Values of Multiple L-functions and Periods of Integrals.

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Abstract

In this paper we define two notions of multiple L-functions of modular forms and study their special values. For one type we show that some of the special values can be expressed as periods of integrals and hence obtain some algebraicity results for these values. We also show that there are relations between them coming from 'shuffle' and 'stuffle' products. Finally, we speculate on the relation between these values and periods of mixed Hodge structures on the fundamental group of modular curves.

Multiple Zeta Values (MZVs) have been studied a great deal lately [Gon01],[KZ01],[Wal00]. A starting point of the study of the space of multiple Zeta values is the expression of an MZV as an iterated integral, due to Kontsevich in the most general form. This expression, combined with the shuffle product structure on the space of iterated integrals, gives relations between different MZVs which are called the *shuffle* relations. There are certain other relations that come from directly expanding the result of multiplying two series. Those are called the *stuffle* relations. These two, along with a third type of relation coming from a combination of the two, conjecturally gives all possible relations.

The situation for Multiple L-values (MLVs) is somewhat different. To start with, there are two kinds of natural definitions of the MLVs. In this paper we find an expression for one kind in terms of iterated integrals. Combining this with the shuffle product gives us *shuffle* relations between these kinds of MLVs. However it is not clear if there is a direct way of expressing the product of two such MLVs in terms of such MLVs.

For the second kind, one can express the result of directly multiplying and expanding the series in terms of similar series, except that one has to add the *naive* Rankin-Selberg product of such series. So the algebra of such MLVs has *stuffle* relations.

From Kontsevich's expression [KZ01] for a multiple Zeta value as an iterated integral, it is clear that those values are periods. For the multiple L values, this is not at all clear. For some of the multiple L-values, a generalization of the notion of *critical L-values* of L-functions of modular forms, we show that this is the case.

Of course there is a lot more to the theory of MZVs - there is a deep link between the algebra of the MZV's and the geometry of the mixed Hodge structure on the fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$. So one might expect there is some geometry underlying the algebra of multiple L-values as well. At the end of this paper we speculate on the possible geometry underlying the algebra of multiple L-values of the first kind. This is being worked out in [DS04].

Arakawa and Kaneko [AK04] study a different type of multiple L-function for Dirichlet characters.

1 Multiple L-Values

Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$. In analogy with Multiple Zeta values (MZVs) we define Multiple L-values for Γ as follows. There are of two kinds.

Let $\underline{s} = (s_1, \dots s_k)$ be a ordered tuple of positive integers and $\underline{F} = (f_1, f_2, \dots f_k)$ be an ordered tuple of modular forms of weight w for Γ with L-functions

$$L(f_k, s) = \sum_{n=1}^{\infty} \frac{a(f_k)_n}{n^s} \tag{1}$$

for $i = \{1, \dots, k\}$. We further demand that $a(f_1)_0 = 0$, or equivalently, f_1 vanishes at ∞.

The Multiple L-Values of the *first kind* are sums of the form

$$L^{\bullet}(\underline{F}, \underline{s}) = \sum_{n_1=1, n_2 \dots, n_k=0}^{\infty} \frac{a(f_1)_{n_1} \dots a(f_{k-1})_{n_{k-1}} a(f_k)_{n_k}}{n_1^{s_1}(n_1+n_2)^{s_2} \dots (n_1+n_2+\dots+n_k)^{s_k}}$$

Multiple L-values of the *second kind* are sums of the form

$$L^{*}(\underline{F}, \underline{s}) = \sum_{n_1, n_2 \dots, n_k=1}^{\infty} \frac{a(f_1)_{n_1} \dots a(f_{k-1})_{n_1+\dots+n_{k-1}} a(f_k)_{n_1+\dots+n_k}}{n_1^{s_1}(n_1+n_2)^{s_2} \dots (n_1+n_2+\dots+n_k)^{s_k}}$$

More generally these two type of definitions can be made in defining multiple Dirichlet series and correspondingly multiple Dirichlet values. In this paper though we will only be concerned with Dirichlet series coming from L-functions of modular forms. Note that the two definitions agree when $k = 1$ or in the case of multiple Zeta values.

Define the *order* of the multiple L-value to be $|\underline{s}| = \sum_i s_i$. Let $\mathcal{M}_r^{\bullet}(\Gamma, w)$ denote the $\mathbb{Q}$-vector space of multiple L-values of the first kind of order r of modular forms of weight k for Γ. Let

$$\mathcal{M}^{\bullet}(\Gamma, w) = \bigoplus_{r=1}^{\infty} \mathcal{M}_r^{\bullet}(\Gamma, w)$$

In the next section, we express such multiple L-values in terms of iterated integrals.

There is also a space of multiple L-values of the second kind. However, we need to consider additional series, so we will leave that for later.

2 Iterated Integrals

Let $F = (f_1, \dots, f_k)$ be an ordered tuple of modular forms as before and $\underline{s} = (s_1, \dots, s_k)$ a tuple of complex numbers. Consider the following iterated integral -

$$\Lambda(\underline{F}, \underline{s}) = \int_0^{\infty} f_k(iz_k) z_k^{s_k-1} \int_{z_{k-1}}^{\infty} f_{k-1}(iz_{k-1}) z_{k-1}^{s_{k-1}-1} \dots \int_{z_2}^{\infty} f_1(iz_1) z_1^{s_1-1} dz_1 \dots dz_k \tag{2}$$

This integral is related to the L-function of certain higher order modular forms studied in more detail in [DS04]. The higher order modular forms, though arise independently, are the analogues of multiple poly-logarithms in this context. A special case of a variant of such a function has been considered by Harris [Har88].

To state the theorem, we need to make a few remarks about k-tuples of integers. Let $\underline{s} = (s_1, \ldots, s_k)$ denote a k-tuple of order $|\underline{s}|$. Let $\mathbf{s}_i$ denote the i^{th} partial sum and $[\underline{s}] = [\mathbf{s}_1, \ldots, \mathbf{s}_k]$ the k-tuple of partial sums. Note that knowing $[\underline{s}]$ is equivalent to knowing $\underline{s}$ so on occasion we will use that. We have a partial ordering on the set of k-tuples of order $|\underline{s}|$ defined as follows. $\underline{r}' < \underline{r}$ if $\mathbf{r}'_i \leq \mathbf{r}_i$ for all i and $\mathbf{r}'_{i_0} < \mathbf{r}_{i_0}$ for some i_0. This is not a total order.

We then have the following theorem, which relates the iterated integrals above to multiple L-values.

Theorem 2.1. *The values of $\Lambda(\underline{F}, \underline{s})$ for $\underline{s} = (s_1, \ldots, s_k)$ a k-tuple of positive integers can be expressed in terms of the multiple L-values $L^\bullet(\underline{F}, \underline{r})$ where $\underline{r} = (r_1, \ldots, r_k)$ runs through all k-tuples of order $|\underline{s}|$ which are less that or equal to $\underline{s}$. Precisely, we have*

$$\Lambda(\underline{F}, \underline{s}) = \frac{1}{(2\pi)^{|\underline{s}|}} \sum_{\underline{r}\, \leq\, \underline{s}} L^\bullet(\underline{F}, \underline{r}) \frac{(\mathbf{s}_1 - 1)!(\mathbf{s}_2 - \mathbf{r}_1 - 1)! \ldots (\mathbf{s}_k - \mathbf{r}_{k-1} - 1)!}{(\mathbf{s}_1 - \mathbf{r}_1)! \ldots (\mathbf{s}_{k-1} - \mathbf{r}_{k-1})!} \tag{3}$$

Proof. The proof is by an explicit computation using properties of the (upper) incomplete Γ-function.

Lemma 2.2. *Let $\Gamma(a, s) = \int_a^\infty e^{-z} z^{s-1} dz$ denote the (upper) incomplete Γ-function. Then, it satisfies the functional equation*

$$\Gamma(a, s) = e^{-a} a^{s-1} + (s-1)\Gamma(a, s-1) \tag{4}$$

In particular, if s is a positive integer, we have

$$\Gamma(a, s) = (s-1)! e^{-a} \left(\sum_{t=1}^{s} \frac{a^{s-t}}{(s-t)!} \right) \tag{5}$$

Proof. Integration by parts. We remark that here we use the convention that $0^0 = 1$. □

To prove the theorem, we first expand the functions f_i into their Fourier series and observe that $\Lambda(\underline{F}, \underline{s})$ is of the form

$$\sum_{i=1}^{k} \sum_{n_i=0}^{\infty} \frac{a(f_1)_{n_1} a(f_2)_{n_2} \ldots a(f_k)_{n_k}}{(2\pi)^{\mathbf{s}_k}} \int_0^\infty e^{-2\pi n_k z_k} z_k^{s_k - 1} \int \ldots \int_{z_2}^\infty e^{-2\pi n_1 z_1} z_1^{s_1 - 1} dz_1 \ldots dz_k$$

The innermost integral can be evaluated using (5) to get

$$\int_{z_2}^\infty e^{-2\pi n_1 z_1} z_1^{s_1 - 1} dz_1 = (s_1 - 1)! e^{-2\pi n_1 z_2} \sum_{r_1=1}^{s_1 - 1} \frac{z_2^{s_1 - r_1}}{(2\pi n_1)^{r_1} (s_1 - r_1)!} \tag{6}$$

We then replace the innermost integral by the expression (6) and repeat the process. We finally end with the expression (3). □

We can 'invert' the situation to get an expression for $L^\bullet(\underline{F},\underline{s})$ in terms of the $\Lambda(\underline{F},\underline{s}')$ and in particular, as an iterated integral. Kontsevich found an expression for the multiple Zeta values as an iterated integral which is the starting point of the study of the algebra of multiple zeta values. This can be viewed as the analogue of that formula for multiple L-values.

Theorem 2.3.

$$\prod_{i=1}^{k}(s_i-1)!L^\bullet(\underline{F},\underline{s}) = (2\pi)^{|\underline{s}|}\Lambda_{k-1}(\underline{F},\underline{s}) \tag{7}$$

where $\Lambda_{k-1}(\underline{F},\underline{s})$, defined below, is certain integral linear combination of the $\Lambda(\underline{F},\underline{r})$'s. Hence this gives an iterated integral expression for the multiple L-value.

Proof. Let $\underline{r}$ be a k-tuple $\leq \underline{s}$. Note that $\underline{r}$ has the property that $\mathbf{r}_k = |\underline{s}|$. The proof is by systematically eliminating those $L^\bullet(\underline{F},\underline{r})$ of k-tuples $\underline{r}$ whose last few partial sums $\mathbf{r}_i$ are not $\mathbf{s}_i$. Finally we will end up with an expression which has only terms $L^\bullet(\underline{F},\underline{r})$ whose partial sums $\mathbf{r}_i$ agree with $\mathbf{s}_i$ for all i. There is only one such term, namely $\underline{s}$.

If $k=1$, there is only one such tuple, namely $\underline{r}=\underline{s}=s_1$ and the theorem is immediate from 2.1.

Now let $k=2$. Let $\underline{r}$ be a 2-tuple of order $|\underline{s}|$ such that $\underline{r}\leq\underline{s}$. For any $\underline{r}'\leq\underline{r}$, From 2.1, observe that the coefficient of $L^\bullet(\underline{F},\underline{r}')$ in $\Lambda(\underline{F},\underline{r})$ depends only on r_1' and

$$\Lambda(\underline{F},\underline{r}) - \frac{1}{(2\pi)^{|\underline{s}|}}\binom{r_1-1}{r_1'}L^\bullet(\underline{F},\underline{r}') \tag{8}$$

has no $L^\bullet(\underline{F},\underline{r}')$ term. Employing this idea systematically, define

$$\Lambda_1(\underline{F},(s_1,s_2)) = \sum_{t=0}^{s_1-1}(-1)^t\binom{s_1-1}{t}\Lambda(\underline{F},(s_1-t,s_2+t))$$

From (8), observe that this has only a term corresponding to $L^\bullet(\underline{F},(s_1,s_2))$ as all the rest have been eliminated. We have

$$\Lambda_1(\underline{F},(s_1,s_2)) = (s_1-1)!(s_2-1)!L^\bullet(\underline{F},(s_1,s_2))$$

Note that this procedure ensures that the *last two* partial sums are the same.

More generally, if $(r_1,\ldots,r_{k-2},*,*)$ is a set of k-tuples with the same first $k-2$ terms, so the sequence of partial sums is $[\mathbf{r}_1,\ldots,\mathbf{r}_{k-2},*,\mathbf{s}_k]$ differing only at one place, then by the above procedure, if $\rho_{k-1}=\mathbf{s_{k-1}}-\mathbf{r_{k-1}}$ we can define

$$\Lambda_1(\underline{F},(r_1,\ldots,r_{k-2},\rho_{k-1},s_k)) = \sum_{t=0}^{\rho_{k-1}-1}(-1)^t\binom{\rho_{k-1}-1}{t}\Lambda(\underline{F},(r_1,\ldots,r_{k-2},\rho_{k-1}-t,s_k+t))$$

This has the property that it has precisely one term of the form $L^\bullet(\underline{F},(r_1',\ldots,r_{k-2}',*,*))$ namely the largest one, for any k-tuple $(r_1',\ldots,r_{k-2}',*,*)\leq(r_1,\ldots,r_{k-2},\rho_{k-1},s_k)$. So it is a sum of terms $L^\bullet(\underline{F},\underline{r})$ with $\mathbf{r}_{k-1}=\mathbf{s}_{k-1}$ and $\mathbf{r}_k=\mathbf{s}_k$.

Finally, let $\Lambda_0(\underline{F}, \underline{r}) = \Lambda(\underline{F}, \underline{r})$ and

$$\Lambda_i(\underline{F}, (r_1, \ldots, r_{k-(i+1)}, \rho_{k-i}, s_{k-i-1}, \ldots, s_k)) = \tag{9}$$
$$= \sum_{t=0}^{\rho_{k-i}-1} (-1)^t \binom{\rho_{k-i}-1}{t} \Lambda_{i-1}(\underline{F}, (r_1, \ldots, r_{k-(i+1)}, \rho_{k-i} - t, s_{k-(i-1)} + t, s_{k-i-2}, \ldots, s_k))$$

where ρ_{k-i} is such that $\rho_{k-i} + \sum_{i=1}^{k-i-1} r_i = \mathbf{s}_{k-i}$. From a repeated application of (8), we have that Λ_i is a sum of terms whose last $i+1$ partial sums are $\mathbf{s}_{k-i}, \ldots, \mathbf{s}_k$. Continuing in this manner, we end up with a Λ_{k-1} such that the surviving term has partial sums $(\mathbf{s}_1, \ldots, \mathbf{s}_k)$. There is only one such L-function, namely $L^\bullet(\underline{F}, \underline{s})$ and one has

$$\Lambda_{k-1}(\underline{F}, \underline{s}) = \prod_{i=1}^{k} (s_i - 1)! L^\bullet(\underline{F}, \underline{s})$$

$\square$

For example, for f_1 and f_2 two forms,

$$\begin{aligned}
\Lambda(f_1, f_2, 1, 3) &= \frac{1}{(2\pi)^4} 2L^\bullet(f_1, f_2, 1, 3) \\
\Lambda(f_1, f_2, 2, 2) &= \frac{1}{(2\pi)^4} \left(2L^\bullet(f_1, f_2, 1, 3) + L^\bullet(f_1, f_2, 2, 2)\right) \\
\Lambda(f_1, f_2, 3, 1) &= \frac{1}{(2\pi)^4} \left(2L^\bullet(f_1, f_2, 1, 3) + 2L^\bullet(f_1, f_2, 2, 2) + 2L^\bullet(f_1, f_2, 3, 1)\right)
\end{aligned}$$

so eliminating, we get

$$\begin{aligned}
2L^\bullet(f_1, f_2, 1, 3) &= (2\pi)^4 \Lambda(f_1, f_2, 1, 3) \\
L^\bullet(f_1, f_2, 2, 2) &= (2\pi)^4 \left(\Lambda(f_1, f_2, 2, 2) - \Lambda(f_1, f_2, 1, 3)\right) \\
2L^\bullet(f_1, f_2, 3, 1) &= (2\pi)^4 \left(\Lambda(f_1, f_2, 3, 1) - 2\Lambda(f_1, f_2, 2, 2) + \Lambda(f_1, f_2, 1, 3)\right)
\end{aligned}$$

3 The Shuffle Product

Iterated integrals have a multiplicative property called a *Shuffle Product.* If k and r are two positive integers a *shuffle of type* (k, r) is a permutation σ of the set $\{1, 2, \ldots, k+r\}$ which preserves the relative order:

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(k)$$
$$\sigma^{-1}(k+1) < \cdots < \sigma^{-1}(k+r)$$

We have the following lemma which shows that one can multiply iterated integrals.

Lemma 3.1 (Ree). *([Hai87], Lemma 2.11) If* $\int w_1 \ldots w_k$ *and* $\int w_{k+1} \ldots w_{k+r}$ *are two iterated integrals, then, for any path* α *one has*

$$\int_\alpha w_1 \ldots w_k \int_\alpha w_{k+1} \ldots w_{k+r} = \sum_\sigma \int_\alpha w_{\sigma(1)} \ldots w_{\sigma(k+r)}$$

where σ *runs through all shuffles of type* (k, r)

Proof. This follows from from observation that an iterated integral of length k over α is the integral over the simplex $\Delta^k \subset [0,1]^k$ of the product of the pullbacks $\alpha^* w_i$ of the 1-forms w_i and the fact that

$$\Delta^k \times \Delta^r = \coprod_\sigma \{\{t_{\sigma(1)}, \ldots, t_{\sigma(k+r)}\} | 0 \leq t_1 \leq \cdots \leq t_{k+r} \leq 1\}$$

where σ runs through all shuffles of type (k, r). $\square$

This product gives relations between iterated integrals. We can use this coupled with Theorem 7 to get some relations between Multiple L-Values. We also observe that from (7), that if L_1 is in $\mathcal{M}^\bullet_{s_1}(\Gamma, w)$ and L_2 is in $\mathcal{M}^\bullet_{s_2}(\Gamma, w)$ then the shuffle product of L_1 and L_2 lies in $\mathcal{M}^\bullet_{s_1+s_2}(\Gamma, w)$.

For example, If f_1 and f_2 are two modular forms of weight w which vanish at ∞, then the shuffle product gives

$$\int_0^\infty f_1(iz_1) z_1^{s_1-1} dz_1 \int_0^\infty f_2(iz_2) z_1^{s_2-1} dz_2 = \int_0^\infty f_1(iz_1) z_1^{s_1-1} f_2(iz_2) z_2^{s_2-1} dz_2 dz_1$$
$$+ \int_0^\infty f_2(iz_2) z_2^{s_2-1} f_1(iz_1) z_1^{s_1-1} dz_1 dz_2$$

However, if s_1 and s_2 are positive integers, then the left hand side is

$$L^\bullet(f_1, s_1) L^\bullet(f_2, s_2)$$

while the right hand side can be expressed in terms of $MZVs$ using 2.1. One then has, for example, if $s_1 = s_2 = 2$,

$$\begin{aligned} L^\bullet(f_1, 2) L^\bullet(f_2, 2) = {} & 6L^\bullet((f_1, f_2), (2,2)) + 2L^\bullet((f_1, f_2)(1,3)) \\ & + 6L^\bullet((f_2, f_1), (2,2)) + 2L^\bullet((f_2, f_1)(1,3)) \end{aligned} \tag{10}$$

If further, f_1 and f_2 are modular forms of level N and weight 2, then they satisfy the functional equation

$$N^{\frac{s_i-1}{2}} L(s_i, f_i) = -N^{\frac{1-s_i}{2}} L(2 - s_i, f_i|_{W_N})$$

where W_N is the Atkin-Lehner involution and $i \in \{1, 2\}$. So one has the following functional equation for the $\Lambda(f_1, f_2, s_1, s_2)$

$$N^{\frac{s_1+s_2-2}{2}} (\Lambda(f_1, f_2, s_1, s_2) + \Lambda(f_2, f_1, s_2, s_1))$$
$$= -N^{\frac{2-(s_1+s_2)}{2}} (\Lambda(f_1|_{W_N}, f_2|_{W_N}, 2 - s_1, 2 - s_2) + \Lambda(f_2|_{W_N}, f_1|_{W_N}, 2 - s_2, 2 - s_1))$$

3.1 The Stuffle relations

Now we study the second kind of MZVs. We can get relations between the L^* functions though this involves the Rankin-Selberg product L-function.

We define the *naive* Rankin-Selberg product of $f_1, f_2, \ldots, f_k$ with Fourier expansions as in (1) to be the series

$$L_{f_1,\ldots,f_k}(s) = \sum_{n=1}^{\infty} \frac{a(f_1)_n \ldots a(f_k)_n}{n^s}$$

Directly multiplying, we observe that

$$L^*(\underline{F}, \underline{s})L^*(\underline{F}', \underline{s}') = \sum_{\underline{s}''} L^*(\underline{F}'', \underline{s}'') \tag{11}$$

where the notation is as follows: $\underline{s}''$ runs through all possible tuples $(s''_1, \ldots, s''_{k''})$ obtained from $\underline{s}$ and $\underline{s}'$ by inserting, in all possible ways, some zeroes in the tuples $\underline{s} = (s_1, \ldots s_k)$ and $\underline{s}' = (s'_1, \ldots, s'_{k'})$ with $max\{k, k'\} \leq k'' \leq k + k'$ and by adding the sequences term by term. So $\underline{s}''$ consists of terms of the form s_i,s'_j or $s_i + s'_j$. $\underline{F}''$ consists of forms such that if the term is s_i, the numerator is given by the coefficients of f_i, if it is s'_j, the numerator is given by the coefficients of f'_j and if it is $s_i + s'_j$, the numerator is given by the coefficients of the Rankin-Selberg product of f_i and f'_j. This is the basic *stuffle* relation.

For example, if f_1 and f_2 are two forms, we have

$$L^*(f_1, s_1)L^*(f_2, s_2) = L^*((f_1, f_2), (s_1, s_2)) + L^*((f_2, f_1), (s_2, s_1)) + L_{f_1,f_2}(s_1 + s_2) \tag{12}$$

Since $L^*(f_i, s_i) = L^\bullet(f_i, s_i) = L(f_i, s_i)$ for $i = \{1, 2\}$, combining this with (10) we get an expression for some of the special values of the Ranking-Selberg convolution in terms of the multiple L-values. If $s_1 = s_2 = 2$, for example, we have

$$\begin{aligned} L_{f_1,f_2}(4) = {} & 6L^\bullet((f_1, f_2), (2, 2)) + 2L^\bullet((f_1, f_2)(1, 3)) \\ & + 6L^\bullet((f_2, f_1), (2, 2)) + 2L^\bullet((f_2, f_1)(1, 3)) \\ & - L^*((f_1, f_2), (2, 2)) + L^*((f_2, f_1), (2, 2)). \end{aligned} \tag{13}$$

This can be generalized to products of several such functions.

To get a set that is closed under multiplication, we let $\mathcal{M}^*_r(\Gamma, k)$ to be the rational vector space generated by multiple L-values of order r along with the the functions $L_{f_1,\ldots,f_k}(r)$. Let

$$\mathcal{M}^*(\Gamma, k) = \bigoplus_r \mathcal{M}^*_r(\Gamma, k)$$

4 Some Functional Equations

In this section we prove a functional equation for the $\Lambda(\underline{F}, \underline{s})$ where $\underline{F} = (f_1, \ldots, f_k)$ is a k-tuple of Hecke eigenforms of weight w and level N such that the constant term in their Fourier expansion around ∞ is 0.

Theorem 4.1. *Let $\underline{F} = (f_1, \ldots, f_k)$ denote a k-tuple of Hecke eigen-forms of weight w and level N. Let τ_N denote the Atkin-Lerner operator, represented by the matrix $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. For $1 \leq j \leq k$ Define*

- $\underline{F}_j = (f_1, \ldots, f_j)$ *and* $\overline{F_j} = (f_{j+1}, \ldots, f_k)$ *and* $\underline{F}_j|_{\tau_N} = (f_1|_{\tau_N}, \ldots, f_j|_{\tau_N})$
- $\underline{s}_j = (s_1, \ldots, s_j)$ *and* $\overline{s_j} = (s_{j+1}, \ldots, s_k)$.

and for an j-tuple $\underline{s}_j = (s_1, \ldots, s_j)$, let $\underline{w} - \underline{s}_j = (w - s_1, \ldots, w - s_j)$. Then one has the following functional equation.

$$\sum_{j=0}^{k} i^{-jw} \sqrt{N}^{(-|\overline{s_j}|+|\underline{w}-\underline{s}_j|)} \Lambda(\overline{F_j}|_{\tau_N}, \overline{s_j}) \Lambda(\underline{F}_j, \underline{w} - \underline{s}_j) = 0 \tag{14}$$

Proof. The idea is the same as the classical case of $k = 1$. We compute

$$\Lambda(\overline{F_0}|_{\tau_N}, \overline{s_0}) = \Lambda(\underline{F}|_{\tau_N}, \underline{s}) = \int_0^\infty f_k|_{\tau_N}(iz_k) z_k^{s_k-1} \int_{z_k}^\infty \cdots \int_{z_2}^\infty f_1|_{\tau_N}(iz_1) z_1^{s_1-1} dz_1 \ldots dz_k \tag{15}$$

Consider the innermost integral

$$\int_{z_2}^\infty f_1|_{\tau_N}(iz_1) z_1^{s_1-1} dz_1 = \int_{z_2}^\infty (\sqrt{N} i z_1)^{-w} f(\frac{i}{N z_1}) (z_1)^{s_1-1} dz_1$$

Using the substitution $u_1 = \frac{1}{N z_1}$ we can simplify the integral to

$$\sqrt{N}^{w-2s_1} i^{-w} \int_0^{1/Nz_2} f(iu_1) u_1^{w-s_1-1} du_1$$

Replacing $\int_0^{1/Nz_2}$ by $\int_0^\infty - \int_{1/Nz_2}^\infty$ we get

$$\sqrt{N}^{w-2s_1} i^{-w} \left(\Lambda(f_1, w - s_1) - \int_{1/Nz_2}^\infty f_1(iu_1) u_1^{w-s_1-1} du_1 \right) \tag{16}$$

Multiplying (15) by $\sqrt{N}^{\,-|\underline{s}|}$ and replacing the innermost integral by (16) we get

$$\begin{aligned} &\sqrt{N}^{\,-|\underline{s}|} \Lambda(\overline{F_0}|_{\tau_N}, \overline{s_0}) = i^{-w} \sqrt{N}^{\,-|\overline{s_1}|+|w-\underline{s}_1|} \Lambda(\overline{F_1}|_{\tau_N}, \overline{s_1}) \Lambda(f_1, w - s_1) \\ &- \int_0^\infty f_k|_{\tau_N}(iz_k) z_k^{s_k-1} \ldots \int_{z_3}^\infty f_2|_{\tau_N}(iz_2) z_2^{s_2-1} \int_{1/Nz_2}^\infty f_1(iu_1) u_1^{w-s_1-1} du_1 dz_2 \ldots dz_k \end{aligned} \tag{17}$$

Now consider the innermost two integrals of the last integral. Simplifying, using $u_2 = 1/Nz_2$ we get

$$\int_{z_3}^\infty f_2|_{\tau_N}(iz_2) z_2^{s_2-1} \int_{z_2/N}^\infty f_1(iu_1) u_1^{w-s_1-1} du_1 dz_2 = \sqrt{N}^{w-2s_2} i^{-w} \int_0^{1/Nz_3} f_2(iu_2) u_2^{w-s_2-1} du_2 \tag{18}$$

Once again, replacing $\int_0^{1/Nz_3}$ by $\int_0^\infty - \int_{1/Nz_3}^\infty$ in (17) the integral becomes

$$\Lambda(f_1, f_2, w - s_1, w - s_2) - \int_{1/Nz_3}^\infty f_2(iu_2) u_2^{w-s_2-1} \int_{u_2}^\infty f_1(iu_1) u_1^{w-s_1-1} du_1 du_2 \tag{19}$$

Putting this in (17) we get

$$\sqrt{N}^{\,-|\underline{s}|}\Lambda(\overline{\underline{F_0}}|_{\tau_N},\overline{\underline{s_0}}) = i^{-w}\sqrt{N}^{\,-|\overline{\underline{s_1}}|+|w-s_1|}\Lambda(\overline{\underline{F_1}}|_{\tau_N},\overline{\underline{s_1}})\Lambda(\underline{F}_1, w-s_1)$$
$$- i^{-2w}\sqrt{N}^{|\underline{s}|-|2w-2s_1-2s_2|}\Lambda(\overline{\underline{F_2}}|_{\tau_N},\overline{\underline{s_2}})\Lambda(\underline{F}_2,(w-s_2,w-s_1)$$
$$+ i^{-2w}\sqrt{N}^{|\underline{s}|-|2w-2s_1-2s_2|}\int_0^\infty f_k|_{\tau_N}(iz_k)z_k^{s_k-1}\dots\int_{1/Nz_3}^\infty f_2(iu_2)u_2^{w-s_2-1}$$
$$\int_{u_2}^\infty f_1(iu_1)u_1^{w-s_1-1}du_1du_2dz_3\dots dz_k \tag{20}$$

Continuing in this manner we finally end up with the expression (14).

□

5 Periods

In light of the work on multiple Zeta values and the work of Kontsevich and Zagier [KZ01] one can speculate whether the multiple L values of modular forms are periods in the sense of Kontsevich and Zagier. From the formula (7) it suffices, at least as far as the the $L^\bullet(\underline{F},\underline{s})$ values go, to show that the values of the functions $\Lambda(\underline{F},\underline{s})$ are periods. In this direction one has that the values $\Lambda(\underline{F},(1,1,\dots,1)$ are periods of the mixed Hodge structure on $\pi_1(X_0(N)-\{cusps\})$ based at ∞ with tangent vector $\vec{0\infty}$ when $\underline{F}$ is a tuple of modular forms of weight 2 .

Conjecture: If $\underline{s}$ is a k-tuple of positive integers and $\underline{F}$ is a k-tuple of modular forms of weight w and level N then the multiple L values $L^\bullet(\underline{F},\underline{s})$ are periods.

Remark 5.1. *In the case when $k=1$, for weight w forms and $0<s<w,\ s\neq w/2$, this is due to Eichler, Shimura and several others. If $s\geq w$ this is due to Beilinson [Beĭ86] for weight 2 and Deninger and Scholl [DS91] for higher weights. For $s=w/2$, that is, at the center of the critical strip, not much is know in general.*

Remark 5.2. *We have no idea how to interpret the values $L^*(\underline{F},\underline{s})$.*

As evidence for the conjecture we have the following observation of an analogue of a theorem of Shimura.

In light of the functional equation for $\Lambda(\underline{F},\underline{s})$ We say a k-tuple of integers $\underline{s}=(s_1,\dots,s_k)$ is *critical* for $\underline{F}$ if $0<s_i<w$ for all i. This is a generalization of the classical definition of critical. At the moment it is not clear how to attach a motive to $\underline{F}$ but perhaps if that were possible, this would be a special case of the definition of Deligne [?] or Scholl [DS91]. We have the following theorem. The proof was suggest to me by M.V. Nori [?].

Theorem 5.3. *Let $\underline{F}=(f_1,\dots,f_k)$ be a k-tuple of modular forms of weight w and $\underline{s}=(s_1,\dots,s_2)$ a k-tuple of integers which are critical for $\underline{F}$.Then the values $L(\underline{F},\underline{s})$ are periods.*

Proof. It suffices to show that the values $\Lambda(\underline{F}, \underline{s})$ are periods.

First let $k = 1$. If f is a modular forms of weight w with coefficients in a number field K, by a theorem of Deligne one has $\omega = 2\pi i f(z)dzdt_1 \dots dt_{w-2}$ is a K-rational differential form on $E^{w-2} \to X_0(N)$ where E^{w-2} is the self product of the universal elliptic curve over the modular curve $X_0(N)$. On each elliptic curve E there are two 1-cycles $(0,1)$ and $(0,z)$. Let $\{(0,\infty), m-1\}$ denote the cycle

$$(0,\infty) \times \underbrace{(0,z) \times \cdots (0,z)}_{m-1 \; times} \times \underbrace{(0,1) \cdots (0,1)}_{w-1-m \; times}$$

Then one has for $m \in \{1, \dots, w-1\}$

$$2\pi i \Lambda(f,m) = \int_0^\infty 2\pi i f(iz) z^{m-1} dz = \int_{\{(0,\infty),m-1\}} 2\pi i f(iz) dz$$

From the de Rham isomorphism theorem, this lies in $K(\pi^{w-1})$

Now let $k > 1$ and assume that the $f_i' s$ are defined over a field K. We first recall that an iterated integral is an integral over a simplex. Let $\upsilon : [0,1] \to X_0(N)$ denote the curve parameterizing $(0,\infty)$ and $\upsilon^k : [0,1]^k \to X_0(N)^k$. Let $\Delta_k \subset [0,1]^k$ denote the k-simplex. Consider the cycle

$$\Upsilon = \{\upsilon^k(\Delta^k), m_1 - 1, \dots, m_k - 1\} \subset (E^{w-2})^k$$

which consists of the image of the k-simplex and $(m_i - 1)$ times the cycle $(0,z)$ and $(w - 1 - m_i)$ times the cycle $(0,1)$ in the universal family over the i^{th} copy of $X_0(N)$. Let $\omega_i = 2\pi i f_i(z) dzdt_1 \dots dt_{w-2}$. Then

$$(2\pi i)^k \Lambda(\underline{F}, \underline{s}) = \int_\Upsilon p_1^*(\omega_1) \dots p_k^*(\omega_k)$$

where $p_i : (E^{w-2})^k \to E^{w-2}$ is the i^{th} projection. This lies in $K(\pi^{(w-1)^k})$.

□

Remark 5.4. *For $k = 1$ and f an eigenform, more is known. The periods of the $\Lambda(f,s)$ for $0 < s < w$ depend only on the parity of s.*

Remark 5.5. *For $s_i \geq w$ the situation appears to be more difficult as even the case of $k = 1$ follows only from the known cases of Beilinson conjectures. One realizes the special value as the regulator of some elements in the K-theory of the universal family of the modular curve and this gives the expression as a period.*

6 Final remarks

The structure of the multiple Zeta values is intimately connected with the geometry of $\mathbb{P}^1 - \{0,1,\infty\}$ - they appear as the periods of the mixed Hodge structure on the group ring of $\pi_1(\mathbb{P}^1 - \{0,1,\infty\}, \vec{01})$, the fundamental group based at a cusp with respect to the tangent vector $\vec{01}$. Analogously one might expect that there is some geometry underlying the structure of the multiple L-values. For $\mathcal{M}^\bullet(\Gamma_0(N), 2)$ the natural candidate is the mixed Hodge structure on the fundamental group of $X_0(N) - \{cusps\}$ based at ∞ with tangent vector $\overrightarrow{0\infty}$. The definition of $\Lambda(\underline{F}, \underline{s})$ in (2) also seems to suggest that. That is still work in progress. A generalization of this may work for higher weights. However, for $\mathcal{M}^*(\Gamma, w)$ we have no idea.

In the case of MZVs one has conversely that any period of the MHS on $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{01})$ can be expressed as a multiple Zeta value. It is not clear what the situation is in this case.

References

[AK04] Tsuneo Arakawa and Masanobu Kaneko. On multiple L-values. *J. Math. Soc. Japan*, 56(4):967–991, 2004.

[Beĭ86] A. A. Beĭlinson. Higher regulators of modular curves. In *Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, volume 55 of *Contemp. Math.*, pages 1–34. Amer. Math. Soc., Providence, RI, 1986.

[DS91] Christopher Deninger and Anthony J. Scholl. The Beĭlinson conjectures. In *L-functions and arithmetic (Durham, 1989)*, volume 153 of *London Math. Soc. Lecture Note Ser.*, pages 173–209. Cambridge Univ. Press, Cambridge, 1991.

[DS04] Nikolaos Diamantis and Ramesh Sreekantan. Higher order modular forms, mixed Hodge theory and multiple L-values. 2004.

[Gon01] Alexander B. Goncharov. Multiple ζ-values, Galois groups, and geometry of modular varieties. In *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, volume 201 of *Progr. Math.*, pages 361–392. Birkhäuser, Basel, 2001.

[Hai87] Richard M. Hain. The geometry of the mixed Hodge structure on the fundamental group. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 247–282. Amer. Math. Soc., Providence, RI, 1987.

[Har88] Bruno Harris. An analytic function and iterated integrals. *Compositio Math.*, 65(2):209–221, 1988.

[KZ01] Maxim Kontsevich and Don Zagier. Periods. In *Mathematics unlimited—2001 and beyond*, pages 771–808. Springer, Berlin, 2001.

[Wal00] Michel Waldschmidt. Valeurs zêta multiples. Une introduction. *J. Théor. Nombres Bordeaux*, 12(2):581–595, 2000. Colloque International de Théorie des Nombres (Talence, 1999).