ESSENTIAL DIMENSION OF ABELIAN VARIETIES OVER NUMBER FIELDS

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Abstract. We affirmatively answer a conjecture in the preprint “Essential dimension and algebraic stacks,” proving that the essential dimension of an abelian variety over a number field is infinite.

Let $k$ be a field and let $\text{Fields}_k$ denote the category whose objects are field extensions $L/k$ and whose morphisms are inclusions $L \hookrightarrow M$ of fields. Let $F : \text{Fields}_k \to \text{Sets}$ be a covariant functor. A field of definition for an element $a \in F(L)$ is a subfield $M$ of $L$ over $k$ such that $a \in \text{im}(F(M) \to F(L))$. The essential dimension of $a \in F(L)$ is $\text{ed} a := \inf\{\text{trdeg}_k M | M$ is a field of definition for $a\}$. The essential dimension of the functor $F$ is $\text{ed} F := \sup\{\text{ed} a | L \in \text{Fields}_k, a \in F(L)\}$.

If $G$ is an algebraic group over $k$, we write $\text{ed} G$ for the essential dimension of the functor $L \mapsto H^1_{\text{fppf}}(L, G)$. That is $\text{ed} G$ is the essential dimension of the functor sending a field $L$ to the set of isomorphism classes of $G$-torsors over $L$. The notion of essential dimension of a finite group was introduced by J. Buhler and Z. Reichstein. The definition of the essential dimension of a functor is a generalization given later by A. Merkurjev. In [3] (which the reader could consult for further background), a notion of essential dimension for algebraic stacks was introduced. In the terminology of that paper, $\text{ed} G$ is the essential dimension of the stack $\mathcal{B}G$.

The purpose of this paper is to generalize the following result.

**Theorem 1** (Corollary 10.4 [3]). Let $E$ be an elliptic curve over a number field $k$. Assume that there is at least one prime $p$ of $k$ where $E$ has semistable bad reduction. Then $\text{ed} E = +\infty$.

Note that another equivalent way of stating the theorem is to say that $\text{ed} E = +\infty$ for all elliptic curves $E$ such that $j(E) \in \mathbb{Q} \setminus \{\text{algebraic integers}\}$. The result was proved by showing that Tate curves have infinite essential dimension. However, this method does not apply to elliptic curves with integral $j$ invariants. Nonetheless, Conjecture 10.5 of [3] guesses that $\text{ed} E = +\infty$ for all elliptic curves over number fields. This conjecture is answered by the following.

**Theorem 2.** Let $A$ be a non-trivial abelian variety over a number field $k$. Then $\text{ed} A = +\infty$.

Note that if $A$ is an abelian variety over $C$, then $\text{ed} A = 2\text{dim} A$. This is the main result of [2].

The theorem is an easy consequence of the following result whose formulation does not involve essential dimension. To state it, for a positive integer $m$, let $\mu_m$ denote the group scheme of $m$-th roots of unity; and, for a rational prime $l$, let $\mu_{l^\infty}$ denote the union $\bigcup_{n \in \mathbb{Z}_+} \mu_{l^n}$. Theorem 2 will follow.
**Theorem 3.** Let $A$ be a non-trivial abelian variety over a number field $k$. Then there is an odd prime $\ell$ and an algebraic field extension $L/k$ such that

1. $\mathbb{Q}_\ell / \mathbb{Z}_\ell \subset A(L)$.
2. $1 < |\mu_{\ell\infty}(L)| < \infty$.

In the first section, we derive Theorem 2 from Theorem 3. The technique used is a result of M. Florence concerning the essential dimension of $\mathbb{Z}/\ell^n$. In section 2, we prove Theorem 3. Here the main results used are those of Bogomolov and Serre on the action of the absolute Galois group $\text{Gal}(k)$ on the Tate module $T/\ell A$.

**Note.** The recent preprint [7] of Karpenko and Merkurjev provides another way to show that the essential dimension of an abelian variety over a number field is infinite. To be precise, by generalizing the results of that paper slightly, one can use them to compute the essential dimension of the group scheme $A[n]$ of $n$-torsion points of an abelian variety. In fact, using this idea one can show that the essential dimension of an abelian variety over a $p$-adic field is also infinite. However, the present proof of Theorem 2 is shorter than a proof using [7] would be and we hope that Theorem 3 is independently interesting.

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1. **Theorem 3 implies Theorem 2**

As mentioned above, we will use the following result [6, Theorem 4.1] of M. Florence.

**Theorem 4.** Let $\ell$ be an odd prime and $r$ a positive integer. Let $L/\mathbb{Q}$ be a field such that $|\mu_{\ell\infty}(L)| = \ell^r < \infty$. Then, for any positive integer $k$,

$$\text{ed}_L \mathbb{Z}/\ell^k = \max\{1, k - r\}.$$  

**Corollary 5.** Let $A$ be an abelian variety over a field $L$ of characteristic 0. Let $\ell$ be an odd prime and suppose that the statements in the conclusion of Theorem 3 are satisfied; i.e:

1. $\mathbb{Q}_\ell / \mathbb{Z}_\ell \subset A(L)$.
2. $1 < |\mu_{\ell\infty}(L)| < \infty$.

Then $\text{ed}A = +\infty$.

**Proof.** Since $L$ satisfies (2), $\text{ed}_L \mathbb{Z}/\ell^n \to \infty$ as $n \to \infty$. By (1), there is an injection $(\mathbb{Z}/\ell^n)_{L} \to A$. Therefore, by [1, Theorem 6.19], $\text{ed}A \geq \text{ed}_L \mathbb{Z}/\ell^n - \dim A$ for all $n$. Letting $n$ tend to $\infty$, we see that $\text{ed}A = +\infty$. \hfill $\Box$

**Proof of Theorem 2 assuming Theorem 3.** Let $A$ be a non-trivial abelian variety over a number field $k$. Using Theorem 3 and Corollary 5, we can find a field extension $L/k$ such that $\text{ed}A_L = +\infty$. This implies that $\text{ed}A = +\infty$ (by [1, Proposition 1.5]).
2. Galois representations and the proof of Theorem 3

Before proving Theorem 3, we fix some (standard) notation. We write $G := \text{Gal}(\overline{k}/k)$ for the absolute Galois group of the number field $k$. For a rational prime $\ell$, we write $T_{\ell}A$ for the Tate-module $\lim_{\rightarrow} A[\ell^n]$ of the abelian variety $A$. We write $V_{\ell}A$ for $T_{\ell}A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. For an integer $n$, we write $\mathbb{Z}/n(1)$ for $\mu_n$, and for $j \in \mathbb{Z}$, $\mathbb{Z}/n(j)$ for $\mu_n^{\otimes j}$. We write $\mathbb{Z}_\ell(j) := \lim_{\rightarrow} \mathbb{Z}/\ell^m(j)$.

For any prime $p$ of $k$ where $A$ has good reduction, write $T_p$ for the corresponding Frobenius torus [4, Definition 3.1 and p. 326]. Suppose that $A$ is non-trivial. Then, by [4, Proposition 3.2], $T_p$ contains a rank 1 torus $D \cong \mathbb{G}_m$ such that, for every rational prime $\ell \not\equiv p$, $D(\mathbb{Q}_\ell) \subset \text{GL}_2(V_{\ell}A)$ is the set of homotheties (i.e. scalar matrices).

**Lemma 6.** Let $p$ be a prime of $k$ such that the reduction $A/p$ of $A$ at $p$ is good but not supersingular and non-trivial. Then the rank of $T_p$ is strictly greater than 1.

**Proof.** This follows directly from [4, Proposition 3.3].

The following proposition was suggested to us by N. Fakhruddin.

**Proposition 7.** Let $V$ be an $n$-dimensional vector space over a field $F$, and let $T$ be an $F$-split torus in $\text{GL}_V$ of rank at least 2 containing the homotheties. Then there is a non-zero vector $v \in V$ and a rank 1 subtorus $S \subset T$ such that

1. $S$ fixes $v$;
2. the determinant map $\det : S \to \mathbb{G}_m$ is surjective.

**Proof.** We can find a basis $e_1, \ldots, e_n$ of $V$ and characters $\lambda_1, \ldots, \lambda_n \in X^*(T)$ such that $i e_i = \lambda_i(t)e_i$ for $t \in T, i \in \{1, \ldots, n\}$. Since $T \subset \text{GL}_V$, the $\lambda_i$ generate $X^*(T)$. Since $T$ contains the homotheties, $\text{det}$ is a non-trivial character of $T$. Moreover, since $T \subset \text{GL}_V$, the $\lambda_i$ generate $X^*(T)$. Since $\dim X^*(T) \otimes \mathbb{Q} \geq 2$, it follows that there exists $i$ such that $\lambda_i \not\subset \text{det}^\perp$. Thus we can find a cocharacter $\nu$ such that $\langle \nu, \lambda_i \rangle = 0$ but $\langle \nu, \text{det} \rangle \neq 0$. Set $S$ equal to the image of $\nu$ in $T$ and $v = e_i$.

**Proof of Theorem 3.** Let $A$ be a non-trivial abelian variety over a number field $k$. We can find a prime $p$ in $k$ such that $A$ has good reduction at $p$ but $A/p$ is not supersingular. (This is well-known if $\dim A = 1$: the case where $A$ has CM is standard and otherwise it follows from the exercise on page IV-13 of [8].) Thus the Frobenius torus $T_p$ has rank at least 2. Using Tchebotarev density, it is easy to see that $T_p \otimes \mathbb{Q}_\ell$ is a split torus for all rational primes $\ell$ in a set of positive density. Thus, we can find an odd rational prime $\ell$ such that $\ell \not\equiv p$ and $T_p \otimes \mathbb{Q}_\ell$ is split. Now, set $F = k(\zeta_\ell)$ where $\zeta_\ell$ is a primitive $\ell$-th root of unity. Note that $T_p$ is the Frobenius torus for $A_F$ as Frobenius tori are invariant under finite extension of the ground field.

Now, using Proposition 7, we can find a rank 1 subtorus $S \subset T_p \otimes \mathbb{Q}_\ell$ and a vector $v \in T(A_F)$ such that $S$ fixes $v$ and $\det : S \to \mathbb{G}_m$ is surjective. Let $\rho : \text{Gal}(F) \to \text{Aut}(V(A_F))$ denote the Galois representation on the Tate module and let $H = \{ g \in \text{Gal}(F) | \rho(g)v = v \}$. By a theorem of Bogomolov [4, Theorem B] (and the fact that $S$ fixes $v$), it follows that the $\text{Lie}(S) \subset \text{Lie}(\rho(H))$. 
where Lie(S) denotes the Lie algebra of S as an algebraic group and Lie(ρ(H)) denotes the Lie algebra as an ℓ-adic group. Therefore the intersection of S(Qℓ) with ρ(H) contains an open neighborhood of the identity in S(Qℓ). In particular, det(H) contains a neighborhood of the identity in Qℓ∗. Set L := F̄H. Then, from the fact that ν is fixed by H, it follows that Qℓ/Zℓ ⊂ A(L). On the other hand, since χ_{2T_ℓA} \cong Z_ℓ(\dim A), the fact that det(H) contains an open subset of the identity in Qℓ∗ implies that μ_{L}(L) is finite. This completes the proof of Theorem 3.

REFERENCES


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