Advanced Foundational School - II
I.S.I.Bangalore, May 7 - June 2, 2007
Algebraic Number Theory Problems
(Only on topics covered in notes but not all of the same level)
B.Sury

Q 1.
Let $[K : Q] = n$ *and* $I \neq 0$ *be an ideal in* $\mathcal{O}_K$. *Prove :*
(a) $I = \sum_{i=1}^{n} Z\alpha_i$ *for some* α_i*'s with* $K = \sum_{i=1}^{n} Q\alpha_i$.
(b) $I \cap Z \neq 0$ *and* $\mathcal{O}_K/I$ *is finite.*

Q 2.
If $I = \prod_{i=1}^{r} P_i^{\alpha_i}, J = \prod_{i=1}^{r} P_i^{\beta_i}$ *are the prime ideal decompositions of two fractional ideals in a Dedekind domain, prove that*

$$GCD(I, J) = I + J = \prod_{i=1}^{r} P_i^{min(\alpha_i,\beta_i)},$$

$$LCM(I, J) = I \cap J = \prod_{i=1}^{r} P_i^{max(\alpha_i,\beta_i)}.$$

Q 3.
(a) Show that $\alpha = \sqrt{2-\sqrt{2}} + i\sqrt{\sqrt{2}-1}$ is an algebraic number (that is, a complex number which is algebraic over $\mathbf{Q}$) with the property $|\alpha| = 1$ but that $|\beta| \neq 1$ for some conjugate β of α.
(b) Give an example of an algebraic number α such that α as well as its conjugates have absolute value 1 but are not roots of unity.

Q 4.
If the ring of integers of a number field is a UFD, prove that it must be a PID.

Q 5. *Prove that* $\mathbf{Z}[e^{2i\pi/23}]$ *is not a PID.*

Q 6. *Use the Minkowski bound to determine the class number of* $\mathbf{Q}[\sqrt{-5}]$. *Using this, show that the equation* $y^2 = x^3 - 5$ *has no solutions in integers.*

Q 7. *Show that the ring of integers of* $\mathbf{Q}[\zeta_n + \zeta_n^{-1}]$ *is* $\mathbf{Z}[\zeta_n + \zeta_n^{-1}]$ *for any* n. *You may use that the ring of integers of any cyclotomic field* $\mathbf{Q}[\zeta_n]$ *is* $\mathbf{Z}[\zeta_n]$.

Q 8.
If K is a cubic extension of $\mathbf{Q}$ with discriminant between -1 and -49, use Minkowski bound to prove that K has class number 1.
Hint : Observe that the discriminant has the sign $(-1)^{r_2}$, where $2r_2$ is the number of non-real embeddings of any number field.

Q 9.
Show that the fundamental unit in $\mathbf{Q}(\sqrt{d^2-1})$ is $d+\sqrt{d^2-1}$ for each square-free integer $d \geq 2$.

Q 10.
For positive integers s, t and a prime p, consider the integers
$f_i = \sum_r (-1)^r \begin{pmatrix} t \\ rp^s + i \end{pmatrix}; 0 \leq i \leq p^s - 1.$
Using the identity

$$(1-\zeta^n)^t = f_0 - \zeta^n f_1 + \zeta^{2n} f_2 \ldots \pm \zeta^{n(p^s-1)} f_{p^s-1}$$

(with $\zeta = e^{2\pi i/p^s}$) or otherwise, show that the GCD of $f_0, \ldots f_{p^s-1}$ is a power of p.

Q 11.
For odd primes p, q consider the cyclotomic fields $K_p = Q(\zeta_p), K_q = Q(\zeta_q)$ and their maximal real subfields L_p, L_q. Look at the numbers $\pi_p \stackrel{d}{=} N_{K_p/L_p}(1-\zeta_p)$ and $\pi_q \stackrel{d}{=} N_{k_q/L_q}(1-\zeta_q)$ as elements of the composite field $L = L_p L_q$. Prove that $N_{L/Q}(\pi_p - \pi_q) = \left(\frac{p}{q}\right)$. Deduce the quadratic reciprocity law of p, q.

Q 12.
Let ζ be a primitive p^n-th root of unity where p is a prime and $n \geq 2$. For $K = Q(\zeta)$, compute $tr_{K/Q}(\zeta^m)$ for any m.

Q 13.
In the following number fields K, show that there is no α in $\mathcal{O}_K$ for which $\mathcal{O}_K = \mathbf{Z}[\alpha]$.
(i) $K = \mathbf{Q}(\sqrt{7}, \sqrt{10})$.
(ii) K is the unique subfield of $L = \mathbf{Q}(\zeta_{31})$ which has degree 6 over $\mathbf{Q}$.

Q 14.
Show that the class number of $\mathbf{Q}(-\sqrt{23})$ is 3 and of $\mathbf{Q}(-\sqrt{247})$ is 5.

Q 15.
For any number field K, show that there is a finite extension L so that every ideal in $\mathcal{O}_K$ becomes principal in $\mathcal{O}_L$.