Annual Foundational School, part II
PROBLEM SET

Instructor: GAUTAM BHARALI **COMPLEX ANALYSIS**

$*$ Problems on Lecture 1

1. (From Remark 1.2) Give an example of a region $\Omega \subset \mathbb{R}^2$, with $0 \in \Omega$, and a function $f : \Omega \to \mathbb{C}$ such that

   $*$ $\left.\dfrac{\partial f}{\partial x}\right|_{z=0}, \left.\dfrac{\partial f}{\partial y}\right|_{z=0}$ exist; but

   $*$ f is not continuous at 0.

2. True or false ? Ω is an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ is complex-analytic on Ω. If $f' \equiv 0$, then $f \equiv$ constant.

   If the above statement is false, how should it be corrected ?

3. (From Theorem 1.6) Let $\Omega \subset \mathbb{C}$ be an open set, and let f be $\mathbb{C}$-differentiable on Ω. Let $z_0 \in \Omega$. Show that if $f'(z_0) \neq 0$, $\exists\ r_0 > 0$ sufficiently small that $D(z_0; r_0) \subset \Omega$ and $f(z) \neq f(z_0)\ \forall\ z : 0 < |z_-z_0| < r_0$.

4. Let $\Omega \subset \mathbb{C}$ be a *domain* — i.e. Ω is an open, connected subset of $\mathbb{C}$. Suppose $f : \Omega \to \mathbb{C}$ is complex-analytic and $f(\Omega) \subset \partial D(0; 1)$. What can you say about f ?

$*$ Problems on Lecture 2

1. (From Remark 2.1) For a given power series $\sum_{n=0}^{\infty} c_n(z-a)^n$, define

   $$R = \left\{ \limsup_{n\to\infty}\ |c_n|^{1/n} \right\}^{-1}.$$

   Show that the series converges absolutely at each $z : |z-a| < R$ and diverges at each $z : |z-a| > R$. Also show that for any $r \in (0, R)$, the series is uniformly convergent on $\overline{D(a; r)}$.

2. Give a rigorous proof that one *cannot* define an analytic branch of the logarithm on $\mathbb{C}\backslash\{0\}$.

3. Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be open subsets. Let $f \in \mathcal{C}(\Omega_1; \mathbb{C}),\ g \in \mathcal{O}(\Omega_2)$ and $f(\Omega_1) \subset \Omega_2$. Suppose

$$g[f(z)] = z \quad \forall z \in \Omega_1,$$

and $g'(w) \neq 0\ \forall w \in \Omega_2$. Show that $f \in \mathcal{O}(\Omega_1)$ and that

$$f'(z) = \frac{1}{g'[f(z)]} \quad \forall z \in \Omega_1.$$

4. Propose a definition for the principal analytic branch of the nth root. Justify why your definition yields a holomorphic function.

5. Show that we can define an analytic branch, say S, of $(1 - z^2)^{1/2}$ such that $S \in \mathcal{O}(\mathbb{C}\backslash\overline{D(0; 1)})$.

6. Using a slightly more sophisticated manipulation, show that the S in problem #5 can be so constructed that $S \in \mathcal{O}(\mathbb{C}\backslash[-1, 1])$.

   Hint: You will have to consider a Möbius transformation.

$*$ Problems on Lecture 3

1. Refer to the definition of reparametrisation (Definition 3.2). Show that if we replace "φ is strictly increasing" by "φ is non-decreasing", then the relation $\sim$ such that $\gamma \sim \sigma \Leftrightarrow \sigma$ is a reparametrisation (suitably redefined) of γ — is not an equivalence relation.

$*$ Problems on Lecture 4

1. Find the set of all points in $\mathbb{C}$ at which the function

$$f(z) := \frac{1}{2i}\ \log\left(\frac{1 + iz}{1 - iz}\right)$$

   is $\mathbb{C}$-differentiable. Is f holomorphic on the resulting set? Find a power-series expansion for f around $z = 0$.

2. Let $f \in \mathcal{O}(D(0;1)) \cap \mathcal{C}(\overline{D(0;1)};\mathbb{C})$. DO NOT ASSUME that f is holomorphic on a larger domain containing $\overline{D(0;1)}$. Show that

$$\int\limits_{\partial D(0;1)} f(z)\, dz = 0.$$

3. Let D be a bounded domain in $\mathbb{R}^2$ with piecewise-smooth boundary. Let $f = (u + iv) : D \to \mathbb{C}$ be continuously differentiable on D and assume that all first-order partial derivatives of u and v extend continuously to ∂D. Show that:

$$f(z) = \frac{1}{2\pi i}\int\limits_{\partial D} \frac{f(w)}{w-z} dw - \frac{1}{2\pi}\int\int\limits_{D} \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f(w)\ \frac{1}{w-z} \qquad dA(w)$$

$$\forall z \in D.$$

∗ Problems on Lecture 5

1. Let F be an entire function. Suppose there exist two positive constants $M > 0$ and $P \geq 1$ such that $|F(z)| \leq M(1 + |z|^p)\ \forall z \in \mathbb{C}$. Describe all functions F that satisfy this growth estimate.

2. Let F be an entire function, and suppose that in every power-series expansion

$$F(z) = \sum_{n=0}^{\infty} c_n^{(a)}(z-a)^n,$$

as a varies in $\mathbb{C}$, at least one coefficient $c_n^{(a)} = 0$. Describe all functions F that satisfy this property.

3. Find all entire functions f such that $f(z) = \cos z\ \forall z \in [0, 2\pi]$.

∗ Problems on Lecture 6

1. Let $\Omega \subset_{\text{open}} \mathbb{C}$ and let $\gamma : [a, b] \to \Omega$ be a piecewise smooth curve. Suppose $f \in \mathcal{C}(\Omega;\mathbb{C})$ be such that it admits a primitive in F on Ω. Show that

$$\int_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a)).$$

2. Fix a point $w_0 \in \mathbb{C}\backslash(-\infty, 0]$. Let $\gamma : [0,1] \to \mathbb{C}$ be a piecewise smooth path in $\mathbb{C}$ from 1 to w_0 that avoids $0 \in \mathbb{C}$.

   (a) Show that $\exists\ k_\gamma \in \mathbb{Z}$ such that

   $$\int_\gamma \frac{1}{z}\ dz = \log(w_0) + 2\pi i k_\gamma.$$

   (b) What is the geometric interpretation of k_γ ?

$*$ Problems on Lecture 7

1. Let Ω be a simply-connected domain in $\mathbb{C}$, and suppose $f \in \mathcal{O}(\Omega)$. Show that f admits a primitive in Ω.

   Hint: Fix $z_0 \in \Omega$. For each $z \in \Omega$, let γ^z represent some piecewise smooth path $\gamma^z : [0,1] \to \Omega$ with $\gamma^z(0) = z_0$ and $\gamma^z(1) = z$. Consider the assignment

   $$z \longmapsto \int_{\gamma^z} f(w)\ dw;$$

   first examining whether it is well-defined.

2. Let $f \in \mathcal{C}(\Omega; \mathbb{C})$ and suppose $f \in \mathcal{O}(\mathbb{C}\backslash[a,b])$, where $[a,b] \subset \mathbb{R}$ is a closed, bounded interval. Show that f is entire.