Omega$ be piecewise smooth curves such that σ is a reparametrisation of γ. Then*

$$\int_\gamma f(z)dz = \int_\sigma f(z)dz.$$

Proof: By hypothesis, $\exists\phi : [c,d] \to [a,b]$ that is strictly increasing, continuosly differentiable and such that $\sigma = \gamma \circ \phi$. Clearly, if $S \in [a,b]$ is a point where γ fails to be differentiable, it is of the form $\phi(T)$, where $T \in [c,d]$ is a point where σ fails to be differentiable. So let

$$c = T_0 < T_1, < \ldots < T_M = d$$

be such that $\sigma|_{[T_{j-1},T_j]} \in \mathcal{C}^1([T_{j-1},T_j])$. Then

$$\begin{aligned}
\int_\sigma f(z)dz &= \sum_{j=1}^M \int_{T_{j-1}}^{T_j} f[\sigma(t)]\sigma'(t)dt \\
&= \sum_{j=1}^M \int_{T_{j-1}}^{T_j} f[\gamma\circ\phi(t)]\gamma'(\phi(t))\phi'(t)dt \\
&= \sum_{j=1}^M \int_{\phi(T_{j-1}}^{\phi(T_j)} f[\gamma(u)]\gamma'(u)du \\
&= \int_\gamma f(z)dz,
\end{aligned}$$

where the third equality follows from the change-of-variables formula — recall that by definition $\phi' > 0$ — with $u = \phi(t)$. $**$

In the end, we mention the following, which should be obvious: the line integral is linear in the integrand, i.e. if $\gamma : [a, b] \to \Omega$ is a piecewise smooth curve and $f, g \in \mathcal{C}(\Omega; \mathbb{C})$, then

$$\int_\gamma (\alpha f + \beta g)(z) dz = \alpha \int_\gamma f(z) dz + \beta \int_\gamma g(z),$$

for constants $\alpha, \beta \in \mathbb{C}$.

4 Lecture 4

This lecture is devoted to proving two powerful results about the behaviour of holomorphic functions. Our first result is Cauchy's integral fheorem. Our second result is a form of converse of Theorem 2.3. Namely: if Ω is an open set in $\mathbb{C}$ and $a \in \Omega$, then any $f \in \mathcal{O}(\Omega)$ can be expressed, in some disc around a, as a convergent power series. To appreciate how special this result is, we remind the reader that there exist functions $F \in \mathcal{C}^\infty(\Omega, \mathbb{C})$ such that the Taylor series of F around $a \in \Omega$ does not converge to F. We begin with Cauchy's theorem.

4.1 Cauchy's integral formula: *Let Ω be an open set in $\mathbb{C}$ and let $f \in \mathcal{O}(\Omega)$. Let $D \subset \Omega$ be any subdomain of Ω such that $\overline{D} \subset \Omega$, and ∂D is piecewise smooth. Then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} dw \quad \forall\ z \in D. \tag{20}$$

Before embarking on a proof we need to make two observations.

Remark 4.2 As a point set, ∂D is a union of disjoint closed curves. But, as the discussion in Lecture 3 shows, we need to specify parametrisations for each component of ∂D. By Proposition 3.3, it would suffice to prescribe directions of traversal — i.e. orientations — for each component of ∂D. For (20) to be true, we require ∂D to be *positively oriented with respect to* ∂D, i.e. each component must be so traversed that D lies to the left as one traverses the curve.

The above orientation of ∂D is necessiated by Green's Theorem, which we shall use to prove Theorem 4.1.

Theorem 4.3 (Green's theorem) *Let D be a bounded domain in $\mathbb{R}^2$ with piecewise-smooth boundary. Let $\mathbf{F} = (P, Q)$ be a continuously differentiable vector field on D such that the partial derivatives of P and Q extend continuously to ∂D. Then*

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

With these elements in place, we can provide

A proof of Cauchy's integral formula:

Let us fix a $z \in D$ and write

$$D_{z,\epsilon} := D \backslash D(z;\epsilon),$$

where $\epsilon > 0$ is so small that $\overline{D(z;\epsilon)} \subset D$. Let us write

$$\frac{f(w)}{w-z} \equiv U(w) + iV(w).$$

Then, referring back to Lecture 3, we can verify that

$$\begin{aligned}\int\limits_{\partial D_{z,\epsilon}} \frac{f(w)}{w-z} dw &= \int\limits_{\partial D_{z,\epsilon}} (U,V) \cdot d\mathbf{r} + i \int\limits_{\partial D_{z,\epsilon}} (V,U) \cdot d\mathbf{r} \\ &= -\int\limits_{D_{z,\epsilon}} \left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right) dA + i \int\limits_{D_{z,\epsilon}} \left(\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}\right) dA \quad (21)\end{aligned}$$

[using Green's Theorem]

Now, note that by construction $(U + iV) \in \mathcal{O}(D_{z,\epsilon})$. The Cauchy-Riemann conditions on $(U + iV)$, when applied to (21) give us

$$\begin{aligned} & -\int\limits_{\partial D_{z;\epsilon}} \frac{f(w)}{w-z} dw + \int_{\partial D} \frac{f(w)}{w-z} dw = 0 \\ \Rightarrow & \int_{\partial D} \frac{f(w)}{w-z} dw = \int_0^{2\pi} f(z + \epsilon e^{i\theta}) i \; d\theta. \end{aligned}$$

Note that the above is true $\forall\ \epsilon > 0$ sufficiently small. It is easy to see that if we define $f_\epsilon(\theta) := f(z + \epsilon e^{i\theta}),\ \theta \in [0, 2\pi]$, then

$$F_\epsilon \longrightarrow f(z) \text{ uniformly on } [0, 2\pi] \text{ as } \epsilon \to 0^+.$$

Hence, we have

$$\begin{aligned}\int_{\partial D} \frac{f(w)}{w-z} dw &= \lim_{\epsilon \to 0^+} \int_0^{2\pi} i f(z + \epsilon e^{i\theta}) d\theta \\ &= \int_0^{2\pi} \lim_{\epsilon \to 0^+} i f(z + \epsilon e^{i\theta}) d\theta \text{ [follows from uniform convergence]} \\ &= 2\pi i f(z),\end{aligned}$$

and the latter holds $\forall\ z \in D$. $**$

The integral formula immediately allows us to prove the following theorem.

Theorem 4.4 *Let Ω be an open set in $\mathbb{C}$ and $a \in \Omega$. Let $R > 0$ be such that $\overline{D(a; R)} \subset \Omega$. Then, f has a power-series representation*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \quad z \in D(a; R),$$

whose radius of convergence $\geq R$.

Proof: The hypothesis of Cauchy's integral formula is met by $D(a; r)$, whence

$$f(z) = \frac{1}{2\pi i} \int_{\partial D(a;R)} \frac{f(w)}{w-z} dw, \quad z \in D(a; R).$$

As $|z-a| < R = |w-a|$ in the above equation, we can write

$$\frac{1}{w-z} = \frac{1}{(w-a)\left(1 - \frac{z-a}{w-a}\right)} = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}}.$$

For a *fixed* $z : |z-a| < R$, the series [viewed as a function in $w \in \partial D(a; R)$]

$$\sum_{n=0}^{\infty} \frac{(z-a)^n}{(\cdot - a)^{n+1}}$$

is, by the Weierstrass M-test, uniformly convergent on $\partial D(a; R)$. Hence

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial D(a;R)} f(w) \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}} dw \\ &= \sum_{n=0}^{\infty} \left\{ \frac{1}{2\pi i} \int_{\partial D(a;R)} \frac{f(w)}{(w-a)^{n+1}} dw \right\} (z-a)^n \qquad (22) \\ &\equiv \sum_{n=0}^{\infty} c_n (z-a)^n. \end{aligned}$$

To complete this proof, we make the following auxiliary observation:

If $f \in \mathcal{C}(\Omega; \mathbb{C})$ and $\gamma : [a, b] \to \Omega$, then

$$\left| \int_\gamma f(z)\ dz \right| \leq \int_\gamma |f(z)|\ d|z|,$$

where the integral on the right is the integral with respect to the arc-length measure.

This follows by applying the Cauchy-Schwarz inequality to the relevant Riemann sum in Lecture 3. If $M := \sup_{w \in \partial D(a;R)} |f(w)|$, then, for any $z \in D(a, R)$:

$$\sum_{n=0}^{\infty} |c_n||z-a|^n \leq M \sum_{n=0}^{\infty} \left(\frac{|z-a|}{R} \right)^n < \infty.$$

By the definition of the radius of convergence, the above absolute-convergence statement tells us that r.o.c.$\left(\sum_{n=0}^{\infty} c_n (z-a)^n \right) \geq R$. $**$

The equation (22) leads to the following

Corollary 4.5 *Let Ω be an open set in $\mathbb{C}$ and $f \in \mathcal{O}(\Omega)$. If $R > 0$ is such that $\overline{D(a;R)} \subset \Omega$ then*

$$f^{(k)}(a) = \frac{k!}{2\pi i} \int\limits_{\partial D(a;R)} \frac{f(w)}{(w-a)^{k+1}} dw.$$

We shall end this lecture with the following observation: the *technique* of proving Theorem 4.1 allows us to prove a much more general theorem — one that does not merely pertain to $\mathcal{O}(\Omega)$-functions. We leave this generalization as an exercise.

Exercise: Let D be a bounded domain in $\mathbb{R}^2$ with piecewise-smooth boundary. Let $f = (u + iv) : D \to \mathbb{C}$ be continuously differentiable on D and assume that all first-order partial derivatives of u and v extend continuously to ∂D. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z} dw - \frac{1}{2\pi} \int \int_D \frac{1}{w-z} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f(w) dA(w)\ \forall z \in D.$$

Finally, note that we get the following Corollary to Cauchy's integral formula:

Theorem 4.6 (Cauchy's integral theorem) *Let the hypotheses on f and D be as in Cauchy's integral formula. Then*

$$\int_{\partial D} f(z)\ dz = 0.$$

Proof: We pick a point a in the interior of D and apply Theorem 4.1 to the holomorphic function $z \mapsto (z - a)f(z)$. $**$

5 Lecture 5

Now that we have the integral formula of Corollary 4.5 at our disposal, as well as the fact that holomorphic functions admit a power-series expansion locally, we can use them to deduce several results which demonstrate that holomorphic functions behave *very* differently from functions that are just finitely differentiable.

We begin with a generalization of an estimate that was already presented in the proof of Theorem 4.4. This is the so-called

5.1 Cauchy's estimate: *Let $\Omega \subset \mathbb{C}$ be an open set, and let $f \in \mathcal{O}(\Omega)$. Suppose $a \in \Omega$ and $R > 0$ is such that $\overline{D(a;R)} \subset \Omega$. Then*

$$|f^{(k)}(a)| \le \frac{k!}{R^k} \sup_{\zeta \in \partial D(a;R)} |f(\zeta)|.$$

Proof: From Corollary 4.5, we have

$$\begin{aligned} |f^{(k)}(a)| &= \frac{k!}{2\pi} \left| \int_{\partial D(a;R)} \frac{f(w)}{(w-a)^{k+1}} dw \right| \\ &\le \frac{k!}{2\pi} \int_{\partial D(a;R)} \frac{|f(w)|}{|w-a|^{k+1}} d|w| \\ &\le \frac{k!}{R^k} \sup_{\zeta \in \partial D(a;R)} |f(\zeta)|. \qquad ** \end{aligned}$$

One of the immediate applications of the above is Liouville's Theorem. Before stating this theorem, we introduce the following term: a function that is holomorphic on $\mathbb{C}$ is called an *entire function*.

5.2 Liouville's theorem: *A bounded entire function must be constant.*

Proof: Let $f \in \mathcal{O}(\mathbb{C})$ and let $M \in \mathbb{R}_+$ be such that

$$|f(z)| \le M \quad \forall z \in \mathbb{C}.$$

Pick any arbitrary $z_0 \in \mathbb{C}$. We have, for any $R > 0$, $\overline{D(z_0; R)} \subset \mathbb{C}$, whence, by Cauchy's estimate

$$|f'(z_0)| \leq \frac{1}{R} \sup_{\zeta \in \partial D(z_0; R)} |f(\zeta)| \leq \frac{M}{R}.$$

As the above is true $\forall R > 0$; it remains true as we take $R \to \infty$. Hence $f' \equiv 0$, since z_0 above was arbitrary. Since $\mathbb{C}$ is connected, this implies that $f \equiv$const. $**$

The power of Liouville's Theorem will be appreciated through the simple proof it provides of:

5.3 The fundamental theorem of algebra: *Let P be a non-constant polynomial having complex coefficients, of degree $N \geq 1$. Then, there exist complex numbers $a_1, \dots, a_N$ such that*

$$P(z) = c \prod_{j=1}^{N} (z - a_j) \quad (c \in \mathbb{C} \backslash \{0\}).$$

Proof: The proof would be complete via induction on degree and the division algorithm if we could show that P — as above — has at least one zero in $\mathbb{C}$. Assume not. Then $1/P \in \mathcal{O}(\mathbb{C})$. Let us write

$$P(z) = c_0 z^N + \sum_{j=1}^{N} c_j z^{N-j}, \quad c_0 \neq 0.$$

It is easy to find an $R > 0$ such that

$$|c_0|\ |z|^N - \sum_{j=1}^{N} |c_j|\ |z|^{N-j} \geq \frac{|c_0|}{2}\ |z|^N \quad \forall z : |z| \geq R.$$

This implies that

$$|1/P(z)| \leq \frac{2}{|c_0||z|^N} \leq \frac{2}{|c_0| R^N} \quad \forall z : |z| \geq R. \tag{23}$$

As $1/P$ is continuous, $\exists M > 0$ such that

$$|1/P(z)| \leq M \quad \forall z : |z| \leq R, \tag{24}$$

where R is as introduced above. But, from (23) and (24), we have

$$|1/P(z)| \leq \max\left(M, \frac{2}{|c_0|R^N}\right) \quad \forall z \in \mathbb{C}.$$

Since $1/P \in \mathcal{O}(\mathbb{C})$, by assumption, P must be constant, by Liouville's Theorem. This contradicts our hypothesis on P. Hence, our assumption is false, and $P(z) = 0$ has at least one root in $\mathbb{C}$. $**$

Of course, polynomials form a very small sub-class of the set of entire functions. And there is certainly no analogue of the previous theorem for non-polynomial entire functions. For example:

$$f(z) = e^{az}, \quad a \neq 0,$$

is entire, but $f(z) \neq 0 \ \forall z \in \mathbb{C}$. Yet, the analogue of the Fundamental Theorem of Algebra for a general $f \in \mathcal{O}(\Omega)$ is that if $f^{-1}\{0\} \neq \emptyset$, then it *must be a discrete set in* Ω or vanish on an entire component of Ω. This is made more precise in the following theorem.

Theorem 5.4 *Let Ω be a connected open set in $\mathbb{C}$ and let $f \in \mathcal{O}(\Omega)$. Then, the following are equivalent:*

a) $f \equiv 0$;

b) $\exists a_0 \in \Omega$ such that $f^{(k)}(a_0) = 0 \ \forall k \in \mathbb{N}$;

c) $f^{-1}\{0\}$ has a limit point in Ω.

Proof: (a) $\Rightarrow$ (c) is obvious.

Let us now show that (c) $\Rightarrow$ (b): Let a be a limit point of $f^{-1}\{0\}$ lying in Ω. Since f is continuous, $f(a) = 0$. Suppose $\exists N \in \mathbb{Z}_+$ such that

$$\begin{aligned} f^{(k)}(a) &= 0 \quad \forall k < N, \\ f^{(N)}(a) &\neq 0. \end{aligned}$$

Let $R > 0$ such that $\overline{D(a; R)} \subset \Omega$. Then, in this disc, we have the power-series development

$$f(z) = \sum_{n=N}^{\infty} c_n (z-a)^n \equiv (z-a)^N g(z), \quad (c_N \neq 0) \quad \forall\, z \in D(a; R). \tag{25}$$

The statements in (25) follow from our assumptions on $f^{(k)}$. In particular, we see that $g \in \mathcal{C}(D(a;R);\mathbb{C})$ and that $\exists r \in (0,R)$ such that $g(z) \neq 0 \ \forall z \in D(a;r)$. We see from (25) thus that

$$f(z) \neq 0 \quad \forall z \in D(a;r) \backslash \{a\},$$

which violates the fact that a is a limit of $f^{-1}\{0\}$. Our assumption about $\{f^{(k)}\}_{k\in\mathbb{N}}$ must hence be wrong. Thus (c) $\Rightarrow$ (b).

It now remains to show that (b) $\Rightarrow$ (a). This is where we use the connectedness of Ω. Define

$$\begin{aligned} A &= \{z \in \Omega : f^{(k)}(z) = 0 \ \forall k \in \mathbb{N}\} \\ &= \bigcap_{k\in\mathbb{N}} [f^{(k)}]^{-1}\{0\}; \end{aligned}$$

the continuity of $f^{(k)}$ for each $k \in \mathbb{N}$ establishes that A is closed in Ω. $A \neq \emptyset$ by (b). Finally, pick $z_0 \in A$. Let $\rho > 0$ such that $\overline{D(z_0;\rho)} \subset \Omega$. Then, f has the power-series development

$$f(z) = \sum_{k\in\mathbb{N}} \frac{f^{(k)}(z_0)}{k!} \, (z - z_0)^k \quad \forall z \in D(z_0;\rho).$$

Consequently, $f|_{D(z_0;\rho)} \equiv 0$, i.e. $D(z_0;\rho) \subset A$. Since z_0 was an arbitrary point in A, we conclude that A is open.

Thus A is a non-empty set that is both Ω-open and Ω-closed. As Ω is connected, $A = \Omega$. This establishes (a). $**$

6 Lecture 6

In this lecture, we introduce an object whose purpose is twofold:

Objective I: The Cauchy Integral Formula, as formulated in Theorem 4.4, involved integration over a disjoint union of simple closed curves. We would like to extend the formula to closed curves that *self-intersect*, with appropriate modifications to the resulting formula.

Objective II: To develop a theory of enumeration of the zeros of a holomorphic function — specifically: if $f \in \mathcal{O}(\Omega)$, $\alpha \in f(\Omega)$, and $a \in \Omega$ is a zero of $f(z) = \alpha$ of multiplicity M_α, to provide a theory for how the number of roots of the equation

$$f(z) = \beta, \quad \beta \in f(\Omega) \text{ and close to } \alpha,$$

are related to M_α.

We will not have the time to explore Objective II within this set of seven lectures, but we must emphasize one of the outcomes of pursuing Objective II — which will be exploited in *other* lectures in this workshop:

Theorem 6.1 (Open Mapping Theorem) *Let Ω be an open connected set in $\mathbb{C}$, and let $f \in \mathcal{O}(\Omega)$. If f is non-constant, then f is an open mapping, i.e. for any open set $U \subset \Omega$, $f(U)$ is open.*

Before providing a formal definition of the object that we have hinted at, we present a short lemma.

Lemma 6.2 *Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth closed curve and let $\alpha \notin \langle\gamma\rangle$. Then,*

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z - \alpha} \in \mathbb{Z}$$

Proof: We use the notation-developed in Lecture 3, i.e. let

$$a = T_0 < T_1 < \cdots < T_M = b,$$

$M \geq 1$ be such that:

- $\gamma|_{(T_{j-1},T_j)}$ is differentiable on $(T_{j-1},T_j),\ j=1,\ldots,M$, with $[\gamma|_{(T_{j-1},T_j)}]' \in \mathcal{C}((T_{j-1},T_j);\mathbb{C})$; and
- $[\gamma|_{(T_{j-1},T_j)}]'$ extends to a continuous function on $[T_{j-1},T_j],\ j=1,2,\ldots,M$.

Define:

$$g(t) = \begin{cases} 0, & \text{if } t=a, \\ \sum\limits_{k=1}^{j-1} \int\limits_{T_{k-1}}^{T_k} \frac{\gamma'(s)}{\gamma(s)-\alpha} ds + \int\limits_{T_{j-1}}^{t} \frac{\gamma'(s)}{\gamma(s)-\alpha} ds, & \text{if } t \in (T_{j-1},T_j), \end{cases}$$

where γ' in each integrand is interpreted appropriately.

If we *fix* $t \in (T_{j-1},T_j)$, by the Fundamental Theorem of Calculus

$$\{g|_{(T_{j-1},T_j)}\}'(t) = [\gamma(t)-\alpha]^{-1}[\gamma|_{(T_{j-1},T_j)}]'(t).$$

Then

$$\begin{aligned} \left.\frac{de^{-g}(\gamma-\alpha)}{dt}\right|_{t\in(T_{j-1},T_j)} &= e^{-g(t)}\left\{\gamma|_{(T_{j-1},T_j)}\right\}'(t) - \left\{g|_{(T_{j-1},T_j)}\right\}'(t)(\gamma-\alpha)(t) \\ &= 0. \end{aligned}$$

This tells us that

$$e^{-g}(\gamma-\alpha)\big|_{(T_{j-1},T_j)} \equiv \text{const}.$$

However, note that g is visibly a continuous function, whence:

$$e^{-g}(\gamma-\alpha) \equiv \text{const}.$$

Hence:

$$e^{-g(b)}(\gamma-\alpha)(b) = e^{-g(a)}(\gamma-\alpha)(a) = (\gamma-\alpha)(a).$$

Since γ is a closed curve and $\alpha \notin \langle\gamma\rangle$, the above gives us

$$e^{-g(b)} = 1 \tag{26}$$

Note that by construction, $g(b)$ is just the desired integral. Thus, by (26), $\exists k \in \mathbb{Z}$ such that

$$\int_\gamma \frac{1}{z-\alpha} dz = 2\pi i k. \qquad **$$

We are now ready to give a definition.

Definition 6.3 Let $\gamma : [0,1] \to \mathbb{C}$ be a piecewise smooth, closed curve in $\mathbb{C}$, and let $a \notin \langle\gamma\rangle$. The *winding number of* γ *around* a, denoted by $\eta(\gamma; a)$, is the integer

$$\eta(\gamma; a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}dz.$$

The question arises: what is the geometric interpretation of the winding number ? To answer this, let us examine several evidences for the answer. First, suppose we fix an $a \in \mathbb{C}$ and let $\widehat{\gamma} : [0,1] \to \mathbb{C}\setminus\{a\}$ be a piecewise smooth, closed curve, *which is allowed to intersect itself*, such that

- $\widehat{\gamma}$ wraps around a anticlockwise exactly once; and
- $\widehat{\gamma}$ crosses the ray $\{z \in \mathbb{C} : (z-a) \in (-\infty, 0]\}$ exactly once.

Now some terminology: If Ω is an open set in $\mathbb{C}$ and $f \in \mathcal{O}(\Omega;\mathbb{C})$, let us say that f *admits a primitive on* Ω if $\exists F \in \mathcal{O}(\Omega)$ such that $F' = f$.

Exercise: Show that if $f \in \mathcal{O}(\Omega;\mathbb{C})$ and admits a primitive F on Ω, then

$$\int_\gamma f(z)dz = F[\gamma(b)] - F[\gamma(a)]$$

for any piecewise smooth $\gamma : [a,b] \to \Omega$.

Let us now return to the $\widehat{\gamma}$ introduced above. Let $\beta \in \mathbb{C}$ be such that

$$\{\beta\} = \langle\widehat{\gamma}\rangle \cap \{z \in \mathbb{C} : (z-a) \in (-\infty, 0]\};$$

and we may assume, without loss of generality, that $\widehat{\gamma}(0) = \widehat{\gamma}(1) = \beta$. Let us write $\widehat{\gamma}_\epsilon := \widehat{\gamma}|_{[\epsilon, 1-\epsilon]}$; whence

$$\langle\widehat{\gamma}_\epsilon\rangle \subset \mathbb{C}\setminus\{z \in \mathbb{C} : (z-a) \in (-\infty, 0]\}.$$

Note that $\mathsf{Log}(z-a)$, defined by $\mathsf{Log}(z-a) := \log|z-a| + i\mathsf{Arg}(z-a)\forall z \in \mathbb{C}\setminus\{z : (z-a) \in (\infty, 0]\}$, is a primitive of $1/(z-a)$ on $\Omega := \mathbb{C}\setminus\{z \in \mathbb{C} : (z-a) \in (-\infty, 0]\}$. Thus, in view of our exercise:

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\widehat{\gamma}} \frac{1}{z-a}dz &= \lim_{\epsilon\to 0^+} \frac{1}{2\pi i}\int_{\widehat{\gamma}_\epsilon} \frac{1}{z-a}dz \\
&= \lim_{\epsilon\to 0^+} \frac{1}{2\pi i}\left[\mathsf{Log}(\gamma(1-\epsilon)) - \mathsf{Log}(\gamma(\epsilon))\right] \\
&= 1,
\end{aligned}$$

i.e. the winding number counts precisely the number of times $\widehat{\gamma}$ maps around a.

The next piece of evidence requires the following constructions: let $\gamma, \sigma : [0,1] \to \mathbb{C}$ be two piecewise smooth closed curves with $\gamma(1) = \sigma(0)$. Then, we write

$$\gamma^{-1}(t) := \gamma(1-t), \quad t \in [0,1],$$

and

$$\gamma * \sigma(t) = \begin{cases} \gamma(2t), & \text{if } t \in [0, 1/2], \\ \sigma(2t-1), & \text{if } t \in [1/2, 1]. \end{cases}$$

Note that $\gamma * \sigma$ is piecewise smooth and is just the juxtaposition of γ and σ, with σ following γ. Then, we have the following easy

Lemma 6.4 *Let $\gamma, \sigma : [0,1] \to \mathbb{C}$ be two piecewise smooth closed paths with $\gamma(1) = \sigma(0)$. Then:*

$$\begin{aligned} \eta(\gamma^{-1}; a) &= -\eta(\gamma; a) \quad \forall a \notin \langle\gamma\rangle, \\ \eta(\gamma * \sigma; a) &= \eta(\gamma; a) + \eta(\sigma; a) \quad \forall a \notin \langle\gamma\rangle \cup \langle\sigma\rangle. \end{aligned}$$

From these two pieces of evidence, we intuit that, at least for a special class of curves:

$\eta(\gamma; a)$ = *the number of times γ wraps around a* (with positive sign indicating anticlockwise traversal and negative sign indicating clockwise traversal), i.e. the number of *complete* rotations executed by the vector $[\gamma(t) - a]$ as t varies from 0 to 1.

The above can be proved in many different ways, but the slickest proof involves the concept of homotopy. We defer this discussion to Lecture 7.

As for Objective I, as stated right at the beginning of this lecture, we can now state the strengthened version of the integral formula.

Theorem 6.5 (Cauchy's integral formula – Version 2) *Let Ω be an open set in $\mathbb{C}$ and let $f \in \mathcal{O}(\Omega)$. Let $\gamma : [0,1] \to \Omega$ be a piecewise smooth closed curve in Ω. Let*

$$\eta(\gamma; w) = 0 \quad \forall w \in \mathbb{C} \setminus \Omega. \qquad (*)$$

Then,

$$\eta(\gamma; a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz \quad \forall a \in \Omega \setminus \langle\gamma\rangle.$$

We will take up the proof of this in our next lecture. However, note the following:

Remark 6.6 If might seem like $(*)$ represents the imposition of an additional hypothesis, which is not needed in Theorem 4.4. But the impression is incorrect. If $D \subset \Omega$ is a subdomain with ∂D piecewise smooth, and $\overline{D} \subset \Omega$ — as in the hypothesis of Theorem 4.4 — then it is easy to check [assuming, provisionally, the geometric interpretation of the winding number stated above] that $(*)$ is automatically true for ∂D. Furthermore, Theorem 6.5 is *false* without the hypothesis $(*)$.

7 Lecture 7

We begin this lecture with a proof of Version 2 of Cauchy's integral formula — stated in the previous lecture. We will then introduce the important concept of homotopy. Among other applications of homotopy: we will provide a rigorous justification of the term "winding number" for $\eta(\gamma; a)$. But first, let us study

7.1 The proof of Theorem 6.5: Let us define the function $\varphi : \Omega \times \Omega \to \mathbb{C}$ as

$$\varphi(z, w) = \begin{cases} \frac{f(z)-f(w)}{z-w}, & \text{if } z \neq w, \\ f'(z), & \text{if } z = w. \end{cases}$$

Since we are given that $f \in \mathcal{O}(\Omega)$, $\varphi \in \mathcal{C}(\Omega \times \Omega; \mathbb{C})$. Furthermore, for each fixed $w \in \Omega$, the function

$$\varphi_w : z \mapsto \varphi(z, w)$$

is holomorphic on Ω. The reader may verify this by explicitly computing the $\mathbb{C}$-derivative

$$(\varphi_w)'(z) = \begin{cases} \frac{f'(z)(z-w)-[f(z)-f(w)]}{(z-w)^2}, & \text{if } z \neq w, \\ \frac{f''(z)}{2}, & \text{if } z = w. \end{cases}$$

Next, define $\mathcal{H} := \{w \in \mathbb{C} : \eta(\gamma; w) = 0\}$. Note that by our hypothesis on γ, we have the following

$$\begin{aligned} \mathbb{C} \backslash \Omega \subset \mathcal{H}&, \\ \Omega \cup \mathcal{H} &= \mathbb{C}. \end{aligned}$$

Now define $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) = \begin{cases} \int_\gamma \varphi(z, w) dw, & \text{if } z \in \Omega \\ \int_\gamma \frac{f(w)}{w-z} dw, & \text{if } z \in \mathcal{H}. \end{cases}$$

Note that

$$\begin{aligned} \int_\gamma \varphi(z, w) dw &= \int_\gamma \frac{f(z) - f(w)}{z - w} \, dw \\ &= -2\pi i f(z)\eta(\gamma; z) + \int_\gamma \frac{f(w)}{w - z} dw \\ &= \int_\gamma \frac{f(w)}{w - z} dw \quad \forall z \in \Omega \cap \mathcal{H}. \end{aligned}$$

This demonstrates that g is well-defined. We now need the following lemma from the real-variable calculus:

Lemma: *Let $\Theta : W \times [a,b] \to \mathbb{C}$ be a continuous function and W an open subset of $\mathbb{R}^N$. Define:*

$$g(x) := \int_a^b \Theta(x,t)\ dt.$$

Then, $g \in \mathcal{C}(W;\mathbb{C})$. If $\partial\Theta/\partial x_j \in \mathcal{C}(W \times [a,b];\mathbb{C})$, then

$$\frac{\partial g}{\partial x_j}(x) = \int_a^b \frac{\partial \Theta}{\partial x_j}(x,t)\ dt.$$

Taking $\Theta(z,t) = \varphi(z,\gamma(t))\gamma'(t)$ OR $\Theta(z,t) = f(\gamma(t))\ \gamma'(t)(\gamma(t)-z)^{-1}$, in the relevant case, over the appropriate regions, and applying the above lemma, we compute that

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) g(z) = 0,$$

i.e. that the Cauchy-Riemann conditions are satisfied. Thus, g is entire. It is evident that $\eta(\gamma;\cdot) \equiv 0$ on the unbounded component of $\mathbb{C}\backslash\langle\gamma\rangle$. Thus,

$$\begin{aligned}\lim_{z\to\infty} |g(z)| &\leq \lim_{z\to\infty} \int_\gamma \frac{|f(w)|}{|w-z|}\ d|w| \\ &\leq \lim_{z\to\infty}\ \sup_{w\in\langle\gamma\rangle} |f(w)| \int_\gamma \frac{d|w|}{|z|-|w|} = 0. \qquad (27)\end{aligned}$$

Thus, $\exists R_0 > 0$ such that $|g(z)| \leq 1\ \forall z : |z| \geq R_0$. So, if we let $M := \sup_{|z|\leq R_0} |g(z)|$, then

$$|g(z)| \leq \max(M,1) \quad \forall z \in \mathbb{C}.$$

By Liouville's Theorem, therefore $g \equiv$ const. By(27), $g \equiv 0$. Hence if we now fix an $a \in \Omega\backslash\langle\gamma\rangle$,

$$\begin{aligned} 0 &= \int_\gamma \frac{f(a)-f(w)}{a-w}\ dw \\ &= -2\pi i\ f(a)\eta(\gamma;a) + \int_\gamma \frac{f(w)}{w-a}\ dw. \qquad ** \end{aligned}$$

For precisely the same reasons as Theorem 4.6, we get

7.2 The Cauchy Integral Theorem (Version II): *Let Ω be an open subset in $\mathbb{C}$ and $f \in \mathcal{O}(\Omega)$. Let $\gamma : [0,1] \to \Omega$ be a piecewise closed curve in Ω such that $\eta(\gamma; w) = 0$ $\forall w \in \mathbb{C} \backslash \Omega$. Then*

$$\int_\gamma f(z)\ dz = 0.$$

Our next refinement involves the concept of homotopy.

Definition 7.3 Let $\gamma_j : [0,1] \to \Omega,\ j = 0, 1,$ be piecewise smooth closed curves in Ω. We say that γ_0 *is homotopic to* γ_1 *in* Ω, written $\gamma_0 \sim_\Omega \gamma_1$, if there exists a continuous function $H : [0,1] \times [0,1] \to \Omega$ such that

$$\begin{aligned} H(\cdot, j) &= \gamma_j, \quad j = 0, 1, \\ H(0, t) &= H(1, t) \quad \forall t \in [0,1]. \end{aligned}$$

Geometrically, the existence of a homotopy between γ_0 and γ_1 signifies that γ_0 can be continuously distorted to γ_1 through a family of of paths $\{H(\cdot, t)\}_{t \in [0,1]}$ which stay in Ω. We say that a piecewise smooth closed path $\gamma : [0,1]$ is *null homotopic in* Ω, written $\gamma \sim_\Omega 0$, if there exists a homotopy $H : [0,1] \times [0,1] \to \Omega$ with $H(\cdot, 0) = \gamma$ and $H(\cdot, 1) \equiv$ const. This concept allows us to deduce the following refined version of Theorem 7.2:

7.4 The homotopy form of Cauchy's integral theorem: *Let Ω be an open subset of $\mathbb{C}$ and let $f \in \mathcal{O}(\Omega)$. Let $\gamma_j : [0,1] \to \Omega, j = 0, 1,$ be piecewise smooth closed curves in Ω such that $\gamma_0 \sim_\Omega \gamma_1$. Then*

$$\int_{\gamma_0} f(z)\ dz = \int_{\gamma_1} f(z)\ dz.$$

In particular, if $\gamma_0 \sim_\Omega 0$, then

$$\int_{\gamma_0} f(z)\ dz = 0.$$

We omit the proof of the above because it will involve several complications:

- The proof of Theorem 7.4 itself is most efficiently given using a compactness+connectedness argument in which the geometric flavour of the result gets effaced; and
- The definition of homotopy *does not* require each curve $H(\bullet, t)$, $t \in (0,1)$ to be piecewise smooth. But one would require this in order to be able to define integrals along $H(\bullet, t)$ while proving Theorem 7.4. In principle, if γ_0 and γ_1 are piecewise smooth and $\gamma_0 \sim_\Omega \gamma_1$, *we can* construct a homotopy $\tilde{H}$ such that $\tilde{H}(\bullet, t)$ are piecewise smooth $\forall t \in (0,1)$. This is rather technical (although not hard).

One of the most useful consequences of Theorem 7.4 requires a preliminary definition:

Definition 7.5 Let Ω be an open subset in $\mathbb{C}$. We say that Ω is *simply connected* if Ω is continuous and *every* continuous path in Ω is null-homotopic in Ω.

With this definition in place, Theorem 7.4 yields the following outcome, which we leave as an

Exercise: Let Ω be a simply connected domain in $\mathbb{C}$. Show that every $f \in \mathcal{O}(\Omega)$ admits a primitive in Ω.

Let us now provide a justification for the geometric interpretation for $\eta(\gamma; a)$ given in the previous lecture. Hence, consider a piecewise smooth closed curve $\gamma : [0,1] \to \mathbb{C}$ and consider a point $a \in \mathbb{C} \setminus \langle\gamma\rangle$. Let $\Omega_a := \mathbb{C} \setminus \{a\}$. We show that $\gamma \sim_{\Omega_a} \gamma_N$, where γ_N is a piecewise smooth path with the property

$*$ $\langle\gamma_N\rangle = \partial D(a; 1)$
$*$ the vector $[\gamma_N(t) - a]$ makes N complete rotations.

The homotopy linking γ and γ_N is just the straight-line homotopy:

$$H(s,t) = (1-t)[\gamma(s) - a] + t\frac{[\gamma(s) - a]}{|\gamma(s) - a|} + a \in \Omega_a \quad \forall (s,t)$$

that continuously moves $\gamma(s)$ to $\{(\gamma(s) - a)/|\gamma(s) - a| + a\}$ along the radial line originating at $a \in \mathbb{C} \setminus \langle\gamma\rangle$. By this construction, we would be done if we could show that $\eta(\gamma, a) = N$.

It is difficult to write down an exact homotopy, but there is a homotopy $\widetilde{H} : [0,1] \times [0,1] \to \partial D(a;1)$ such that

$$\begin{aligned}
\widetilde{H}(\cdot, 0) &= \gamma_N, \\
\widetilde{H}(s, 1) &= a + e^{2\pi i N s} \quad \forall s \in [0,1], \\
\widetilde{H}(0, t) &= \widetilde{H}(1, t) \quad \forall t \in [0,1].
\end{aligned}$$

In summary: $\gamma \sim_{\Omega_a} \widetilde{\gamma}_N$, where $\widetilde{\gamma}_N := \tilde{H}(\cdot, 1)$. But now, by the homotopy form of Cauchy's theorem

$$\begin{aligned}
\eta(\gamma; a) &= \frac{1}{2\pi i} \int_\gamma \frac{1}{z-a}\, dz \\
&= \frac{1}{2\pi i} \int_{\widetilde{\gamma}_N} \frac{1}{z-a}\, dz \\
&= \frac{1}{2\pi i} \int_0^1 \frac{2\pi i N e^{2\pi i N s}}{[a + e^{2\pi i N s}] - a} ds = N,
\end{aligned}$$

which is what we needed to show.

We mention, in conclusion of this lecture, the significance of the exercise above. For any $f \in \mathcal{O}(\Omega)$ to admit a primitive in Ω is one of two crucial ingredients in the proof of the Riemann Mapping Theorem. Moreover the property that *every* $f \in \mathcal{O}(\Omega)$ admits a primitive in Ω characterises simple-connectedness of Ω.