INTRODUCTION TO ALGEBRAIC GEOMETRY NOTES

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Contents

Notations and Conventions

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote respectively the ring of integers, the field of rational numbers, the field of real numbers and the field of complex numbers.

Unless otherwise stated a ring will always be commutative with an identity element. Any ring homomorphism must take the identity element to the identity element.

1. Rings and Ideals

1.1. **Basics.**

Definition 1.1.1. A *ring* A is a set with two binary operations (addition and multiplication) such that :

(i) A is an abelian group with respect to addition (denoted by $+$);
(ii) multiplication (denoted by $\cdot$) is associative and distributive over addition.

We will only consider rings with an identity element (denoted by 1) with respect to multiplication and further the rings are *commutative* i.e. $x \cdot y = y \cdot x$ for all $x, y \in A$.

Remark 1.1.2. If we have $1 = 0$ in a ring, then we get the so called "zero" ring.

Definition 1.1.3. A *ring homomorphism* between two rings is a map between the two sets which respects addition, multiplication and the identity element.

An isomorphism between two rings is defined in the usual way.

Remark 1.1.4. We have a unique ring homomorphism from $\mathbb{Z}$ to any ring. On the other hand we have a unique map from any ring to the zero ring.

Lemma 1.1.5. *Composition of two ring homomorphisms is a ring homomorphism.*

Definition 1.1.6. A subset of a ring is called a *subring* if it is closed under addition and multiplication and contains the identity element.

Remark 1.1.7. If a subset is a subring then inclusion map is a ring homomorphism.

Definition 1.1.8. An *ideal* of a ring is a subset which is closed under addition and closed under multiplication by any element of the ring.

Lemma 1.1.9. *Let I be an ideal of a ring A. Then the quotient group A/I inherits a unique multiplicative structure from A such that the quotient map $\pi : A \to A/I$ is a surjective ring homomorphism.*

Definition 1.1.10. Rings of the form A/I with the induced ring structure from A are called *quotient rings*.

Proposition 1.1.11. *There is a one-to-one order preserving correspondence between the ideals of A which contain I and the ideals of A/I.*

Proposition 1.1.12. *Let $f : A \to B$ be a ring homomorphism. Then $ker(f)$ is an ideal of A, $im(f)$ is a subring of B and f induces a ring isomorphism $A/ker(f) \cong im(f)$.*

Theorem 1.1.12.1. *Let A be a ring and I be an ideal of A. Then the quotient ring A/I satisfies the following universal property : for any ring homomorphism $f : A \to B$ such that $f(I) = 0$, there exists a unique ring homomorphism $g : A/I \to B$ such that $f = g \circ \pi$. This property defines the quotient ring upto unique isomorphism.*

Definition 1.1.13. Let A be a ring.

(i) An element $x \in A$ is called a *zero-divisor* if there exists a non zero element $y \in A$ such that $xy = 0$. A ring with no zero-divisors $\neq 0$ is called an *integral domain*.
(ii) An element $x \in A$ is called *nilpotent* if for some integer $n > 0$ we have $x^n = 0$.
(iii) An element of a ring is called a *unit* if it has a multiplicative inverse.
(iv) A *field* is a ring in which $1 \neq 0$ and every non-zero element is a unit.

Proposition 1.1.14. *Let A be a ring $\neq 0$. Then TFAE :*

(i) *A is a field,*
(ii) *the only ideals of A are 0 and A,*
(iii) *any ring homomorphism from A to a non-zero ring is injective.*

1.2. **Prime and Maximal Ideals.**

Definition 1.2.1. An ideal $\mathfrak{p}$ of A is called prime if $\mathfrak{p} \neq A$ and $xy \in \mathfrak{p} \Rightarrow x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.
An ideal $\mathfrak{m}$ of A is called maximal if $\mathfrak{m} \neq A$ and there is no ideal $\mathfrak{a}$ of A such that $\mathfrak{m} \subset \mathfrak{a} \subsetneq A$.

Proposition 1.2.2. *Let $\mathfrak{p}, \mathfrak{m}$ be ideals of A. Then :*

(i) *$\mathfrak{p}$ is a prime ideal $\Leftrightarrow$ $A/\mathfrak{p}$ is an integral domain.*
(ii) *$\mathfrak{m}$ is a maximal ideal $\Leftrightarrow$ $A/\mathfrak{m}$ is a field.*

Remark 1.2.3. Maximal ideals are always prime, but the converse is not true in general.

Lemma 1.2.4. *Let $f : A \to B$ be a ring homomorphism and let $\mathfrak{q}$ be a prime ideal of B. Then $f^{-1}(\mathfrak{q})$ is a prime ideal of A.*

Remark 1.2.5. Inverse image of a maximal ideal need not be maximal.

Theorem 1.2.5.1. *Every ring $A \neq 0$ has a maximal ideal.*

Corollary 1.2.6.

(i) *Let $\mathfrak{a} \neq A$ be an ideal. Then there is a maximal ideal of A containing $\mathfrak{a}$.*
(ii) *Every non-unit of A is contained in a maximal ideal.*

Definition 1.2.7. A *local ring* is a ring with exactly one maximal ideal. It is usually denoted by $(A, \mathfrak{m})$.
For a local ring $(A, \mathfrak{m})$, the quotient field $A/\mathfrak{m}$ is called the *residue field*.

Proposition 1.2.8.

(i) *Let $\mathfrak{m} \neq A$ be an ideal of A such that every element of $A - \mathfrak{m}$ is a unit. Then A is a local ring with $\mathfrak{m}$ being the maximal ideal.*
(ii) *Let $\mathfrak{m}$ be a maximal ideal of A such that every element of $1 + \mathfrak{m}$ is a unit. Then A is a local ring.*

Definition 1.2.9. An ideal generated by a single element is called a *principal* ideal.
An integral domain in which every ideal is principal is called a *principal ideal domain*.

Definition 1.2.10. The set of nilpotent elements of a ring is called the *nilradical* of the ring.
For a ring A we denote its nilradical by $Nil(A)$ or $\sqrt{0}$.

Proposition 1.2.11. *Let A be a ring. Then $Nil(A)$ is an ideal of A and $A/Nil(A)$ has no nilpotent elements.*

Proposition 1.2.12. *Let A be a ring. Then $Nil(A)$ is the intersection of all the prime ideals of A.*

Definition 1.2.13. The intersection of all the maximal ideals of a ring is called its *Jacobson radical*.
For a ring A we denote its Jacobson radical by $J(A)$.

Proposition 1.2.14. *$x \in J(A) \Leftrightarrow 1 - xy$ is a unit for all $y \in A$.*

1.3. **Operations on Ideals.**

Definition 1.3.1. Given any family $\{\mathfrak{a}_i\}_{i\in I}$ (possibly infinite) of ideals of A we define their *sum* to be :

$$\sum_{i\in I}\mathfrak{a}_i = \{\sum_{i\in I} x_i : x_i \in \mathfrak{a}_i \;\; \forall i \in I \;\text{ and almost all of the }\; x_i \;\text{ are zero}\}.$$

It is the smallest ideal of A which contains all the ideals $\mathfrak{a}_i$.

Remark 1.3.2. The *intersection* of any family $\{\mathfrak{a}_i\}_{i\in I}$ of ideals is an ideal.

Definition 1.3.3. Given any finite family $\{\mathfrak{a}_i\}_{i\in I}$ of ideals of A we define their *product* to be :

$$\prod_{i\in I}\mathfrak{a}_i = \{\prod_{i\in I} x_i : x_i \in \mathfrak{a}_i \;\; \forall i \in I\}$$

Remark 1.3.4. We can talk about the powers $\mathfrak{a}^n \;\; (n > 0)$ of any ideal $\mathfrak{a}$. By convention $\mathfrak{a}^0 = A$.

Proposition 1.3.5.

(i) *The three operations defined above are all commutative and associative.*
(ii) $\mathfrak{a}(\mathfrak{b}+\mathfrak{c}) = \mathfrak{a}\mathfrak{b}+\mathfrak{a}\mathfrak{c}$ *for any ideals* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$.

Definition 1.3.6. Two ideals $\mathfrak{a}, \mathfrak{b}$ are said to be *coprime* (or *comaximal*) if $\mathfrak{a}+\mathfrak{b} = A$.

Definition 1.3.7. Given a family $\{A_i\}_{i\in A}$ of rings we define their *direct product* to be :

$$\prod_{i\in I} A_i = \{(x_i)_{i\in I} : x_i \in A_i \;\; \forall i \in I\}.$$

We define addition and multiplication to be componentwise. Then $\prod_{i\in I} A_i$ is a commutative ring with identity element $(1)_{i\in I}$.

Theorem 1.3.8. *Let A be a ring and $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ ideals of A. We can define a homomorphism*

$$\phi : A \to \prod_{i=1}^{n} A/\mathfrak{a}_i$$

by the rule $\phi(x) = (x+\mathfrak{a}_1, \dots, x+\mathfrak{a}_n)$. Then :

(i) *if $\mathfrak{a}_i, \mathfrak{a}_j$ are coprime whenever $i \neq j$, we have $\prod_{i=1}^{n}\mathfrak{a}_i = \bigcap_{i=1}^{n}\mathfrak{a}_i$;*
(ii) *ϕ is injective $\Leftrightarrow \bigcap_{i=1}^{n}\mathfrak{a}_i = (0)$;*
(iii) *ϕ is surjective $\Leftrightarrow \mathfrak{a}_i, \mathfrak{a}_j$ are coprime whenever $i \neq j$.*

Proposition 1.3.9. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be prime ideals and let $\mathfrak{a}$ be an ideal such that $\mathfrak{a} \subseteq \bigcup_{i=1}^{n}\mathfrak{p}_i$. Then $\mathfrak{a} \subseteq \mathfrak{p}_i$ for some i.*

Proposition 1.3.10. *Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be ideals and let $\mathfrak{p}$ be a prime ideal such that $\mathfrak{p} \supseteq \bigcap_{i=1}^{n}\mathfrak{a}_i$. Then $\mathfrak{p} \supseteq \mathfrak{a}_i$ for some i. If $\mathfrak{p} = \bigcap_{i=1}^{n}\mathfrak{a}_i$ then $\mathfrak{p} = \mathfrak{a}_i$ for some i.*

Definition 1.3.11.

(i) Let $\mathfrak{a}, \mathfrak{b}$ be ideals. Their *ideal quotient* is defined as $(\mathfrak{a} : \mathfrak{b}) = \{x \in A : x\mathfrak{b} \subseteq \mathfrak{a}\}$.
(ii) Let $\mathfrak{a}$ be an ideal. The *annihilator* of $\mathfrak{a}$ is defined as $\mathrm{Ann}(\mathfrak{a}) = \{x \in A : x\mathfrak{a} = 0\}$.
(iii) The *radical* of $\mathfrak{a}$ is defined to be $r(\mathfrak{a}) = \{x \in A : x^n \in \mathfrak{a} \;\text{ for some }\; n > 0\}$.

Remark 1.3.12.

(i) Note that $r(\mathfrak{a})$ is an ideal. It is also denoted by $\sqrt{\mathfrak{a}}$.

(ii) $\text{Ann}(\mathfrak{a}) = (0 : \mathfrak{a})$.
(iii) $r(\mathfrak{a})$ is an ideal.

Proposition 1.3.13.

(1) $\mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{b})$;
(2) $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subseteq \mathfrak{a}$;
(3) $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c}) = ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b})$;
(4) $(\bigcap_i \mathfrak{a}_i : \mathfrak{b}) = \bigcap_i (\mathfrak{a}_i : \mathfrak{b})$;
(5) $(\mathfrak{a} : \sum_i \mathfrak{b}_i) = \bigcap_i (\mathfrak{a} : \mathfrak{b}_i)$.

Proposition 1.3.14.

(1) $r(\mathfrak{a}) \supseteq \mathfrak{a}$;
(2) $r(r(\mathfrak{a})) = r(\mathfrak{a})$;
(3) $r(\mathfrak{a}\mathfrak{b}) = r(\mathfrak{a} \bigcap \mathfrak{b}) = r(\mathfrak{a}) \bigcap r(\mathfrak{b})$;
(4) $r(\mathfrak{a}) = (1) \Leftrightarrow \mathfrak{a} = (1)$;
(5) $r(\mathfrak{a} + \mathfrak{b}) = r(r(\mathfrak{a}) + r(\mathfrak{b}))$;
(6) *if $\mathfrak{p}$ is a prime, then $r(\mathfrak{p}^n) = \mathfrak{p}$ for all $n > 0$.*

Proposition 1.3.15. *The radical of an ideal $\mathfrak{a}$ is the intersection of the prime ideals which contain $\mathfrak{a}$.*

Proposition 1.3.16. *Let D be the set of zero-divisors of A. Then $D = \bigcup_{x \neq 0} \text{Ann}(x) = \bigcup_{x \neq 0} r(\text{Ann}(x))$.*

Proposition 1.3.17. *Let $\mathfrak{a}, \mathfrak{b}$ be ideals of A such that $r(\mathfrak{a}), r(\mathfrak{b})$ are coprime. Then $\mathfrak{a}, \mathfrak{b}$ are coprime.*

1.4. Extension and Contraction.

Definition 1.4.1. Let $f : A \to B$ be a ring homomorphism. For an ideal $\mathfrak{a}$ of A we define its *extension* $\mathfrak{a}^e$ to be the ideal generated by $f(\mathfrak{a})$ in B. For an ideal $\mathfrak{b}$ of B we define its *contraction* $\mathfrak{b}^c$ to be the ideal $f^{-1}(\mathfrak{b})$.

Lemma 1.4.2. *Contraction of a prime ideal is prime.*

Remark 1.4.3.

(1) Extension of a prime ideal need not be prime.
(2) Contraction of a maximal ideal need not be maximal.

Proposition 1.4.4.

(1) $\mathfrak{a} \subseteq \mathfrak{a}^{ec}, \mathfrak{b} \supseteq \mathfrak{b}^{ce}$;
(2) $\mathfrak{a}^e = \mathfrak{a}^{ece}, \mathfrak{b}^c = \mathfrak{b}^{cec}$;
(3) *There is a bijective correspondence between the set of contracted ideals of A and the set of extended ideals of B given by $\mathfrak{a} \mapsto \mathfrak{a}^e$ and $\mathfrak{b} \mapsto \mathfrak{b}^c$ respectively.*

Proposition 1.4.5. *Let $\mathfrak{a}_1, \mathfrak{a}_2$ be ideals of A. Then :*

(1) $(\mathfrak{a}_1 + \mathfrak{a}_2)^e = \mathfrak{a}_1^e + \mathfrak{a}_2^e$;
(2) $(\mathfrak{a}_1 \bigcap \mathfrak{a}_2)^e \subseteq \mathfrak{a}_1^e \bigcap \mathfrak{a}_2^e$;
(3) $(\mathfrak{a}_1\mathfrak{a}_2)^e = \mathfrak{a}_1^e\mathfrak{a}_2^e$;
(4) $(\mathfrak{a}_1 : \mathfrak{a}_2)^e \subseteq (\mathfrak{a}_1^e : \mathfrak{a}_2^e)$;
(5) $r(\mathfrak{a})^e \subseteq r(\mathfrak{a}^e)$.

Proposition 1.4.6. *Let $\mathfrak{b}_1, \mathfrak{b}_2$ be ideals of B. Then :*

(1) $(\mathfrak{b}_1 + \mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^c + \mathfrak{b}_2^c$;
(2) $(\mathfrak{b}_1 \bigcap \mathfrak{b}_2)^c = \mathfrak{b}_1^c \bigcap \mathfrak{b}_2^c$;
(3) $(\mathfrak{b}_1\mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^C\mathfrak{b}_2^c$;

(4) $(\mathfrak{b}_1 : \mathfrak{b}_2)^c \subseteq (\mathfrak{b}_1^c : \mathfrak{b}_2^c)$;
(5) $r(\mathfrak{b})^c = r(\mathfrak{b}^c)$.

2. MODULES

2.1. **Basics.**

Definition 2.1.1. Let A be a ring. An *A-module* is an abelian group M (written additively) along with a multiplication map $A \times M \to M$, $(a, x) \mapsto a \cdot x$, satisfying the following axioms :

$$a \cdot (x + y) = a \cdot x + a \cdot y, (a + b) \cdot x = a \cdot x + b \cdot y, (ab) \cdot x = a \cdot (b \cdot x), 1 \cdot x = x$$

$(a, b \in A; x, y \in M)$.

Equivalently, an A-module M is an abelian group together with a ring homomorphism $A \to End_{\text{Ab}}(M)$, the ring of endomorphisms of the abelian group M.

Remark 2.1.2.

(1) An ideal $\mathfrak{a}$ of A is an A-module.
(2) If $A = k$, a field, then A-module $= k$-vector space.
(3) $A = \mathbb{Z}$, then $\mathbb{Z}$-module $=$ abelian group.
(4) $A = k[x]$ where k is a field; the A-module $= k$-vector space with a linear transformation.

Definition 2.1.3. Let M, N be A-modules. A map $f : M \to N$ is an *A-module homomorphism* (or is *A-linear*) if $f(x + y) = f(x) + f(y)$, $f(a \cdot x) = a \cdot f(x)$ for all $a \in A, x, y \in M$.

Proposition 2.1.4.

(1) *Composition of two A-module homomorphisms is again an A-module homomorphism.*
(2) *The set* $\text{Hom}_A(M, N)$ *of all A-module homomorphisms from* M *to* N *has the structure of an A-module.*
(3) *There is a natural isomorphism* $\text{Hom}_A(A, M) \cong M$.
(4) *If* $u : M \to M'$ *is an A-module homomorphism, then we have an induced morphism of A-modules* $u^* : \text{Hom}_A(M', N) \to \text{Hom}_A(M, N)$ *given by* $u^*(f) = f \circ u$.
(5) *If* $v : N \to N'$ *is an A-module homomorphism, then we have an induced morphism of A-modules* $v_* : \text{Hom}_A(M, N) \to \text{Hom}_A(M, N')$ *given by* $v_*(g) = v \circ g$.

Definition 2.1.5. Let M be an A-module. A *submodule* M' of M is a subgroup which is closed under multiplication by elements of A.

Lemma 2.1.6. *Let* M' *be a submodule of* M. *Then the quotient group* M/M' *inherits a natural A-module structure from* M.

Definition 2.1.7. Modules of the form M/M' with the inherited module structure are called quotient modules.

Proposition 2.1.8. *Let* M, N *be A-modules and let* M' *be a submodule of* M.

(1) *The natural map* $\pi : M \to M/M'$ *is a surjective A-module homomorphism.*
(2) *There is a one-to-one order preserving correspondence between submodules of* M *which contain* M' *and submodules of the quotient module* M/M'.
(3) *Given any A-module homomorphism* $f : M \to N$ *such that* $f(M') = 0$, *there exists a unique morphism* $f' : M/M' \to N$ *of A-modules satisfying* $f = f' \circ \pi$. *Quotient modules are characterized by this universal property.*

Definition 2.1.9. Let $f : M \to N$ be an A-module homomorphism.

(1) The *kernel* of f is the set $\text{Ker}(f) = \{x \in M : f(x) = 0\}$.
(2) The *image* of f is the set $\text{Im}(f) = f(M)$.
(3) The *cokernel* of f is the set $\text{Coker}(f) = N/\text{Im}(f)$.

Lemma 2.1.10. $\text{Ker}(f)$ *is a submodule of* M. $\text{Im}(f)$ *is a submodule of* N. $\text{Coker}(f)$ *is a quotient module of* N.

Proposition 2.1.11. *Let* $f : M \to N$ *be an* A*-module homomorphism and let* M' *be a submodule of* M *such that* $M' \subseteq \text{Ker}(f)$. *Then we have an induced* A*-module morphism* $f' : M/M' \to N$ *with* $\text{Ker}(f') = \text{Ker}(f)/M'$. *Further, we have an isomorphism of* A*-modules* $M/\text{Ker}(f) \cong \text{Im}(f)$.

Proposition 2.1.12. *Let* $L \supseteq M \supseteq N$ *be* A*-modules. Then*

$$(L/N)/(M/N) \cong L/M$$

2.2. Operations on Modules.

Definition 2.2.1. Let M be an A-module and let $\{M_i\}_{i\in I}$ be a family of submodules of M. Their *sum* $\sum_{i\in I} M_i$ is defined as the set of all sums $\sum_{i\in I} x_i$ where $x_i \in M_i$ for all $i \in I$ and almost all the x_i are zero.

$\sum_{i\in I} M_i$ is the smallest submodule of M containing all of the M_i's.

Remark 2.2.2. The intersection $\bigcap M_i$ is again a submodule of M.

Proposition 2.2.3. *Let* M_1, M_2 *be submodules of* M. *Then* $(M_1 + M_2)/M_1 \cong M_2/(M_1 \bigcap M_2)$.

Definition 2.2.4. Let $\mathfrak{a}$ be an ideal of A and let M be an A-module. We define the *product* $\mathfrak{a}M$ to be the set of all finite sums $\sum_{i\in I} a_i x_i$ where $a_i \in \mathfrak{a}, x_i \in M$ for all $i \in I$.

It is a submodule of M.

Definition 2.2.5.

(1) Let N, P be submodules of M. We define $(N : P)$ to be the set of all $a \in A$ such that $aP \subseteq N$. It is an ideal of A.
(2) We define the *annihilator* of M to be the set of all $a \in A$ such that $aM = 0$. It is an ideal denoted by $\text{Ann}(M)$. In fact $\text{Ann}(M) = (0 : M)$.
(3) An A-module M is called *faithful* if $\text{Ann}(M) = 0$.

Lemma 2.2.6.

(1) *Let* $\mathfrak{a}$ *be an ideal of* A *such that* $\mathfrak{a} \subseteq \text{Ann}(M)$. *Then* M *has a natural structure of* $A/\mathfrak{a}$*-module. If* $\text{Ann}(M) = \mathfrak{a}$, *then* M *is faithful as an* $A/\mathfrak{a}$*-module.*
(2) $\text{Ann}(M + N) = \text{Ann}(M) \bigcap \text{Ann}(N)$, $(N : P) = \text{Ann}((N + P)/N)$.

Definition 2.2.7. Let $\{M_i\}_{i\in I}$ be a family of A-modules.

(1) We define their *direct product* $\prod_{i\in I} M_i$ to be the set of all families $(x_i)_{i\in I}$ such that $x_i \in M_i$ for all $i \in I$. $\prod_{i\in I} M_i$ has a natural A-module structure given by componentwise addition and scalar multiplication.
(2) We define their *direct sum* $\bigoplus_{i\in I} M_i$ to be the set of all families $(x_i)_{i\in I}$ such that $x_i \in M_i$ for all $i \in I$ and almost all x_i are zero. $\bigoplus_{i\in I} M_i$ has a natural A-module structure given by componentwise addition and scalar multiplication.

Remark 2.2.8. Direct product and direct sum are the same if the index set I is finite. But it is not true in general.

2.3. **Finitely Generated Modules.**

Definition 2.3.1. Let M be an A-module. A set of elements $\{x_i\}_{i\in I}$ is said to *generate* M if every element of M can be expressed as a finite linear combination of the x_i's with coefficients in A.

M is said to be *finitely generated* if it has a finite set of generators.

Definition 2.3.2. A *free* A-module is one which is isomorphic to an A-module of the form $\bigoplus_{i\in I} M_i$, where each M_i is isomorphic to A as an A-module. It is sometimes denoted by A^I.

Remark 2.3.3. A finitely generated free A-module is therefore isomorphic to $A \bigoplus \cdots \bigoplus A$ (n summands), which is denoted by A^n. By convention A^0 is the zero module.

Proposition 2.3.4. *M is a finitely generated A-module if and only if M is isomorphic to a quotient of A^n for some $n \in \mathbb{N}$.*

Definition 2.3.5. An A-module M is said to be *finitely presented* if M is a quotient of A^n for some $n \in \mathbb{N}$ and the kernel of the quotient map is a finitely generated A-module.

Proposition 2.3.6. *Let M be a finitely generated A-module, let $\mathfrak{a}$ be an ideal of A, and let ϕ be an A-module endomorphism of M such that $\phi(M) \subseteq \mathfrak{a}M$. Then ϕ satisfies an equation of the form $\phi^n + a_1\phi^{n-1} + \ldots + a_{n-1}\phi + a_n = 0$ where the $a_i \in \mathfrak{a}$.*

Corollary 2.3.7. *Let M be a finitely generated A-module and $\mathfrak{a}$ be an ideal of A such that $\mathfrak{a}M = M$. Then there exists $x \equiv 1 (\text{mod } \mathfrak{a})$ such that $xM = 0$.*

Corollary 2.3.8. *Let M be a finitely generated A-module and $\mathfrak{a}$ be an ideal of A contained in the Jacobson radical of A. Then $\mathfrak{a}M = M$ implies $M = 0$.*

Corollary 2.3.9. *Let M be a finitely generated A-module, N a submodule of M and $\mathfrak{a}$ be an ideal of A contained in $J(A)$. Then $M = \mathfrak{a}M + N \implies M = N$.*

Proposition 2.3.10. *Let $(A, \mathfrak{m})$ be a local ring with with residue field k. Let M be a finitely generated A-module. Then*

(i) *$M/\mathfrak{m}M$ is naturally a finite dimensional k-vector space.*
(ii) *Let $\{x_i\}_{i=1}^n$ be elements of M whose image in $M/\mathfrak{m}M$ form a k-basis for this vector space. Then $\{x_i\}_{i=1}^n$ generate M.*

2.4. **Exact Sequences.**

Definition 2.4.1. A sequence of A-modules and A-morphisms

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \longrightarrow \cdots$$

is said to be a *complex* if $f_i \circ f_{i-1} = 0$, or in other words $\text{Im}(f_{i-1}) \subseteq \text{Ker}(f_i)$.

It is said to be *exact* at M_i if $\text{Im}(f_{i-1}) = \text{Ker}(f_i)$. The sequence is called *exact* if it is exact at each M_i.

Lemma 2.4.2.

(1) *$0 \longrightarrow M' \xrightarrow{f} M$ is exact iff f is injective;*
(2) *$M \xrightarrow{g} M'' \longrightarrow 0$ is exact iff g is surjective;*
(3) *$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is exact iff f is injective, g is surjective and $\text{Im}(f) = \text{Ker}(g)$.*

Remark 2.4.3. A sequence of the type as the last sequence in the above Lemma is called a *short exact sequence*.

Any long exact sequence can be split up into short exact sequences.

Proposition 2.4.4.

(1) *Let* $M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$ *be a sequence of* A*-modules and homomorphisms. Then it is exact iff for all* A*-modules* N*, the sequence* $0 \longrightarrow \mathrm{Hom}(M'', N) \xrightarrow{v^*} \mathrm{Hom}(M, N) \xrightarrow{u^*} \mathrm{Hom}(M', N)$ *is exact.*

(2) *Let* $0 \longrightarrow N' \xrightarrow{u} N \xrightarrow{v} N''$ *be a sequence of* A*-modules and morphisms. Then it is exact iff for all* A*-modules* M*, the sequence* $0 \longrightarrow \mathrm{Hom}(M, N') \xrightarrow{u_*} \mathrm{Hom}(M, N) \xrightarrow{v_*} \mathrm{Hom}(M, N'')$ *is exact.*

Definition 2.4.5. An A-module M is called a *projective module* if for any short exact sequence $0 \longrightarrow N' \xrightarrow{u} N \xrightarrow{v} N'' \longrightarrow 0$ of A-modules, the induced sequence $0 \longrightarrow \mathrm{Hom}(M, N') \xrightarrow{u_*} \mathrm{Hom}(M, N) \xrightarrow{v_*} \mathrm{Hom}(M, N'') \longrightarrow 0$ is exact.

An A-module N is called a *injective module* if for any short exact sequence $0 \longrightarrow M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$ of A-modules, the induced sequence $0 \longrightarrow \mathrm{Hom}(M'', N) \xrightarrow{v^*} \mathrm{Hom}(M, N) \xrightarrow{u^*} \mathrm{Hom}(M', N) \longrightarrow 0$ is exact.

Proposition 2.4.6. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{u} & M & \xrightarrow{v} & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & N' & \xrightarrow{u'} & N & \xrightarrow{v'} & N'' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of A*-modules and morphisms, with exact rows. Then there exists an exact sequence*

$$0 \longrightarrow \mathrm{Ker}(f') \xrightarrow{\bar{u}} \mathrm{Ker}(f) \xrightarrow{\bar{v}} \mathrm{Ker}(f'') \xrightarrow{\delta} \mathrm{Coker}(f') \xrightarrow{\bar{u'}} \mathrm{Coker}(f) \xrightarrow{\bar{v'}} \mathrm{Coker}(f'') \longrightarrow 0$$

where $\bar{u}, \bar{v}$ *are restrictions of* u, v *and* $\bar{u'}, \bar{v'}$ *are induced by* u', v'*.*

Remark 2.4.7. This is the so called **Snake Lemma**. The morphism δ is called the *boundary morphism* or *connecting morphism*.

Proposition 2.4.8. *We have natural isomorphism of* A*-modules :*

(1) $\mathrm{Hom}_A(M, N_1 \oplus N_2) \cong \mathrm{Hom}_A(M, N_1) \times \mathrm{Hom}_A(M, N_2)$*;*
(2) $\mathrm{Hom}_A(M_1 \oplus M_2, N) \cong \mathrm{Hom}_A(M_1, N) \times \mathrm{Hom}_A(M_2, N)$*;*
(3) $\mathrm{Hom}_A(M, \prod_{i \in I} N_i) \cong \prod_{i \in I} \mathrm{Hom}_A(M, N_i)$*;*
(4) $\mathrm{Hom}_A(\bigoplus_{i \in I} M_i, N) \cong \prod_{i \in I} \mathrm{Hom}_A(M_i, N)$*.*

Definition 2.4.9. A short exact sequence $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ of A-modules is said to be *split exact* if there exists a morphism $h : M'' \to M$ such that $g \circ h = \mathbb{1}_{M''}$.

Proposition 2.4.10. *Let* $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ *be a short exact sequence. TFAE :*

(1) *the above sequence is split exact;*
(2) *there exists a morphism* $\pi : M \to M'$ *such that* $f \circ \pi = \mathbb{1}_{M'}$*;*
(3) *we have a natural direct sum decomposition* $M \cong f(M') \oplus \mathrm{Ker}(M'')$ *such that the inclusion map induces* f *and the projection map induces* g*.*

2.5. **Tensor Product.**

Definition 2.5.1. Let M, N, P be A-modules. A mapping $f : M \times N \to P$ is said to be A-*bilinear* if for each $x \in M$ the map $y \mapsto f(x, y)$ of $N \to P$ is A-linear, and for each $y \in N$ the map $x \mapsto f(x)$ of $M \to P$ is A-linear.

Theorem 2.5.2. *Let M, N be A-modules. Then there exits a pair (T, θ) consisting of an A-module T and an A-bilinear map $\theta : M \times N \to T$ with the following universal property :*

given any A-module P and any bilinear map $f : M \times N \to P$, there exists a unique A-linear mapping $f^{'} : T \to P$ such that $f = f^{'} \circ \theta$.

Such a pair (T, θ) is unique upto unique isomorphism.

Definition 2.5.3. The pair (T, θ) as described in the above Theorem is called the *tensor product* of M and N. It is denoted by $M \otimes_A N$.

Corollary 2.5.4. *Let $x_i \in M, y_i \in N$ ($1 \leq i \leq n$) be such that $\sum_i x_i \otimes y_i = 0$ in $M \otimes_A N$. Then there exist finitely generated submodules M_0 of M and N_0 of N such that $\sum_i x_i \otimes y_i = 0$ in $M_0 \otimes N_0$.*

Proposition 2.5.5. *Let M, N, P be A-modules. Then there exist unique isomorphisms :*

(1) $A \otimes_A M \cong M$;
(2) $M \otimes_A N \cong N \otimes_A M$;
(3) $(M \oplus N) \otimes_A P \cong (M \otimes_A P) \oplus (N \otimes P)$;
(4) $(M \otimes_A N) \otimes_A P \cong M \otimes_A (N \otimes_A P) \cong M \otimes_A N \otimes_A P$.

3. Localisation

4. Chain Conditions

5. Primary Decomposition

6. Integral Extensions

7. Dimension Theory

8. Affine Varieties

9. Projective Varieties

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