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Lectures on Complex analysis

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Lecture 1

Let Ω be an open subset of $\mathbb{C}$. We recall that a function $f : \Omega \to \mathbb{C}$ is said to be *holomorphic* if f is complex differentiable at all points of Ω. That is, $f'(z) := \lim\limits_{\substack{w \to z \\ w \in \Omega}} \frac{f(w)-f(z)}{w-z}$ exists for all $z \in \Omega$. It is important to understand that holomorphicity is defined on open sets. However, sometimes we find it convenient to say that a function $f : K \to \mathbb{C}$ on a compact set $K \subseteq \mathbb{C}$ is holomorphic on K if there is an open set $\Omega \supset K$ such that f extends to a holomorphic function on Ω.

Exercise 1: Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \begin{cases} e^{-1/|z|} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$

Show that f is complex differentiable at 0 but nowhere else. Thus f is not holomorphic at 0 (or anywhere else, for that matter!)

If $z_0 \in \mathbb{C}$ and $\{a_n : n = 0, 1, 2, \ldots\}$ is a sequence of complex numbers then the series $\sum\limits_{n=0}^{\infty} a_n (z - z_0)^n$ is called a power series around z_0. There is an $r \geq 0$, such that the series converges for $|z - z_0| \leq r$. The supremum of all such r is called the radius of convergence of the power series. The radius of convergence R is given by the formula due to Hadamard:

$$\frac{1}{R} = \limsup_{n \to \infty} \; |a_n|^{1/n}.$$

If $\Omega \subseteq \mathbb{C}$ is an open set, then we shall say that a function $f : \Omega \to \mathbb{C}$ is analytic if f is locally given by a power series. That is, for each $z_0 \in \Omega$ there is an $r > 0$ (depending on z_0) and a power series $\sum\limits_{n=0}^{\infty} a_n (z - z_0)^n$ around z_0 (again depending on z_0) such that the disc $\{z \in \mathbb{C} : |z - z_0| < r\}$ is contained in Ω and we have $f(z) = \sum\limits_{n=0}^{\infty} a_n (z - z_0)^n$ whenever $|z - z_0| < r$.

You may be surprised by the two notions: holomorphic and analytic. You may even protest that these two notions are identical! And of course, you would be right. But it is extremely difficult to see that these two definitions yield the same class of functions. Indeed, all the mysterious tools one encounters in a first course of complex analysis - path integrals, Cauchy's fundamental theorem and integral formulae - are introduced in order to establish this equivalence. But one direction is easy:

Exercise 2: Show that any (convergent) power series can be complex differentiated term by term within its disc of convergence. Conclude that if $f : \Omega \to \mathbb{C}$ is analytic then it is holomorphic, and for $z_0 \in \Omega$ the mysterious power series $\sum a_n(z - z_0)^n$, whose existence is asserted in the definition of analyticity, is actually given by $a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n = 0, 1, 2, \dots$.

The two definitions have their respective advantages. For instance, from the definition of analyticity, it is easy and elementary to deduce most (but not all !) of the strange properties of analytic functions that we have learnt about. Try these (do not use Cauchy's theorem!) :

Exercise 3:

Let $\Omega \subseteq \mathbb{C}$ be a connected open set.

(a) If $f : \Omega \to \mathbb{C}$ is analytic then show that all the derivatives $f^{(n)}, \ n = 0, 1, 2, \dots$ exist and are analytic.

(b) Also show that locally f has an analytic "primitive". That is, for each $z_0 \in \Omega$, there is a neighbourhood $U \subseteq \Omega$ of z_0 and an analytic function $g : U \to \mathbb{C}$ such that $g' = f$ on U. Conclude that for any simple closed curve γ in U, $\int\limits_{\gamma} f = 0$.

(c) Prove that either $f \equiv 0$ or the zeroes of f are isolated. That is, if $f \not\equiv 0$ is analytic on Ω and $f(z_0) = 0$ for some $z_0 \in \Omega$ then there is an $r > 0$ such that the disc $\{z : \ |z - z_0| < r\}$ is contained in Ω and $f(z) \neq 0$ for $0 < |z - z_0| < r$. Conclude that the zeroes of a non-constant analytic function are countable in number.

(d) Use (c) to prove the principle of analytic continuation: if $f_1, f_2 : \Omega \to \mathbb{C}$ are analytic functions and there is an open (or just uncountable) non-empty set $U \subseteq \Omega$ such that $f_1 = f_2$ on U, then $f_1 = f_2$ on Ω. Thus, if $U \subset \Omega$ is open, and $g : U \to \mathbb{C}$ is analytic, then there is at most one analytic function $f : \Omega \to \mathbb{C}$ which extends g. (f is called the analytic continuation of g to Ω).

(e) If $f : \Omega \to \mathbb{C}$ is a non-constant analytic function then show that the continuous function $|f|$ has no local maximum (i.e, if $z_0 \in \Omega$ and $r > 0$ such that $\{|z - z_0| \leq r\} \subseteq \Omega$ then $|f(z)| \leq |f(z_0)|$ whenever $|z - z_0| \leq r$ implies that f must be constant.) Deduce that if Ω is bounded and f is analytic on $\bar{\Omega}$ (recall that this means f extends analytically to an open set containing $\bar{\Omega}$) then $|f|$ is maximised on the boundary $\partial\Omega = \bar{\Omega}\backslash\Omega$. (Maximum Modulus Principle).

Unfortunately, given a function, it is hard to verify (in general) whether f is analytic. On the other hand, it is often easier to verify holomorphicity of f. This makes it important to establish the equivalence of holomorphicity with analyticity. In view of the above discussion, it suffices to show that holomorphic $\Rightarrow$ analytic.

Throughout, Ω is an open set in $\mathbb{C}$. A curve in Ω is a continuously differentiable function $\gamma : [a, b] \to \Omega$ (for some $a < b$ in $\mathbb{R}$). $\gamma(a)$ and $\gamma(b)$ are called the initial and end point of γ respectively. If $f : \Omega \to \mathbb{C}$ is continuous, then the path integral $\int\limits_\gamma f$ is defined to be $\int\limits_a^b f(\gamma(t))\gamma'(t)dt$ (Riemann Integral). A path in Ω is a finite sequence $\gamma_1, \ldots, \gamma_n$ of curves such that the end point of γ_j coincides with the initial point of γ_{j+1} for $1 \leq j < n$. If $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a path and $f : \Omega \to \mathbb{C}$ is continuous, one defines the path integral $\int\limits_\gamma f$ to be $\sum\limits_{j=1}^n \int\limits_{\gamma_j} f$. The initial point of γ_1 (respectively, the end point of γ_n) is called the initial (respectively, end) point of γ. A path is said to be closed if its initial point and end point coincide.

If γ is a curve then the length of γ is defined to be the number $\int\limits_a^b |\gamma'(t)|dt$.

If γ is a path, its length is $\sum\limits_{j=1}^n$ length (γ_j).

(If the curve γ is thought of as describing a moving point which occupies position $\gamma(t)$ at time t, then $|\gamma'(t)|$ is its speed at time t. This justifies our definition of length).

Exercise 4: If γ is a path in Ω and $f : \Omega \to \mathbb{C}$ is continuous then show that $|\int\limits_\gamma| \leq$ length $(\gamma) \cdot \|f\|_\gamma$ when $\|f\|_\gamma$ is the maximum of $|f|$ on $Im(\gamma) :=$

$\bigcup_{j=1}^{n} Im(\gamma_j)$.

By a "rectangle" we shall mean a rectangle with sides parallel to the axes, including all points on the boundary and interior. Thus a rectangle is a compact set with no holes. The path describing the boundary of a rectangle R in an anti-clockwise manner will be denoted ∂R.

(Goursat's) Lemma 1: If R is a rectangle and f is holomorphic on R then $\int\limits_{\partial R} f = 0$.

Sketch of Proof: We inductively define a sequence $\{R_n\}$ of rectangles in R satisfying

(1) $R = R_1 \supset R_2 \supset \dots$

(2) Length of $\partial R_{n+1} = \frac{1}{2}$ length (∂R_n) $\forall n$ (Hence length $(\partial R_n) = 2^{-n}$ length $(\partial R) \to 0$.

(3)$|\int\limits_{\partial R_{n+1}} f| \geq \frac{1}{4}|\int\limits_{\partial R_n} f| \quad \forall n$ and hence $|\int\limits_{\partial R_n} f| \geq \frac{1}{4^n}|\int\limits_{\partial R} f| \quad \forall n$.

From (1) and (2) one sees that there is a point $z_0 \in R$ such that $\{z_0\} = \bigcap\limits_{n=1}^{\infty} R_n$. Put $g(z) = f(z) - f(z_0) - (z - z_0)f'(z_0)$ and observe that g is holomorphic on R. Observe that $\int\limits_{\partial R_n} (f(z_0) + (z - z_0)f'(z_0)) = 0$ (why ?). Hence it suffices to show that $\int\limits_{\partial R} g = 0$. Further, we have the estimates (2) and (3) with g in place of f.

Now, since $g'(z_0) = g(z_0) = 0$, there is a neighbourhood $U \subset R$ of z_0 such that $|g(z)| \leq \varepsilon |z - z_0|$ for all $z \in U$.

Take n so large that $R_n \subseteq U$.

Then $\frac{1}{4^n}|\int\limits_{\partial R} g| \leq |\int\limits_{\partial R_n} g| \leq \frac{\text{length}(R)}{2^n} \sup\limits_{z \in R_n} |z - z_0|\epsilon \leq \frac{c}{4^n}\epsilon$ where $c > 0$ is an absolute constant. Since $\epsilon > 0$ was arbitrary, this completes the proof. $\square$

Lemma 2: Let $f : \Omega \to \mathbb{C}$ be holomorphic. Then, locally, f has a holomorphic primitive. That is, for any closed disc $D \subset \Omega$, there is a holomorphic function $g : D \to \mathbb{C}$ such that $g' = f$ on D.

Sketch of Proof: Say z_0 is the centre of D. For $z \in D$, define $g(z) = \int\limits_{z_0}^{z} f$ where the integral is along two of the sides of a rectangle whose opposite vertices are z_0 and z.

In view of Goursat's lemma, it does not matter whether we go horizontal and then vertical, or the other way round. Verify that $g' = f$ on D. $\square$

Local Cauchy Theorem: For holomorphic $f : \Omega \to \mathbb{C}$ and any closed path γ in a closed disc $D \subset \Omega$, we have $\int\limits_{\gamma} f = 0$.

Sketch of Proof: By Lemma 1, $f = g'$ on D. Hence $\int\limits_{\gamma} f = \int\limits_{\gamma} g' = 0$. $\square$

Local Cauchy Integral Formula: Let D be a closed disc in Ω and $f : \Omega \to \mathbb{C}$ be holomorphic. Let ∂D denote the path traversing the boundary of D in an anticlockwise manner.

Then $f(w) = \frac{1}{2\pi i} \int\limits_{\partial D} \frac{f(z)}{z-w}$ for $w \in D^0$.

Sketch of Proof: Define g on Ω by $g(z) = \frac{f(z)-f(w)}{z-w}$ if $z \neq w$ and $g(z) = f'(w)$ if $z = w$. It is easy to prove that g is continuous on Ω and analytic on $\Omega \backslash \{w\}$. A limiting argument shows that the local Cauchy Theorem applies to g (even though we do not know that g is analytic at w), and hence $\int\limits_{\partial D} g = 0$. But $f(z) = f(w) + (z-w)g(z)$ for z on ∂D, so that

$$\begin{aligned} \int\limits_{\partial D} \frac{f(z)}{z-w} &= f(w) \int\limits_{\partial D} \frac{dz}{z-w} + \int\limits_{\partial D} g \\ &= 2\pi i f(w). \end{aligned}$$

$\square$

Theorem: Every holomorphic function is analytic. Thus holomorphicity and analyticity are equivalent notions.

Proof: Given holomorphic $f : \Omega \to \mathbb{C}$ and $w \in \Omega$, take any disc D with center w such that $\bar{D} \subseteq \Omega$. Then use the local integral formula, expand $\frac{1}{z-w}$ in a power series in z, and integrate term by term to get a power series around w which converges on D and represents f. $\square$

Lecture 2

Note that our proof of the equivalence of analyticity and holomorphicity actually yields something more:

If $f : \Omega \to \mathbb{C}$ is holomorphic and $z_0 \in \Omega$ then the power series around z_0 which represents f near z_0 actually converges on (any and hence) the largest disc centered at z_0 and contained in Ω. In particular, recalling that a holomorphic function on the whole of $\mathbb{C}$ is called entire, we have:

Corollary: If $f : \mathbb{C} \to \mathbb{C}$ is entire then the power series around any point $z_0 \in \mathbb{C}$ representing f converges on the whole of $\mathbb{C}$. Thus, if f is entire, then we have $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{C}$.

Exercise 5: Use the above corollary to give an elementary proof of Liouville's theorem: Only bounded entire functions are constants.
(Hint: If $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{C}$; and $a_k \neq 0$ for some k, then show that $\limsup\limits_{|z| \to \infty} \frac{|f(z)|}{|z|^k} > 0$).

Exercise 6: Since holomorphic $\Rightarrow$ analytic, it follows that if $f : \Omega \to \mathbb{C}$ is holomorphic then f has local primitives. (For any disc $D \subseteq \Omega$, $\exists\, g : D \to \mathbb{C}$ such that $g' = f$ on D). Give an example to show that f need not have a global primitive: there may not exist $h : \Omega \to \mathbb{C}$ such that $h' = f$ on Ω.

Winding numbers: If D is a disc with centre z_0 and ∂D denotes the path which surrounds the boundary of D once in the anticlockwise orientation, then an elementary calculation shows that $\frac{1}{2\pi i} \int\limits_{\partial D} \frac{dz}{z - z_0} = 1$. More generally, if γ goes around the boundary n times in the anticlockwise direction, then this integral equals $+n$, while if γ goes around the boundary n times in the clockwise direction, then the integral equals $-n$. This motivates the following:

Definition: If γ is a closed path and z_0 is a point outside $Im(\gamma)$, then the winding number $W(\gamma, z_0)$ is defined by

$$W(\gamma, z_0) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{dz}{z - z_0}.$$

Intuitively, this counts (with a sign for the direction), the net number of times γ goes around z_0. For this to be correct, the following lemma better be correct:

Lemma 3: $W(\gamma, z_0)$ is an integer.

Proof: Let $\gamma : [a, b] \to \mathbb{C}$. Define $F : [a, b] \to \mathbb{C}$ by $F(t) = \int\limits_a^t \frac{\gamma'(t)}{\gamma(t)-z_0}$.

Since γ is piecewise continuously differentiable, it follows that F is continuous on $[a, b]$ and $F'(t) = \frac{\gamma'(t)}{\gamma(t)-z_0}$ for all but finitely many points (where γ is not differentiable) in the interval $[a, b]$. Now define $G : [a, b] \to \mathbb{C}$ by $G(t) = \frac{e^{F(t)}}{\gamma(t)-z_0}$. Logarithmic differentiation yields $\frac{G'(t)}{G(t)} = F'(t) - \frac{\gamma'(t)}{\gamma(t)-z_0} = 0$ at all but finitely many points. Thus $G'(t) = 0$ at all these points. So G is a piecewise constant function. But G is continuous. So G is a constant function. In particular, $G(a) = G(b)$. That is, $\frac{e^{F(a)}}{\gamma(a)-z_0} = \frac{e^{F(b)}}{\gamma(b)-z_0}$. But we have $\gamma(a) = \gamma(b)$. Also, from its definition $F(a) = 0,\ F(b) = 2\pi i W(\gamma, z_0)$. Hence we get $e^{2\pi i W(\gamma,z_0)} = 1$. Thus $W(\gamma, z_0)$ is an integer. $\square$

Exercise 7: If γ is a closed path in $\mathbb{C}$ then show that the function $z \mapsto W(\gamma, z)$ from $\mathbb{C}\backslash Im(\gamma)$ into $\mathbb{Z}$ is continuous. Hence conclude that $W(\gamma, z)$ is a constant on each connected component of $\mathbb{C}\backslash Im(\gamma)$. Also show that $W(\gamma, z) = 0$ for z in the unbounded component of $\mathbb{C}\backslash Im(\gamma)$ [Hint: Fix $z_0 \in \mathbb{C}\backslash Im(\gamma)$. Since $Im(\gamma)$ is compact, the distance of z_0 from points on $Im(\gamma)$ is bounded away from 0. That is, $\exists\ r > 0$ such that $|\gamma(t) - z_n| \geq r\ \forall\ t$. It follows that if z is sufficiently close to z_0 then $|\gamma(t) - z \geq r/2$. Hence bound the absolute difference between the two integrands defining $W(\gamma, z)$ and $W(\gamma, z_0)$.]

Definition: We shall say that two paths $\gamma : [a, b] \to \Omega$ and $\eta : [a, b] \to \Omega$ are *close together* if they have the same initial point, same end point (i.e., $\gamma(a) = \eta(a), \gamma(b) = \eta(b)$) and there is a partition $a = a_1 < a_2 < \ldots < a_n = b$ and closed discs $D_1, \ldots, D_{n-1} \subseteq \Omega$ such that $\gamma([a_i, a_{i+1}]) \subseteq D_i,\ \ \eta([a_i, a_{i+1}]) \subseteq D_i$ for $1 \leq i < n$.

Lemma 4: Let γ, η be two paths in Ω which are close together. Then for any holomorphic function $f : \Omega \to \mathbb{C}$, we have $\int\limits_\gamma f = \int\limits_\eta f$.

Proof: With notations as in the definition of "close together", Lemma 2 gives a primitive g_i of f on D_i, $1 \leq i < n$. Write γ_i for $\gamma|_{[a_i,a_{i+1}]}, \eta_i$ for $\eta|_{[a_i,a_{i+1}]}, 1 \leq i < n$. Also put $z_i = \gamma(a_i)$, $w_i = \eta(a_i)$. Then $\int\limits_\gamma f = \sum\limits_{i=1}^{n-1} \int\limits_{\gamma_i} f = \sum\limits_{i=1}^{n-1} \int\limits_{\gamma_i} g_i' = \sum\limits_{i=1}^{n-1}(g_i(z_{i+1}) - g_i(z_i))$ and similarly $\int\limits_\eta f = \sum\limits_{i=1}^{n-1}(g_i(w_{i+1}) - g_i(w_i))$. But g_i and g_{i+1} are both primitives of f on the connected open set $D_i \cap D_{i+1}$. Hence $g_{i+1} - g_i$ is a constant on $D_i \cap D_{i+1}$. Also, $D_i \cap D_{i+1}$ contains both z_{i+1} and w_{i+1}. Therefore $g_{i+1}(w_{i+1}) - g_i(w_{i+1}) = g_{i+1}(z_{i+1}) - g_i(z_{i+1})$. That is $g_{i+1}(w_{i+1}) - g_{i+1}(z_{i+1}) = g_i(w_{i+1}) - g_i(z_{i+1})$ for $1 \leq i < n$.

Therefore,

$$\begin{aligned}
\int\limits_\gamma f - \int\limits_n f &= \sum_{i=1}^{n-1}((g_i(z_{i+1}) - g_i(z_i)) - (g_i(w_{i+1}) - g_i(w_i))) \\
&= \sum_{i=1}^{n-1}((g_i(z_{i+1}) - g_i(w_{i+1})) - (g_i(z_i) - g_i(w_i))) \\
&= \sum_{i=1}^{n-1}((g_{i+1}(z_{i+1}) - g_{i+1}(w_{i+1})) - (g_i(z_i) - g_i(w_i))) \\
&= \sum_{i=1}^{n-1}(g_{i+1}(z_{i+1}) - g_i(z_i)) - \sum_{i=1}^{n-1}(g_{i+1}(w_{i+1}) - g_i(w_i))
\end{aligned}$$

Hence, by telescoping.

$$\begin{aligned}
\int\limits_\gamma f - \int\limits_\eta f &= g_n(z_n) - g_1(z_1) - g_n(w_n) + g_1(w_1) \\
&= (g_n(z_n) - g_n(w_n)) - (g_1(z_1) - g_1(w_1))
\end{aligned}$$

But $z_n = w_n =$ the common end point of γ and η and $z_1 = w_1 =$ the common initial point of γ and η. Hence $\int\limits_\gamma f - \int\limits_\eta f = 0$. □

Definiton: Let $\gamma, \eta : [a, b] \to \Omega$ be two paths in Ω with common initial and end points. Then we say that γ and η are homotopic in Ω (with initial and end points held fixed) if there is a "continuous" one parameter family $\gamma_s : [a, b] \to \Omega$, $0 \leq s \leq 1$ such that $\gamma_s(a) = \gamma(a), \gamma_s(b) = \gamma(b)$ for all

$s \in [0,1]$, and $\gamma_0 = \gamma, \gamma_1 = \eta$. (More precisely, the "continuity" requirement means that the "homotopy map" $(s,t) \mapsto \gamma_s(t)$ from $[0,1] \times [a,b]$ into Ω is continuous.)

Exercise 8: If γ, η are homotopic in Ω then use the uniform continuity of the homotopy map to show that there is an $\epsilon > 0$ such that for $s_1, s_2 \in [0,1]$ with $|s_1 - s_n| < \epsilon, \gamma_{s_1}$ and γ_{s_2} are close together. Hence conclude that there is a partition $0 = s_1 < s_2 < \ldots < s_n = 1$ such that $\gamma_{s_{i+1}}$ and γ_{s_i} are close together for $1 \leq i < n$. Therefore, Lemma 4 implies:

Homotopy Version of Cauchy's Theorem: If γ, η are homotopic paths in Ω then for any holomorphic $f : \Omega \to \mathbb{C}$, we have

$$\int_\gamma f = \int_\eta f.$$

A closed path γ in Ω is said to be *null-homotopic* if it is homotopic to a point (constant path). It is easy to see that the above theorem is equivalent to:

Alternative homotopy version of Cauchy's Theorem: If γ is a null-homotopic closed path in Ω then for any holomorphic $f : \Omega \to \mathbb{C}$, $\int_\gamma f = 0$.

Definition: Ω is said to be *simply connected* if every closed path in Ω is null homotopic in Ω (intuitively, this means that Ω has no holes). An immediate consequence of the theorem is:

Corollary: If Ω is simply connected, then for any closed path γ in Ω and any holomorphic $f : \Omega \to \mathbb{C}$, we have $\int_\gamma f = 0$.

Corollary: If Ω is simply connected then any holomorphic $f : \Omega \to \mathbb{C}$ has a global primitive $g : \Omega \to \mathbb{C}$ such that $g' = f$.

Exercise 9: Show that this conclusion is false if Ω is not simply connected.

Lecture 3

Definition: Let γ and η be two paths in Ω with the same initial and end points ($\gamma(a) = \eta(a), \gamma(b) = \eta(b)$, where $\eta, \gamma : [a, b] \to \mathbb{C}$). We say that γ, η are homologous in Ω if $W(\gamma, z_0) = W(\eta, z_0) \forall z_0 \notin \Omega$. If γ is a closed path homologous to a point (constant path) then we say that γ is null-homologous.

For technical reasons, it is good to extend this definition as follows. We say that γ is a *chain* if it is a formal sum of finitely many paths. If γ is a formal sum of finitely many closed paths then we say that γ is a closed chain.

If $\gamma = \gamma_1 + \ldots + \gamma_n$, then $Im(\gamma) \overset{\text{def}}{=} \bigcup\limits_{i=1}^{n} Im(\gamma_i))$. If $z_0 \notin Im(\gamma)$, we define $W(\gamma, z_0) := \sum\limits_{i=1}^{n} W(\gamma_i, z_0)$. If each γ_i is a path in Ω we say that γ is a *chain in* Ω. For a chain $\gamma = \gamma_1 + \ldots + \gamma_n$ in Ω and $f : \Omega \to \mathbb{C}$, we define $\int\limits_{\gamma} f := \sum\limits_{i=1}^{n} \int\limits_{\gamma_i} f$. Two chains γ, η in Ω are called homologous in Ω if $W(\gamma, z_0) = W(\eta, z_0)\ \forall\ z_0 \notin \Omega$. A closed chain γ in Ω is called null-homologous in Ω if $W(\gamma, z_0) = 0\ \forall\ z_0 \notin \Omega$.

With this definition, the most general version of Cauchy's fundamental theorem is:

Global Cauchy Theorem: If γ is a null-homologous closed chain in Ω then $\int\limits_{\gamma} f = 0$ for all holomorphic f on Ω. Equivalently, if γ, η are homologous closed chains in Ω then $\int\limits_{\gamma} f = \int\limits_{\eta} f$ for all holomorphic f on Ω.

This is the most general version of Cauchy in the sense that if a closed chain γ is not null-homologous in Ω then there is a holomorphic function f on Ω such that $\int\limits_{\gamma} f \neq 0$ (namely $f(z) = \frac{1}{z - z_0}$ for a suitable $z_0 \notin \Omega$).

To prove Cauchy's Global Theorem, we need the following two lemmas.

We shall say that a path is *rectangular* if it is a concentration ("union") of horizontal and vertical line segments.

Lemma 5: If γ is a path in Ω then there is a rectangular path η in Ω such that γ and η are close together in Ω. In consequence (Lemma 4) γ and η are homologous, and $\int\limits_\gamma f = \int\limits_\eta f$ for any holomorphic f on Ω.

In view of this Lemma, to prove the Global Cauchy Theorem, it is enough to prove it for "rectangular" closed chains (i.e, formal sums of rectangular closed paths).

Proof of Lemma 5: Given $\gamma : [a,b] \to \Omega$ take a partition $a = a_1 < a_n < \ldots < a_n = b$ of $[a,b]$ such that $\gamma([a_i, a_{i+1}]) \subseteq D_i$, a closed disc in Ω. Put $\gamma_i = \gamma|_{[a_i,a_{i+1}]}, z_i = \gamma(a_i)$. Take a rectangular path η_i lying inside D_i and joining z_i to z_{i+1} $(1 \leq i < n)$. Let η be the concatenation of $\eta_1, \ldots, \eta_{n-1}$. This clearly works.

If $\gamma : [a,b] \to \mathbb{C}$ is a path and $a = a_1 < \ldots < a_n = b$, $\gamma_i = \gamma|_{[a_i,a_{i+1}]}$, $1 \leq i < n$ then the chain $\gamma = \gamma_1 + \ldots + \gamma_{n-1}$ will be called a subdivision of γ. More generally, if $\eta = n_1 + \ldots + n_m$ is a chain, $n_{i1} + \ldots + n_{in}$ is a subdivision of the path n_i for each i, then the chain $\sum\limits_{i,j} n_{ij}$ will be called a subdivision of the chain η. Clearly if the chain η' is a subdivision of the chain η then η, η' are homologous (in any domain Ω such that η, η' are in Ω).

Lemma 6 (Artin) : If γ is a rectangular closed chain in Ω which is null-homologous in Ω then there exist rectangles $R_1, \ldots R_N$ in Ω and integers $\alpha_1, \ldots, \alpha_N$ such that the rectangular closed chain $\eta = \sum\limits_{i=1}^{N} d_i \partial R_i$ is a subdivision of γ. (Here ∂R_i is the closed path traversing the boundary of R_i anticlockwise).

Sketch of Proof: Draw all the lines which contain one of the line-segments constituting γ. These are (finitely many) horizontal and vertical lines. They partition the complex plane into finitely many regions, some of which are rectangles and others unbounded. If R is one of these rectangles then the interior R^0 is inside a connected component of $Im(\gamma)$, and hence the winding number $W(\gamma, \cdot)$ is a constant, say α_R, on R^0. If $\alpha_R \neq 0$ for some R then $R^0 \subseteq \Omega$ (since $W(\gamma, \cdot)$ is zero in the complement of Ω). Hence it is easy to see that the closed chain $\eta = \sum\limits_{R:\alpha_R \neq 0} \alpha_R \cdot \partial R$ is a subdivision of γ. $\square$.

Now, Lemma 5 and Lemma 6 together show that: if γ is a null-homologous closed chain in Ω then there are rectangles $R_1, \ldots, R_N$ in Ω and integers

$\alpha_1, \ldots, \alpha_N$ such that $\int\limits_\gamma f = \int\limits_{\sum \alpha_i \partial R_i} = \sum\limits_{i=1}^N \alpha_i \int\limits_{\partial R_i} f = 0$, proving Cauchy's Global Theorem.

One application of Cauchy's Global Theorem is to reduce the calculation of integrals (of holomorphic functions) over complicated paths to those over simple paths. Namely, we have:

Theorem: Let γ be a null-homologous closed chain in Ω. Let $z_1, \ldots, z_N$ be finitely many distinct points in Ω and let $D_1, \ldots, D_N$ be pairwise disjoint closed discs in Ω with centres $z_1, \ldots, z_N$. Assume γ does not pass through any of the points z_i. Put $m_i = W(\gamma, z_i)\ \ 1 \leq i \leq n$. Then γ is homologous in $\Omega \setminus \{z_1, \ldots, z_N\}$ to the chain $\sum\limits_{i=1}^N m_i \partial D_i$. Hence, for any holomorphic function $f : \Omega \setminus \{z_1, \ldots, z_N\} \to \mathbb{C}$, we have

$$\int\limits_\gamma f = \sum_{i=1}^N m_i \int\limits_{\partial D_i} f.$$

Proof : If $\alpha \notin \Omega$ then $W(\gamma, \alpha) = 0$ and $W(\partial D_i, \alpha) = 0$. Hence $W(\sum m_i \partial D_i, \alpha) = 0 = W(\gamma, \alpha)$. On the other hand if $\alpha = z_i$ for some i then $W(\gamma, \alpha) = m_i$, while $W(\partial D_j, \alpha) = \delta_{ij}$. Hence $W(\sum m_j \partial D_j, \alpha) = \sum m_j\ \delta_{ij} = m_i = W(\gamma, \alpha)$. Thus $W(\gamma, \alpha) = W(\sum m_i \partial D_i, \alpha)$ for all α outside $\Omega \setminus \{z_1, \cdots, z_N\}$. this proves the first statement. The second statement follows from the global Cauchy theorem. $\square$

Another consequence is:

The Global version of Cauchy's integral formula: Let γ be a null-homologous closed chain in Ω. Let $z_0 \in \Omega$ be such that γ does not pass through z_0. Then for any holomorphic $f : \Omega \to \mathbb{C}$, we have $\frac{1}{2\pi i} \int\limits_\gamma \frac{f(z)}{z - z_0} = W(\gamma, z_0) f(z_0)$.

Proof: In view of the definition of $W(\gamma, z_0)$, this formula may be written as $\frac{1}{2\pi i} \int\limits_\gamma g(z) dz = 0$ where $g : \Omega \to \mathbb{C}$ is defined by $g(z) = \frac{f(z) - f(z_0}{z - z_0}$ if $z \neq z_0$, $g(z_0) = f'(z_0)$. Since γ is null-homologous and g is (analytic and hence) holomorphic, this follows from Global Cauchy Theorem. $\square$

If f is a complex-valued function defined on a subset of Ω and $z_0 \in \Omega$, we say that z_0 is a singular point (or singularity) of f if f is not holomorphic on any neighbourhood of z_0. We say that z_0 is regular otherwise. Thus the set Ω_0 of regular points is the largest open subset of Ω on which f is holomorphic. A point $z_0 \in \Omega$ is said to be an isolated singularity of f if there is an open disc B centred at z_0 ($B \subseteq \Omega$) such that f is holomorphic on $B \setminus \{z\}$. Then f has a convergent *Laurent expression* on $B \setminus \{z_0\}$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad z \in B \setminus \{z_0\}.$$

This statement is a special case of the following :

Theorem: Let $A = \{z \in \mathbb{C} : r \leq |z| \leq R\}$ be an annulus $(0 < r < R)$. Let f be holomorphic on A. Then f has a Laurent expansion which converges absolutely and uniformly on A.

Proof: There is an open $\Omega \supseteq A$ on which f is holomorphic. Let C_r and C_R denote the bounding circles of A, oriented anti-clockwise. Then the closed chain $C_R - C_r$ is null-homologous in Ω (easy calculation). Hence by Cauchy's integral formula, we have

$$f(z) = \frac{1}{2\pi i} \int\limits_{C_R} \frac{f(w)}{w - z} dw - \frac{1}{2\pi i} \int\limits_{C_r} \frac{f(w)}{w - z} dw, \quad z \in A^0.$$

Now, for $w \in C_R$, $|\frac{z}{w}| < 1$ and for $w \in C_r$, $|\frac{w}{z}| < 1$. Therefore we may expand the first integrand as a power series in $\frac{z}{w}$ and the second as a power series in $\frac{w}{z}$. Interchanging sum and integral yields the Laurent expansion.

Definition: Let z_0 be an isolated singularity of the function f. Let $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n (z - z_0)^n$ be the Laurent expansion of f around z_0.

1) If $a_n = 0 \;\; \forall n < 0$ then we say that z_0 is a removable singularity of f.

2) If $\exists k < 0$ such that $a_k \neq 0$ but $a_n = 0 \; \forall n < k$ then we say that z_0 is a pole of order $|k|$ for f.

3) If $a_n \neq 0$ for infinitely many $n < 0$ then z_0 is said to be an essential singularity of f.

This is a mutually exclusive and exhaustive classification of isolated singularities.

Exercise 10: Let z_0 be an isolated singularity of f, holomorphic on $\Omega \setminus \{z_0\}$. Then

(a) show that z_0 is removable iff f extends to a holomorphic function on Ω iff f is bounded on some punctured neighbourhood of z_0 in Ω.

(b) If z_0 is an isolated essential singularity of f then show that for every neighbourhood U of z_0 in Ω, $f(U \setminus \{z_0\})$ is dense in $\mathbb{C}$.
(Hint : If α is a point outside the closure of $f(U \setminus \{z_0\})$, then $\frac{1}{f(z)-\alpha}$ is holomorphic and bounded on $U \setminus \{z_0\}$.

(c) Use (b) to show that z_0 is a pole of f iff $\lim\limits_{z \to z_0} |f(z)| = \infty$.

Lecture 4

Definition:
Let z_0 be an isolated singularity of f. Let $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n(z-z_0)^n$ $(0 < (z - z_0| < \epsilon)$ be the Laurent expansion of f around z_0. Then the coefficient a_{-1} is called the residue of f at z_0 and is denoted Res (f, z_0)

Residue Theorem: Let f be holomorphic on Ω except for finitely many singularities $z_1, \ldots, z_N$. Let γ be a null-homologous closed chain in Ω not passing through any z_i. Then $\int\limits_{\gamma} f = 2\pi i \sum\limits_{j=1}^{N} W(\gamma, z_j)$ Res (f, z_j).

Proof : In view of a previous theorem, it suffices to show that if D_j is a small disc with centre z_j then $\int\limits_{\partial D_j} f = 2\pi i$ Res (f, z_j). This is easily seen by plugging in the Laurent expansion for f around z_j. □

If z_0 is a zero or pole of f, then looking at the power series/Laurent series expansion of f around z_0, one sees that there is an integer $k \neq 0$ such that we have $f(z) = (z - z_0)^k g(z)$ for some non-vanishing analytic function g in a neighbourhood of z_0. Then $|k|$ is called the order or multiplicity of the zero/pole. One thinks of a zero/pole of multiplicity m as a zero/pole which is repeated m times.

Note that if $f(z) = (z - z_0)^k g(z)$ with g as above, then $\frac{f'(z)}{f(z)} = \frac{k}{z-z_0} + \frac{g'(z)}{g(z)}$. and $\frac{g'}{g}$ is analytic in a neighbourhood of z_0.
Hence Res $\left(\frac{f'}{f},\ z_0\right) = k = \begin{cases} m & \text{if } z_0 \text{ is a zero} \\ -m & \text{if } z_0 \text{ is a pole.} \end{cases}$

(Here m is the multiplicity of z_0 as a zero/pole of f).

Recall that a function is called *meromorphic* if all its singularities are poles. In view of the above observation, one applies the residue theorem to $\frac{f'}{f}$ for a meromorphic f, to obtain:

Theorem: Let γ be a null-homologous closed chain in Ω. Let f be a meromorphic function on Ω such that γ does not pass through any zero or pole of f. Then

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'}{f} = \sum_{z\in\Omega,\ z \text{ zero}} \text{mult }(z)W(\gamma,z) - \sum_{w\in\Omega,\ w \text{ pole}} \text{mult }(w)W(\gamma,w).$$

Here mult$(\cdot)$ denotes the multiplicity of the zero or pole.

A simple and useful case of this theorem is where γ is a simple closed path null-homotopic in Ω. Recall that $\gamma : [a,b] \to \Omega$ is said to be a simple closed path if $\gamma(t_1) = \gamma(t_2) \Leftrightarrow (t_1 = t_2 \text{ or } \{t_1,t_2\} = \{a,b\})$. By a famous theorem of Jordan, the complement of the image of a simple closed path has just two components, one bounded and the other unbounded. Every point z_0 in the bounded (respectively unbounded) component has $W(\gamma,z_0) = 1$ (respectively $W(\gamma,z_0) = 0$). Such a γ is null-homotopic in Ω precisely when Ω contains the bounded component of $\mathbb{C}\backslash Im(\gamma)$. In view of these results, a special case of the above theorem is:

Theorem (The argument principle) : Let γ be a null-homotopic simple closed curve in Ω and let f be a meromorphic function on Ω such that γ does not pass through any zero or pole of f. Then $\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f'}{f}$ = Number of zeroes of f enclosed by γ - Number of poles of f enclosed by γ (both counting multiplicity).

Exercise 11: If g is a continuous function on an interval $[a,b]$ with values in $\mathbb{C}$ such that $g(t) \neq 0\ \forall\ t$, then show that there is a continuous branch of $\arg g(t)$, i.e, a continuous function $t \mapsto \arg(g(t))$ where $\arg(\cdot)$ denotes one of the values of the usual (multi-valued) argument. Any two such continuous branches differ by a constant (an integer multiple of $2\pi i$). Applying this observation to $f\circ\gamma = g$ where f & γ are as above, show that $\int\limits_{\gamma}\frac{f'}{f} = \arg f(\gamma(b)) - \arg f(\gamma(a))$ = Net change in argument of $f(w)$ as w goes around the simple closed curve γ. This explains the name "Argument Principle". Notice that $\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f'}{f} = W(f\circ\gamma, 0)$.

Let us recall:

Morera's Theorem: Let $f : \Omega \to \mathbb{C}$ be a continuous function. Suppose for every rectangle $R \subseteq \Omega$, $\int\limits_{\partial R} f = 0$. Then f is holomorphic on Ω.

Proof: During the proof of equivalence of analyticity and holomorphicity, we have seen that the hypothesis implies the existence of local primitives of f. But each such local primitive is holomorphic (after all f is its complex derivative!). But the derivative of a holomorphic function is holomorphic. So f is holomorphic on each disc $\subseteq \Omega$. Then f is holomorphic on Ω. $\square$

Definition: Let $\{f_n\}$ be a sequence of complex-valued functions on Ω. Let f be a complex-valued function on Ω. We shall say that $f_n \to f$ *locally uniformly* if for each $z_0 \in \Omega$, there is a neighbourhood U of z_0 $(U \subseteq \Omega)$ such that $f_n(z) \to f(z)$ uniformly for $z \in U$.

Exercise 12 : Show that $f_n \to f$ locally uniformly on Ω iff for every compact set $K \subseteq \Omega$, $f_n(z) \to f(z)$ uniformly for $z \in K$.

Theorem: If f_n, $n = 1, 2, 3, \dots$ is a sequence of holomorphic functions on Ω such that $f_n \to f$ locally uniformly on Ω, then f is holomorphic on Ω.

Proof: Let $R \subseteq \Omega$ be any rectangle. Then $f_n \to f$ uniformly on R. Hence $\int\limits_{\partial R} f_n \to \int\limits_{\partial R} f$. But by local Cauchy theorem, $\int\limits_{\partial R} f_n = 0 \; \forall \; n$. So $\int\limits_{\partial R} f = 0$. Hence by Morera's theorem, f is holomorphic. $\square$

In short, "local uniform" limits of holomorphic functions are holomorphic. This shows that local uniform convergence is the right notion of convergence for holomorphic functions. Another evidence for this is the following result, which says that "complex differentiation" is a continuous function on the space of holomorphic functions.

Theorem: If f_n, f are holomorphic functions on Ω such that $f_n \to f$ locally uniformly on Ω as $n \to \infty$, then $f_n' \to f'$ locally uniformly on Ω.

Proof: Fix $z_0 \in \Omega$. Let D be a closed disc contained in Ω with centre z_0. Then we have the Cauchy integral formula $f(w) = \frac{1}{2\pi i} \int\limits_{\partial D} \frac{f(z)}{z-w} dz,\; w \in D^0$. Differentiating with respect to w, we get

$$f'(w) = \frac{1}{2\pi i} \int\limits_{\partial D} \frac{f(z)}{(z-w)^2} dz,\; w \in D^0.$$

Similarly, $f_n'(w) = \frac{1}{2\pi i} \int\limits_{\partial D} \frac{f_n(z)}{(z-w)^2} dz,\; w \in D^0$. Now, let E be a closed disc with centre z_0 such that $E \subseteq D^0$. As $f_n \to f$ locally uniformly, we have

$\frac{f_n(z)}{(z-w)^2} \to \frac{f(z)}{(z-w)^2}$ uniformly for $z \in \partial D,\ w \in E$ (why ?). Integrating with respect to z, we get $f_n'(w) \to f'(w)$ uniformly for $w \in E$. $\square$

Exercise 13 : Using this theorem and the argument principle, prove that if $\{f_n\}$ is a sequence of holomorphic functions on Ω converging to f locally uniformly on Ω, and γ is a null-homotopic closed path not passing through any zero of f then $\exists N$ such that for $n \geq N$, the number of zeroes of f_n (counting multiplicity) enclosed by γ equals the number of zeroes of f (counting multiplicity) enclosed by γ. Conclude that if all the f_ns are non-vanishing on Ω and Ω is connected, then either $f \equiv 0$ or f is non-vanishing on Ω.

Lecture 5

Recall that the usual stereographic projection provides a bijection between the two-dimensional sphere S^2 minus a point (north pole) and the complex plane $\mathbb{C}$. We may say that the north pole maps to ∞ under this projection. Then we have a bijection between S^2 and $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Using this bijection, we identify $\hat{\mathbb{C}}$ with S^2. Thus viewed, $\hat{\mathbb{C}}$ is called the *Riemann Sphere*. Using the stereographic projection, we transfer the topology (motion of convergence, openness) from S^2 to $\hat{\mathbb{C}}$. Note that a sequence $\{z_n\}$ in $\mathbb{C}$ (viewed as a sequence in $\hat{\mathbb{C}}$) converges to ∞ iff $|z_n| \to \infty$. A typical neighbourhood of ∞ in $\hat{\mathbb{C}}$ is $\{z \in \mathbb{C} : \ |z| > R\} \cup \{\infty\}$.

If f is a meromorphic function on $\Omega \subseteq \mathbb{C}$, then $z_0 \in \Omega$ is a pole of f iff $\lim\limits_{z \to z_0} |f(z)| = \infty$, i.e, with the above understanding of convergence to ∞, iff $\lim\limits_{z \to z_0} f(z) = \infty$. Thus it is natural to put $f(z_0) = \infty$ for all poles z_0 of f, and view a meromorphic function f on Ω as a "holomorphic" function $f : \Omega \to \hat{\mathbb{C}}$.

If f is a meromorphic function on $\mathbb{C}$, then we may think of ∞ as a singular point of f (where f is not defined, but f is defined on the punctured neighbourhood $\mathbb{C}$ of ∞). We shall say that ∞ is a removable singularity, pole or essential singularity of f accordingly as 0 is a removable singularity, pole or essential singularity of the meromorphic function $g(z) := f\left(\frac{1}{z}\right)$. (We are using the map $z \mapsto \frac{1}{z}$ on $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$ to bring infinity to the point 0 in the "finite part". Bring infinity close and study the behaviour of f at ∞).

Exercise 14: If f is meromorphic on $\mathbb{C}$ then ∞ is a removable singularity/ pole/essential singularity according as $\lim\limits_{z \to \infty} f(z)$ exists and is a complex number, exists and is $= \infty$, or does not exist, respectively.

For any open set $\Omega \subseteq \hat{\mathbb{C}}$, a meromorphic function f on Ω may be thought of as a (everywhere defined) function $f : \Omega \to \hat{\mathbb{C}}$ which takes the poles of f to ∞. With this understanding a meromorphic function is a $\hat{\mathbb{C}}$-valued holomorphic function. (This statement may be made precise by formally introducing a complex structure - the structure of a Riemannian manifold - on the sphere $\hat{\mathbb{C}}$).

What are the meromorphic functions $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$? Well, such a function can have only finitely many zeroes and poles in $\hat{\mathbb{C}}$ (unless it is a constant function) counting multiplicity - or else they would have a limit point which would be an essential singularity of f. Hence we can find a rational function R (ratio of two polynomials) with the same zeroes and poles of f, counting multiplicity. Then $\frac{f}{R}$ is a meromorphic function on $\hat{\mathbb{C}}$ with no zeroes or poles Liouville's Theorem then shows that $\frac{f}{R}$ is a constant function. Then f is rational. This shows :

Theorem: The only meromorphic functions on the whole of $\hat{\mathbb{C}}$ are rational functions.

Exercise 15: If f is a rational function then show that, counting multiplicity, number of zeros of f in $\hat{\mathbb{C}}$ = number of poles of f in $\hat{\mathbb{C}}$ and sum of residues of the poles of $f = 0$. (Here the multiplicity of ∞ as a zero/pole of f is to be defined as the multiplicity of 0 as a zero/pole of $z \to f\left(\frac{1}{z}\right)$. Similarly the residue of f at ∞ is the residue of $z \mapsto f\left(\frac{1}{z}\right)$ at 0.)

Definition: A biholomorphic automorphism of $\hat{\mathbb{C}}$ is a meromorphic function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which is a bijection. Clearly the set of all biholomorphic automorphisms of $\hat{\mathbb{C}}$ form a group under composition, denoted bihol($\hat{\mathbb{C}}$).

If $f \in$ bihol ($\hat{\mathbb{C}}$), then in particular, f is a rational function. Hence we can write $f = \frac{P}{Q}$ where P & Q are polynomials with no common zero. Since f is a bijection, it takes the value 0 exactly once. Hence P is linear. Also f takes the value infinity exactly once, hence Q takes the value 0 exactly once, so that Q is linear. Thus f has the form $f(z) = \frac{az+b}{cz+d}$. The numbers a, b, c, d must satisfy $ad - bc \neq 0$, or else f would be constant.

Any function $z \mapsto \frac{az+b}{cz+d}$, with $ad-bc \neq 0$, is called a *fractional linear transformation.* Thus all elements of bihol($\hat{\mathbb{C}}$) are fractional linear transformations. Conversely, it is easy to see that any fractional linear transformation is a biholomorphic automorphism of $\hat{\mathbb{C}}$. Thus,

Theorem: bihol($\hat{\mathbb{C}}$) is the group of all fractional linear transformations, with composition as the group operation.

Exercise 16: $GL(2, \mathbb{C})$ is defined to be the group of all 2×2 non-singular matrices with complex entries. Its centre is the subgroup $\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C}\backslash\{0\} \right\}$

of non-singular scalar matrices. The quotient is called $PGL(2,\mathbb{C})$. Show that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az+b}{cz+d}$ is a group homomorphism from $GL(2,\mathbb{C})$ onto bihol$(\hat{\mathbb{C}})$ and its kernel is precisely the centre of $GL(2,\mathbb{C})$. Hence bihol$(\hat{\mathbb{C}})$ is isomorphic to $PGL(2,\mathbb{C})$.

Exercise 17: The special fractional linear transformations $z \mapsto z+a$ are called the translations, $z \mapsto bz (b \neq 0)$ are called the dilations and $z \mapsto \frac{1}{z}$ is called the inversion (across the unit circle). Show that any fractional linear transformation is a composition of finitely many of these special transformations. (In fact, four of them suffice). Also show that each of these special transformations maps circles in $\hat{\mathbb{C}}$ to circles in $\hat{\mathbb{C}}$ (where a circle in $\hat{\mathbb{C}}$ is defined to be either a circle in $\mathbb{C}$ or a line in $\mathbb{C}$ together with ∞). Hence all elements of bihol$(\hat{\mathbb{C}})$ map circles in $\hat{\mathbb{C}}$ to circles in $\hat{\mathbb{C}}$.

Exercise 18: Show that bihol$(\hat{\mathbb{C}})$ is sharply 3-transitive on $\hat{\mathbb{C}}$ in the sense that if z_1, z_2, z_3 are those distinct points in $\hat{\mathbb{C}}$ and w_1, w_2, w_3 are three distinct points in $\hat{\mathbb{C}}$ then there is a unique element $g \in$ bihol $(\hat{\mathbb{C}})$ such that $g(z_i) = w_i, \quad 1 \leq i \leq 3$.

Let $\mathbb{D}$ denote the open unit disc $\{z : \quad |z| < 1\}$. A holomorphic $f : \mathbb{D} \to \mathbb{D}$ is called a biholomorphic automorphism of $\mathbb{D}$ if it is a bijection. The set of all biholomorphic automorphisms of $\mathbb{D}$ form a group under composition, denoted bihol$(\mathbb{D})$. We wish to determine this group.

Exercise 19: A Möbius map is a fractional linear transformation of the form $z \mapsto \alpha \cdot \frac{z-\beta}{1-\bar{\beta}z}$ where $|\alpha| = 1, \ |\beta| < 1$. Show that the set of all Möbius maps forms a subgroup of bihol$(\hat{\mathbb{C}})$, denoted Möb, and called the Möbius group. Show that Möb maps $\mathbb{D}$ to $\mathbb{D}$ bijectively, and hence Möb is a subgroup of bihol$(\mathbb{D})$. (Enough to show that Möb maps ∂D to ∂D.) Finally, show that Möb is transitive on $\mathbb{D}$ in the sense that if $z_1, z_2 \in \mathbb{D}$, there is a $g \in$ Möb such that $g(z_1) = z_2$.

Theorem: bihol$(\mathbb{D})$=Möb. That is, the Möbius maps are the only biholomorphic automorphisms of $\mathbb{D}$.

Proof: Take a biholomorphic automorphism $\varphi : \mathbb{D} \to \mathbb{D}$. Put $\alpha = \varphi(0) \in \mathbb{D}$. By the exercise above, there is a $\varphi_1 \in$ Möb such that $\varphi_1(0) = \alpha$. Put $\varphi_2 = \varphi_1^{-1} \circ \varphi$. Then $\varphi_2 \in$ bihol$(\mathbb{D})$ and $\varphi_2(0) = 0$. It is enough to show that $\varphi_2 \in$

Möb (then $\varphi = \varphi_1 \circ \varphi_2 \in$ Möb since Möb is a group). Indeed, we shall show that $\varphi_2(z) = \alpha z$ for some α with $|\alpha| = 1$.

Put

$$\begin{aligned} f(z) &= \frac{\phi_2(z)}{z},\ z \in \mathbb{D}\backslash\{0\}, \\ f(0) &= \phi_2'(0). \end{aligned}$$

By the maximum modulus principle,

$$\begin{aligned} \sup |f(z)| &= \limsup_{z\to\partial\mathbb{D}} |f(z)| = \limsup_{z\to\partial\mathbb{D}} \left|\frac{\varphi_2(z)}{z}\right| \\ &= \limsup_{z\to\partial\mathbb{D}} |\varphi_2(z)| = 1. \end{aligned}$$

Thus $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. That is, $|\varphi_2(z)| \leq |z|$ for $z \in \mathbb{D}$. Since φ_2^{-1} is also in bihol($\mathbb{D}$) sending 0 to 0, the same argument applies to φ_2^{-1}, yielding $|\varphi_2^{-1}(w)| \leq |w|$ for $w \in \mathbb{D}$. Setting $w = \varphi_2(z)$, we get $|z| \leq |\varphi_2(z)|,\quad z \in \mathbb{D}$. Thus $|\varphi_2(z)| = |z|,\ z \in \mathbb{D}$. So $|f(z)| = 1$ for $z \in \mathbb{D}$. Hence f must be a constant function α. So $\varphi_2(z) = \alpha z,\ z \in \mathbb{D}$. Since $\varphi_2 \in$ bihol ($\mathbb{D}$), must have $|\alpha| = 1$. So $\varphi_2 \in$ Möb. Then $\varphi = \varphi_1 \circ \varphi_2 \in$ Möb. $\square$

Lecture 6

A family $\mathcal{F}$ of complex valued functions on Ω is said to be *normal* (or pre-compact) if every sequence $\{f_n\} \subseteq \mathcal{F}$ has a subsequence which converges locally uniformly on Ω (to some function on Ω). A family $\mathcal{F}$ is said to be *locally* uniformly bounded (respectively locally uniformly equicontinuous) if any point $z_0 \in \Omega$ has a neighbourhood $U \subseteq \Omega$ on which $\mathcal{F}$ is uniformly bounded : $\exists\ c > 0 \ni |f(z)| \leq c \ \ \forall\ z \in U\ \forall\ f\ \in \mathcal{F}$ (respectively, on which f is uniformly equicontinuous : $\forall\ z_0 \in U,\ \ \forall\ \ \epsilon > 0,\ \exists\ \delta > 0$ such that $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon\ \forall\ z \in U,\ \forall\ f \in \mathcal{F}$.) Recall:

Arzela-Ascoli Theorem : If $\mathcal{F}$ is a locally uniformly bounded and locally uniformly equicontinuous family of (continuous) complex-valued function on Ω then $\mathcal{F}$ is normal.

Sketch of Proof: Fix a countable dense subset $D \subseteq \Omega$ (For instance, D may consist of all points of Ω with rational real and imaginary parts.) Use Cantor's diagonal argument to show that any sequence $\{f_n\} \subseteq \mathcal{F}$ has a subsequence $\{g_n\}$ which converges at all points of D. Use local boundedness and locally uniform equicontinuity of $\{g_n\}$ to conclude that $\{g_n\}$ converges locally uniformly on Ω.

Montel's Theorem: If $\mathcal{F}$ is a locally uniformly bounded family of holomorphic function on Ω, then $\mathcal{F}$ is normal.

Proof: By the Arzela-Ascoli theorem, it is enough to show that $\mathcal{F}$ is locally uniformly equicontinuous. Fix $z_0 \in \Omega$. Let $D \subseteq \Omega$ be a closed disc with centre z_0. Then we have $f(z) = \frac{1}{2\pi i} \int\limits_{\partial D} \frac{f(w)}{w-z},\ z \in D^0,$ for each $f \in \mathcal{F}$. Differentiating under the integral sign, we get

$$f'(z) = \frac{1}{2\pi i} \int\limits_{\partial D} \frac{f(w)}{(w-z)^2} \quad z \in D^0.$$

Therefore, if D_1 is a slightly smaller disc with centre z_0 then there is an absolute constant c_1 such that

$$|f'(z)| \leq c_1 \sup_{w \in D} |f(w)|,\ z \in D_1.$$

Now, as $\mathcal{F}$ is locally uniformly bounded there is a constant c_2 such $\sup_{w\in D} |f(w)| \leq c_2$ for all $f \in \mathcal{F}$. Hence we get $|f'(z)| \leq c \ \forall\, z \in D_1, \ \forall\, f \in \mathcal{F}$ where $c = c_1 c_2$. Therefore $|f(z) - f(z_0)| = |\int\limits_{z_0}^{z} f'(w)dw| \leq c|z - z_0| \ \ \forall z \in D_1, \ \forall f \in \mathcal{F}$. Then $\mathcal{F}$ is locally uniformly equicontinuous on D_1. $\square$

If $\Omega_1, \ \Omega_2$ are open subsets of $\mathbb{C}$, then a map $f : \Omega_1 \to \Omega_2$ is said to be a biholomorphic isomorphism if f is a bijection and both f and f^{-1} are holomorphic.

Exercise 20: Let $f : \Omega \to \mathbb{C}$ be holomorphic and injective. Then show that $f'(z) \neq 0 \ \forall \ z \in \Omega$, and f is a biholomorphic isomorphism between Ω and $f(\Omega)$.

Exercise 21: Let $\{f_n\}$ be a sequence of injective holomorphic functions from Ω to $\mathbb{C}$. Suppose $f_n \to f$ locally uniformly on Ω. Then show that f is either a constant function or injective. (Hint: Argument principle).

Exercise 22: If $\mathcal{F}$ is a normal family of holomorphic functions on Ω then show that the family $\mathcal{F}' := \{f' \ : \ f \in \Omega\}$ is again normal.

Let $\mathbb{D}$ denote the disc with radius 1 and centre 0.

Lemma: Let Ω be a simply connected, connected, proper, open subset of $\mathbb{C}$. Then there is an injective holomorphic function $f : \Omega \to \mathbb{D}$.

Proof: Take $\alpha \in \mathbb{C}\backslash\Omega$. Since Ω is simply connected, there is a (global) primitive g of $z \mapsto \frac{1}{z-\alpha}$ on Ω. Then $e^{g(z)} = z - \alpha$. In consequence g is injective on Ω, and further, for any $z_0 \in \Omega$, g does not assume the value $g(z_0) + 2\pi i$. Hence there is an $r > 0$ such that the closed disc D with centre $z_0 + 2\pi i$ and radius r is disjoint from $g(\Omega)$. (Else there would be a sequence $\{z_n\}$ in Ω such that $g(z_n) \to g(z_0) + 2\pi i$, whence, exponentiating, $z_n \to z_0$ and hence $g(z_n) \to g(z_0)$. Hence $g(z_0) = g(z_0) + 2\pi i$ and $2\pi i = 0$, a contradiction.) Therefore the function $f(z) = \frac{1}{g(z)-g(z_0)-2\pi i}$ on Ω is injective, holomorphic and bounded. Scaling, f can be made to map into $\mathbb{D}$. $\square$

Riemann Mapping Theorem: Every simply connected and connected proper open subset Ω of $\mathbb{C}$ is biholomorphically equivalent to $\mathbb{D}$, i.e., there is a biholomorphic isomorphism $g : \Omega \to \mathbb{D}$.

Proof: Without loss, we may assume that $0 \in \Omega$. By the lemma, there is a holomorphic injection $f : \Omega \to \mathbb{D}$. Since the Möbius group is transitive on $\mathbb{D}$, there is a Möbius map $\phi : \mathbb{D} \to \mathbb{D}$ such that ϕ takes $f(0)$ to 0. Then $\tilde{f} = \phi \circ f$ is a holomorphic injection from Ω to $\mathbb{D}$ such that $\tilde{f}(0) = 0$.

Then the family $\mathcal{F} = \{f : \Omega \to \mathbb{D}$ such that f is a holomorphic injection, $f(0) = 0\}$ is non-empty. By Montel's Theorem, $\mathcal{F}$ is a normal family. Therefore, there is an $f \in \mathcal{F}$ which maximises $|f'(0)|$ among all elements of $\mathcal{F}$. (If $\alpha = \sup\limits_{f \in \mathcal{F}} |f'(0)|$, then there is a sequence $\{f_n\} \subseteq \mathcal{F}$ such that $|f'_n(0)| \to \alpha$. Since $\mathcal{F}$ is normal, we may replace $\{f_n\}$ by a suitable subsequence and assume $f_n \to f$ locally uniformly on Ω. Then $f'_n \to f'$ locally uniformly. In particular, $f'_n(0) \to f'(0)$. Thus $|f'_n(0)| \to |f'(0)|$. Hence $|f'(0)| = \alpha > 0$. Since $f'(0) \neq 0$, f is not a constant function. Therefore by Exercise 21, f is injective. Since f is non-constant, the maximum modulus principle implies that f maps Ω into $\mathbb{D}$. So $f \in \mathcal{F}$).

Thus we may choose $f \in \mathcal{F}$ which maximises $|f'(0)|$. (This is a typical application of compactness: $\bar{\mathcal{F}}$ is compact and $f \mapsto |f'(0)|$ is a "continuous" function on $\bar{\mathcal{F}}$, hence is maximised somewhere in $\bar{\mathcal{F}}$. Then the argument principle comes in handy to show that the maximising "point" f is actually in $\mathcal{F}$.). We claim that this function f maps Ω onto $\mathbb{D}$ and hence is the required biholomorphic isomorphism between Ω and $\mathbb{D}$. Otherwise, there exists $\alpha \in \mathbb{D}$ such that f does not take the value α. Then there is a Möbius map φ such that $\varphi(\alpha) = 0$. Hence $\varphi \circ f$ is a holomorphic injection from Ω into an open subset of $\mathbb{D}$ which does not contain 0. Since $\varphi \circ f$ is a holomorphic injection onto its image, this subset is also simply connected. Hence there is a holomorphic branch of the square root function on this subset ($z \mapsto \exp(\frac{1}{2} log z)$). Composing it with the function $\varphi \circ f$, we get a holomorphic function $z \mapsto \sqrt{\varphi(f(z))}$ from Ω into $\mathbb{D} \backslash \{0\}$. Let ψ be a Möbius map sending $\sqrt{\phi(f(0))}$ to 0. Let $\tilde{f}(z) = \psi(\sqrt{\phi(f(z)}),\ z \in \Omega$. Then $\tilde{f}$ is a holomorphic map from Ω into $\mathbb{D}$, such that $\tilde{f}(0) = 0$. Also, $\tilde{f}$ is an injection ($\tilde{f}(z_1) = \tilde{f}(z_2) \Rightarrow \psi(\sqrt{\phi(f(z_1)})) = \psi(\sqrt{\phi f(z_2)}) \Rightarrow \sqrt{\phi(f(z_1))} = \sqrt{\phi(f(z_2))} \Rightarrow \phi(f(z_1)) = \phi(f(z_2)) \Rightarrow f(z_1) = f(z_2) \Rightarrow z_1 = z_2$.)

Then $\tilde{f} \in \mathcal{F}$. Let S be the squaring function. Then we have $f = (\varphi^{-1} \circ S \circ \psi^{-1}) \circ \tilde{f}$. Hence $|f'(0)| = |(\varphi^{-1} \circ S \circ \psi^{-1})'(0)| \cdot |\tilde{f}'(0)|$. But the following exercise shows that $|(\varphi^{-1} \circ S \circ \psi^{-1})'(0)| < 1$. Therefore $|f'(0)| < |\tilde{f}'(0)|$ and $\tilde{f} \in \mathcal{F}$. This contradicts the choice of f. So f must have been a surjection.

$\square$

Exercise 23: If $h : \mathbb{D} \Rightarrow \mathbb{D}$ is a holomorphic map such that $h(0) = 0$ and h is not an injection, then $|h'(0)| < 1$.

Remark: This theorem shows, in particular, that any connected and simply connected proper open subset of $\mathbb{C}$ is homeomorphic to $\mathbb{D}$.

Exercise 24: Show that $\mathbb{D}$ is homeomorphic to $\mathbb{C}$ but is not biholomorphically equivalent to $\mathbb{C}$.

Exercise 25: If $\Omega \subseteq \mathbb{C}$, $0 \notin \Omega$ and Ω admits a holomorphic branch of the square root function, then show that Ω is simply connected.