Non-Archimedean regulator maps and special values of L-functions.

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September 4, 2008

Abstract

We define an analogue of the 'Real' Deligne cohomology group at a prime of semi-stable or good reduction of a variety. We also define regulator maps to this group and formulate a conjecture about the image. This allows us to formulate a non-Archimedian version of Beilinson's Hodge-$\mathcal{D}$-conjecture and S-integral and function field versions of Beilinson's global conjectures as well as a precise special value conjecture in the function field case. Finally we give a few examples where these conjectures are known to be true.

MSC 2000 classification:11G40,11R42,14G25,11M38

1 Introduction

Let X be a smooth projective variety over $\mathbb{Q}$. Beilinson [Beĭ84] formulated conjectures relating special values of L-functions to the K-theory of such varieties in terms of the 'Real' Deligne cohomology of the varieties. Roughly speaking, he constructed a regulator map from a higher Chow group of the variety to a real vector space, the Real Deligne cohomology, and conjectured that the image gives a $\mathbb{Q}$-structure. The dimension of this vector space is the order of pole of the Archimedean factor of a cohomological L-function of the variety. Finally the real vector space has another lattice structure induced by the Betti and de Rham cohomology groups and he conjectured that the determinant of the the change of basis matrix is related to a special value of this L-function.

In this paper we define an analogue of the Deligne cohomology group for a finite prime of good or strict semi-stable reduction. We show there is a regulator map from the higher Chow group to this Deligne cohomology group and show that it has similar properties for known or conjectural reasons.

One aim of this paper is to formulate a version of Beilinson's conjectures in the case of varieties over function fields of characteristic p so as to put the results of [Kon02], [Ṕ07] and [Sre] in a general framework. Since in this case all the primes are finite we do obtain such a formulation. Further, we can formulate an S-integral version of the Beilinson conjectures in general. In the function field case can be interpreted as a statement that the regulator map gives an isomorphism rationally. Finally, in this case. we can also formulate a precise conjecture for the special value which suggests that it can be thought of measuring the obstruction to the regulator map being an isomorphism integrally.

As a by-product we can also formulate a local conjecture which can be viewed as a non-Archimedean analogue of the Beilinson's Hodge-$\mathcal{D}$-conjecture. While the Hodge-$\mathcal{D}$-conjecture is known to be false it is still expected to be true under certain circumstances. As we remark, the direct analogue of the Hodge-$\mathcal{D}$-conjecture in our case is false, but the modified version is implied by the general conjecture and is interesting in its own right. In some other papers [Sre08],[RS] we have established special cases of this conjecture.

It is generally believed (see [Man91], for example) that a variety should be considered to have totally degenerate reduction at an Archimedean place. In particular it has semi-stable reduction, so the usual conjectures are just the statements in the case of an Archimedean prime. Our methods, however, do not give the Archimedean regulator. If the non-Archimedean prime has split semi-stable reduction it appears that one has very similar formulas - due to the theory of p-adic uniformization.

In the final section we show some examples where these conjectures are known to be true.

We have only dealt with the higher Chow groups. As we use the boundary map in the localization sequence to define our regulator map, our methods do not allow us to formulate anything about the usual Chow groups. The paper of Raskind and Xarles [RX07] suggests how one may be able to formulate a conjecture in some of these cases. Further, there is the recent work of Kato and Trihan [KT03] on the B-SD conjecture for elliptic curves over function fields.

There are also several people who have formulated conjectures regarding special values of L-functions in the number field case, including Bloch-Kato [BK90], Lichtenbaum [Lic72] and Burns-Flach [BF01]. In some of these cases the conjectures can be formulated in the function field case as well - but we have not investigated the relationship between our conjectures and theirs.

Acknowledgements: I would like to thank Spencer Bloch for suggesting this line of thought, Patrick Brosnan, Caterina Consani, Najmuddin Fakhruddin, Tom Haines, Jonathan Korman, Niranjan Ramachandran, G.V. Ravindra and Xavier Xarles for their comments and the University of Toronto and the TIFR Centre for Applicable Mathematics for their hospitality.

2 The Archimedean Case

Let X be a smooth projective variety defined over $\mathbb{Q}$. The usual Real Deligne cohomology $H^q_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(q-a))$, with $q>2a+1$ has the following properties:

- (i). It is a finite dimensional real vector space with
$$\dim_{\mathbb{R}} H^q_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(q-a)) = -\operatorname{ord}_{s=a} L_\infty(H^{q-1}(X),s)$$
the L-factor at the Archimedean place.
- (ii). There is a regulator map $r_{\mathcal{D}}: CH^{q-a}(X,q-2a)\otimes\mathbb{Q}\longrightarrow H^q_{\mathcal{D}}(X_{/\mathbb{R}},\mathbb{R}(q-a))$
- (iii). There is a $\mathbb{Q}$ structure on this real vector space induced by the Betti cohomology group $H^{q-1}_B(X(\mathbb{C}),\mathbb{Q})$ and piece of the de Rham cohomology group $F^{q-a}H^{q-1}_{dR}(X_{/\mathbb{R}})$.
- (iv). The image of the regulator map is conjecturally another $\mathbb{Q}$-lattice in the real vector space.
- (v). Assuming (iv) one can compute the determinant of the change of basis matrix with respect to these two lattices. Let $c_\infty(X,q,a)$ be that number. Then conjecturally
$$L^*(X,a)\sim_{\mathbb{Q}^*} c_\infty(X,q,a)$$
where $L^*(X,a)$ denotes the first non-zero value of the Laurent expansion of $L(X,s)$ at $s=a$.

When $q-2a=1$ the situation is slightly different. The regulator map on the higher chow group does not suffice to give a $\mathbb{Q}$-lattice. One has to 'thicken' it using a cycle class map. Our formulation takes this in to consideration as well. A good reference for all the facts stated here is the paper by P. Schneider in [Sch88].

We will define a $\mathbb{Q}$ vector space for a prime p of semi-stable or good reduction which has property (i) owing to the work of Consani, [Con98]. We will then define a regulator map to this vector space and will speculate on analogues of properties (iii),(iv) and (v).

Remark 2.1. *The restriction that X is defined over $\mathbb{Q}$ is not a very serious one. If X is defined over a number field then by reduction of scalars we can assume it is defined over $\mathbb{Q}$.*

3 Preliminaries

Let X be a smooth proper variety over a global field K, v a closed point and K_v the completion of K at v. Let Λ_v denote the ring of integers of K_v which is a discrete valuation ring with closed point v and generic point $\eta = Spec(K_v)$.

By a model $\mathcal{X}$ of X we mean a flat proper scheme $\mathcal{X} \to Spec(\Lambda_v)$ together with an isomorphism of the generic fibre X_η with X.

Let Y be the special fibre $\mathcal{X} \times Spec(k(v))$. We will always also make the assumption that the model is strictly semi-stable, which means that it is a regular model and the fibre Y is a divisor with normal crossings, the components have multiplicity one and they intersect transversally.

4 Consani's Double Complex

In [Con98], Consani defined a double complex of Chow groups of the components of the special fibre following the work of Steenbrink [Ste76] and Bloch-Gillet-Soulé [BGS95]. We need to use this complex to define the Deligne cohomology in the case of strict semi-stable reduction.

Let $Y = \bigcup_{i=1}^{t} Y_i$ be the special fibre of dim n with Y_i its irreducible components. For $I \subset \{1, 2, ...t\}$, define

$$Y_I = \cap_{i \in I} Y_i$$

Let $r = |I|$ denote the cardinality of I. Define

$$Y^{(r)} := \begin{cases} \mathcal{X} & \text{if } r = 0 \\ \coprod_{|I|=r} Y_I & \text{if } 1 \leq r \leq n \\ \emptyset & \text{if } r > n \end{cases}$$

For u and t with $1 \leq u \leq t < r$ define the map

$$\delta(u) : Y^{(t+1)} \to Y^{(t)}$$

as follows. Let $I = (i_1, i_2, ...i_{t+1})$ with $i_1 < i_2 < ... < i_{t+1}$. Let $J = I - \{i_u\}$. This gives an embedding $Y_I \to Y_J$. Putting these together induces the map $\delta(u)$. Let $\delta(u)_*$ and $\delta(u)^*$ denote the corresponding maps on Chow homology and cohomology respectively. They further induce the Gysin and restriction maps on the Chow groups.

Define

$$\gamma := \sum_{u=1}^{r} (-1)^{u-1} \delta(u)_*$$

and

$$\rho := \sum_{u=1}^{r} (-1)^{u-1} \delta(u)^*$$

These maps have the properties that

- $\gamma^2 = 0$
- $\rho^2 = 0$

- $\gamma \cdot \rho + \rho \cdot \gamma = 0$

Let $i, j, k \in \mathbb{Z}$. Define, following [Con98](3.1)

$$K^{i,j,k} := \begin{cases} CH^{\frac{i+j-2k+n}{2}}(Y^{(2k-i+1)}) \otimes \mathbb{Q} & \text{if } k \geq max(0,i) \text{ and } i+j+n \equiv 0(2) \\ 0 & \text{otherwise} \end{cases}$$

and let

$$K^{i,j} = \oplus_k K^{i,j,k} \text{ and } K^n = \oplus_{i+j=n} K^{i,j}$$

The maps ρ and γ induce differentials

$$\partial' : K^{i,j,k} \to K^{i+1,j+1,k+1} \qquad \partial'(a) = \rho(a)$$

$$\partial'' : K^{i,j,k} \to K^{i+1,j+1,k} \qquad \partial''(a) = -\gamma(a)$$

Further define

$$N : K^{i,j,k} \to K^{i+2,j,k+1}(-1) \qquad N(a) = a$$

Let $\partial = \partial' + \partial''$ on $K^{i,j}$. From the definition we have $[\partial, N] = 0$ and $\partial^2 = 0$.

Let $Cone(N) : K^* \to K^*$ be the complex $K^* \oplus K^*[-1]$ with differential

$$D(a,b) = (\partial(a), N(a) - \partial(b))$$

Consani [Con98][Prop 3.2] shows that this cone complex is quasi-isomorphic to a complex of Chow groups of the fibre:

Proposition 4.1 (Consani)**.** *Let $*$ be a fixed integer. The complex, for $q \in \mathbb{Z}$,*

$$Cone(N : K^{q-2*,q-n} \to K^{q-2*+2,q-n})$$

is quasi-isomorphic to the following complex

$$\mathcal{C}^q(*) = \begin{cases} CH^{q-*}(Y^{(2*-q)}) \otimes \mathbb{Q} & \text{if } q \leq * - 1 \\ CH^*(Y^{(q-2*)}) \otimes \mathbb{Q} & \text{if } q \geq * \end{cases}$$

The differential $\partial_{\mathcal{C}}$ is given by

$$\partial_{\mathcal{C}}(a) = \begin{cases} \partial''(a) & \text{if } q < * - 1 \\ -i^* i_*(a) & \text{if } q = * - 1 \\ \partial'(a) & \text{if } q \geq * \end{cases}$$

5 The 'Deligne cohomology' at a finite prime

We define the v**-adic Deligne Cohomology group** as follows. Let

$$PCH^a(Y) := \frac{Ker(i^* i_* : CH_{n-a}(Y^{(1)}) \to CH^{a+1}(Y^{(1)}))}{Im(\gamma : CH_{n-a}(Y^{(2)}) \to CH_{n-a}(Y^{(1)}))}$$

Define

$$H^q_{\mathcal{D}}(X_{/v}, \mathbb{Q}(q-a)) := \begin{cases} CH^{q-a-1}(Y, q-2a-1) \otimes \mathbb{Q} & \text{if } q - 2a > 1 \\ PCH^a(Y) \otimes \mathbb{Q} & \text{if } q - 2a = 1 \end{cases}$$

Here n is the dimension of Y. This is a $\mathbb{Q}$ vector space which we will show has the expected properties assuming certain conjectures. Note that if Y is non-singular and $q - 2a > 1$ the Parshin-Soulé [Sou84] conjecture asserts that the higher Chow group is finite, hence this space is 0. When $q - 2a = 1$ the group is $CH^a(Y) \otimes \mathbb{Q}$.

Remark 5.1. *In [Con98](Section 3), Consani shows that the groups appearing above can be computed as the cohomology of a complex related to the double complex $K^{i,j}$. Further, in (Section 4), she shows that if one uses a certain group of differential forms instead of the chow groups of the components to define the $K^{i,j}$ the cohomology of the complex gives the real Deligne cohomology of the variety. So this, along with the properties described below, motivated calling it the Deligne cohomology at a finite prime.*

6 Properties of the Deligne Cohomology

In this section we show that the v-adic Deligne cohomology has, or is expected to have, properties similar what the usual Deligne cohomology listed in Section 2.

6.1 Dimension:

The usual Real Deligne cohomology has the property that its dimension is the order of the pole of the Archimedean part of the L-function at a certain point on the left of the critical point. Here we have a similar property. Let F^* be the geometric Frobenius and $N(v)$ the number of elements of $k(v)$. The local L-factor of the $(q-1)^{st}$-cohomology group is then

$$L_v(H^{q-1}(X), s) = (det(I - F^* N(v)^{-s} | H^{q-1}(\bar{X}, \mathbb{Q}_\ell)^I))^{-1}$$

Theorem 6.1 (Consani)**.** *Let v be a place of semi-stable reduction. Assuming the weight-monodromy conjecture [Del80], the Tate conjecture for the components and the injectivity of the cycle class map on the components Y_I, the Parshin-Soulé conjecture and that F^* acts semisimply on $H^*(\bar{X}, \mathbb{Q}_\ell)^I$. we have*

$$\dim_{\mathbb{Q}}(H^q_{\mathcal{D}}(X_{/v}, \mathbb{Q}(q-2a))) = -\operatorname{ord}_{s=a} L_v(H^{q-1}(X), s) := d_v$$

Proof. [Con98], Cor 3.6. □

Remark 6.2. *Since the L-factor at a prime of good reduction does not have a pole at $s = a$ when $q - 2a > 1$, the Parshin-Soulé conjecture can be interpreted as the statement that the v-adic Deligne cohomology has the correct dimension, namely 0, even at a prime of good reduction.*

Remark 6.3. *In the function fields setting and more recently, in the setting of p-adically uniformized varieties, the weight-monodromy conjecture is a theorem [Del80], [Ito05]. Further, in the p-adically uniformized case, the variety is totally degenerate so the Tate conjecture for the components is trivial, so assuming semi-simplicity of the action of the Frobenius and injectivity of the cycle class map on the components, Consani's theorem holds in this case. The referee of an earlier version of this manuscript informed me that the injectivity of the cycle class map is also known for p-adically uniformized varieties.*

6.2 Regulator maps

To define the regulator map we use the localization sequence [Blo86]. It is as follows. If $X, \mathcal{X}$ and Y are as before, and $q, a \in \mathbb{Z}, q - 2a > 0$ we have

$$\cdots \to CH^{q-a}(\mathcal{X}, q-2a) \to CH^{q-a}(X, q-2a) \xrightarrow{\partial} CH^{q-a-1}(Y, q-2a-1)$$
$$\to CH^{q-a}(\mathcal{X}, q-2a-1) \to CH^{q-a}(X, q-2a-1) \to \cdots$$

There are two cases that have to be considered.

6.2.1 Case 1: $q - 2a > 1$

We define the v**-adic regulator map** to be the map ∂.

$$r_{\mathcal{D},v} : CH^{q-a}(X, q-2a) \xrightarrow{\partial} H^q_{\mathcal{D}}(X_{/v}, \mathbb{Q}(q-a))$$

In analogy with the Beilinson conjectures, we have the following conjecture

CONJECTURE A1: The image

$$Im(r_{\mathcal{D},v}(CH^{q-a}(X, q-2a))) \subset H^q_{\mathcal{D}}(X_{/v}, \mathbb{Q}(q-a))$$

is a full sub-lattice.

6.2.2 Case 2: $q - 2a = 1$

In this case the Chow groups of the special fibre are not torsion, so the conjecture has to be slightly modified. However, we get a conjecture which is non trivial even in the case of good reduction.

One has a presentation of the Chow group

$$CH^*(Y) = Coker(\gamma : CH_{n-*}(Y^{(2)}) \to CH_{n-*}(Y^{(1)}))$$

Recall that $i : Y \hookrightarrow \mathcal{X}$ is the inclusion map of the special fibre into the model. From the exactness of the localization sequence and the fact that the model $\mathcal{X}$ is regular, the image of ∂ lies in the kernel of

$$i_* : CH_{n-*}(Y) \to CH_{n-*}(\mathcal{X}) = CH^{*-1}(\mathcal{X})$$

In particular, it lies in the kernel of $i^*i_* : CH_{n-*}(Y^{(1)}) \to CH_{n-*}(Y^{(1)})$.

We define the v**-adic regulator map** as before as map ∂

$$r_{\mathcal{D},v} : CH^{a+1}(X, 1) \xrightarrow{\partial} H^{2a+1}_{\mathcal{D}}(X_{/v}, \mathbb{Q}(a+1))$$

and we have the following conjecture.

CONJECTURE A2: The map

$$Im(r_{\mathcal{D},v}(CH^{a+1}(X, 1))) \subset H^{2a+1}_{\mathcal{D}}(X_{/v}, \mathbb{Q}(a+1))$$

is a full sublattice.

Remark 6.4. *For a variety defined over $\mathbb{R}$, Beilinson formulated what is known as the Hodge-$\mathcal{D}$-conjecture, which can be found in Jannsen's article in [Sch88]. Our* **Conjecture A** *can be viewed as a non-Archimedean analogue of that conjecture. However the Hodge-$\mathcal{D}$-conjecture is known to be false in general [MS97] but is still expected to be true if the variety is defined over a number field. We call this the* non-Archimedean Hodge-$\mathcal{D}$-conjecture.

Remark 6.5. *One can similarly formulate* **Conjecture A** *for any smooth projective variety over a local field but it was suggested by N.Fakhruddin [Fak03] that perhaps similar methods to Jannsens's can be used to show that it is false over local fields. Further, work of Asakura and Saito [AS07] suggest that the conjecture is false in general for varieties over a local field. However it may still hold over discrete valuation rings whose quotient field is contained in the algebraic closure of a number field.*

Remark 6.6. Conjecture A *is implied by* **Conjecture B** *below so provides a first test of the validity of* **Conjecture B***. In fact,* **Conjecture B** *implies* **Conjecture A** *for* all *finite primes but it is not clear if knowing* **Conjecture A** *for all primes would imply* **Conjecture B**.

7 The S-integral Beilinson conjecture

The conjectures above can be combined with the Beilinson conjectures to formulate an S-integral version of the Beilinson conjectures. Let X be a variety defined over a number field K which has either good or semi-stable reduction at all places (this restriction may not be so important but at the moment it is necessary). Let $\mathcal{O}_K$ be the ring of integers of K and let ∞ denote the set of Archimedean places. Let $L_v(H^{q-1}(X), s)$ be a cohomological L-function at the place v for the $(q-1)^{st}$ cohomology group as defined above, where for v Archimedean it is defined as in [Sch88] pg 4. Define the completed L-function $\Lambda(X, s)$ as follows:

$$\Lambda(X,s) = A^{\frac{s}{2}} \prod_v L_v(X,s)$$

where A is a generalized conductor.

The **Standard Conjectures** of Grothendieck, generalizing those of Hasse-Weil and Serre, asserts that this function has a meromorphic continuation to the entire complex plane and satisfies the functional equation

$$\Lambda(X,s) = \pm\Lambda(X, q-s)$$

For a set of places S containing the Archimedean places we define

$$L_S(X,S) = \prod_{v \notin S} L_v(X,s) \quad \text{and} \quad L^S(X,s) = \prod_{v \in S} L_v(X,s)$$

so

$$\Lambda(X,s) = A^{\frac{s}{2}} L_S(X,s) L^S(X,s)$$

We can formulate the S-integral Beilinson conjectures as follows. As in the usual Beilinson conjectures there are two cases – when $q-2a > 1$ and when $q-2a = 1$.

Let $c_\infty(X,q,a)$ be the number appearing the usual Beilinson conjectures, coming from isomorphism of the two $\mathbb{Q}$-structures. For a finite set of places S containing the Archimedean places, let $\mathcal{X}_S$ be a model over $\mathcal{O}_K[S]$ with semi-stable reduction at the finite places of S and good reduction outside S.

CONJECTURE B1: If $q - 2a > 1$, then

- The map

$$\oplus_{v \in S \backslash \infty} r_{\mathcal{D},v} \otimes \mathbb{Q} \bigoplus \oplus_{v | \infty} r_{\mathcal{D},v} \otimes \mathbb{R} : CH^{q-a}(\mathcal{X}_S, q-2a) \otimes \mathbb{Q} \longrightarrow \bigoplus_{v \in S} H^q_{\mathcal{D}}(X_{/v}, \mathbb{Q}(q-a))$$

  is an isomorphism.

- $L_S^*(X,a) \sim_{\mathbb{Q}^*} c_\infty(X,q,a) \prod_{v \in S \backslash \infty} (\log(N(v)))^{d_v}$

- In particular $\text{ord}_{s=a} L_S(X,s) = \dim_{\mathbb{Q}} CH^{q-a}(\mathcal{X}_S, q-2a) \otimes \mathbb{Q}$, since $\Lambda(X,s)$ is regular and non-vanishing at a.

To formulate the conjecture in the case when $q - 2a = 1$ we need another map

$$z_v^a : CH^a(X) \longrightarrow H^{2a+1}_{\mathcal{D}}(X_{/v}, \mathbb{Q}(a+1))$$

which is defined as follows. The map $j^* : CH^a(\mathcal{X}) \to CH^a(X)$ induced by restriction is surjective and we have the map $i^* : CH^a(\mathcal{X}) \to CH^a(Y)$. This induces a map $i^* \circ (j^*)^{-1} : CH^a(X) \to CH^a(Y)$ which is well defined up to an element of the image of $i^* i_* : CH^{a-1}(Y) \to CH^a(Y)$. So we have a well defined map

$$\xi_v^a : CH^a(X) \to \frac{Ker(\rho : CH^a(Y^{(1)}) \to CH^a(Y^{(2)}))}{Im(i^* i_*)} \otimes \mathbb{Q}$$

There is always a morphism τ from the group on the right to the Deligne cohomology induced by the cap product $\cap[Y]$ [BGS95], pg. 454., and we define

$$z_v^a := \tau \circ \xi_v^a$$

In some instances, for example, if Y is smooth [BGS95][Prop 2.3.3], the map τ is known to be an isomorphism. In several other instances [BGS95][Section 6], the map is known to be an isomorphism after going to cohomology.

The Archimedean version of the map z^a comes from the usual cycle class map to Betti cohomology and the long exact sequence relating the Betti, de Rham and Deligne cohomologies.

CONJECTURE B2: If $q - 2a = 1$. Let $B^a(X)$ denote the group $(CH^a(X)/CH^a_{hom}(X))$ and z^a the cycle class map to Deligne cohomology induced by the cycle class map to $H_B^{2a}(X, \mathbb{R}(a))^{(-1)^a}$

Then

- The map

  $$\oplus_{v \in S \setminus \infty} r_{\mathcal{D},v} \otimes \mathbb{Q} \oplus_{v|\infty} r_{\mathcal{D},v} \otimes \mathbb{R} \oplus z^a : CH^{q-a}(\mathcal{X}_S, 1) \oplus B^a(X) \otimes \mathbb{Q} \to \oplus_{v \in S} H^q_{\mathcal{D}}(X/v, \mathbb{Q}(q-a))$$

  is an isomorphism.

- $L_S^*(X, a) \sim_{\mathbb{Q}^*} c_\infty(X, q, a) \prod_{v \in S \setminus \infty} (\log(N(v)))^{d_v}$

  In particular

- $\mathrm{ord}_{s=a} L_S(X, s) = \dim_{\mathbb{Q}} CH^{q-a}(\mathcal{X}_S, 1) \otimes \mathbb{Q}$

- $\mathrm{ord}_{s=a+1} L_S(X, s) = -\dim_{\mathbb{Q}} B^a(X)$ (Tate's Conjecture)

A few remarks are in order.

Remark 7.1. *In a sense the conjecture of Parshin and Soulé which states that the higher Chow groups of smooth projective varities are finite explains to a certain extent why there are two conjectures. When $q - 2a > 1$ the conjectures assert that the higher Chow group $CH^{q-a}(X, q-2a)$ is finite dimensional - whose dimension is determined by the Archimedean primes and the primes of semi-stable reduction. This is because the finite primes of good reduction make no contribution to the rank as the Chow group of a special fibre is still a 'higher' Chow group hence is finite.*

This is evidently false for $q - 2a = 1$. In fact in this case the higher Chow group $CH^{a+1}(X, 1)$ is always expected to be infinitely generated and this is because the Chow group of the special fibre can be of non-zero rank at primes of good reduction. Note that here one has to include the Tate cycles in order to get an isomorphism.

Remark 7.2. *Our regulator maps come from the boundary map of the localization sequence. Beilinson's regulator map should appear as the boundary map in the 'arithmetic' localization sequence.*

$$\cdots \to \widehat{CH}^{q-a}(\mathcal{X}, q-2a) \to CH^{q-a}(\mathcal{X}, q-2a) \xrightarrow{r_{\mathcal{D}}} H_{\mathcal{D}}^{2a+1}(X_{/\mathbb{R}}, \mathbb{R}(a+1)) \to \cdots$$

Recently we became aware of the work of [Gon05] and [Fel] where higher arithmetic Chow groups were defined and they are shown to satisfy such a long exact sequence.

8 Function Fields

In the case of function fields, we can formulate a conjecture along the lines of **Conjecture B** using the fact that the primes at ∞ are just finite primes.

Let $\mathbb{F}_\ell$ be the finite field of order ℓ and K a function field over $\mathbb{F}_\ell$ of transcendence degree 1. Let S be a non empty finite set of primes of K. Then there exists an element x in K whose poles are precisely the elements of S. Let $\mathcal{O}_S$ denote the integral closure of $\mathbb{F}_\ell[x]$ in K. The prime ∞ is the prime $1/x$ of $\mathbb{F}_\ell[x]$ and the set S is the set of primes lying over ∞. Let $\mathcal{X}_S$ be a model of X over $\mathcal{O}_S$ with at worst semi-stable reduction. Define $R_{\mathcal{D},S} := \bigoplus_{v\in S} r_{\mathcal{D},v}$.

CONJECTURE B1FF: Let S be a finite set of places and $\mathcal{X}_S$ as above and $q-2a>1$.
Then

- $\operatorname{ord}_{s=a} L_S(X,s) = \dim_{\mathbb{Q}} CH^{q-a}(\mathcal{X}_S, q-2a)\otimes\mathbb{Q}$
- The map
$$R_{\mathcal{D},S}\otimes\mathbb{Q} : CH^{q-a}(\mathcal{X}_S, q-2a) \to \bigoplus_{v\in S} H^q_{\mathcal{D}}(X_{/v}, \mathbb{Q}(q-a))$$
is an isomorphism.
- $L^*_S(X,a) \sim_{\mathbb{Q}^*} \prod_{v\in S}(\log(N(v)))^{d_v}$

In the case $q-2a=1$, let $B^a(X)$ denote the group $(CH^a(X)/CH^a_{hom}(X))$ and let z^a_S denote the map
$$z^a_S := \oplus_{s\in S} z^a_v : B^a(X) \to \oplus_{s\in S} H^{2a+1}_{\mathcal{D}}(X/v, \mathbb{Q}(a+1))$$
induced by the restriction maps $CH^a(X)\to CH^a(Y_v)$. Define $R_{\mathcal{D},S} := \bigoplus_{v\in S} r_{\mathcal{D},v}\oplus z^a_v$

CONJECTURE B2FF: If $q-2a=1$, Let S be a finite set of places and $\mathcal{X}_S$ as above. Then

- $\operatorname{ord}_{s=a} L_S(X,s) = \dim_{\mathbb{Q}} CH^{q-a}(\mathcal{X}_S, 1)\otimes\mathbb{Q}$
- $\operatorname{ord}_{s=a+1} L_S(X,s) = -\dim_{\mathbb{Q}} B^a(X)\otimes\mathbb{Q}$ (Tate's Conjecture)
- The map
$$R_{\mathcal{D},S}\otimes\mathbb{Q} : CH^{a+1}(\mathcal{X}_S, 1)\oplus B^a(X) \to \bigoplus_{v\in S} H^{2a+1}_{\mathcal{D}}(X/v, \mathbb{Q}(a+1))$$
is an isomorphism.
- $L^*_S(X,a) \sim_{\mathbb{Q}^*} \log(q)^{\dim(CH^{a+1}(\mathcal{X}_S,1))}$

Notice that the only transcendental term is a power of $\log(q)$ which comes from the residue of the local L-function.

9 A special value conjecture

The considerations of the previous sections allow us to formulate a special value conjecture in the function field case. Formulating the conjecture here is a little easier than in the number field case as in that case Beilinson's regulator map is only defined up to $\mathbb{Q}^*$.

9.1 $\mathbb{Z}$-structures

To conjecture an expression for the special value, we use the fact that the Deligne cohomology at a finite prime has two $\mathbb{Z}$ structures, one natural and the other conjectural. The first $\mathbb{Z}$ structure comes from the integral Chow group $CH^{q-a-1}(Y_v, q-2a-1)$. **Conjecture A** asserts that the image of the regulator map gives a second $\mathbb{Z}$ structure, which is a subgroup of the integral Chow group.

When S is the set of all primes and $q-2a>1$ the earlier conjectures assert that the map

$$R_{\mathcal{D}} \otimes \mathbb{Q} : CH^{q-a}(X, q-2a) \otimes \mathbb{Q} \longrightarrow \bigoplus_v CH^{q-a-1}(Y_v, q-2a-1) \otimes \mathbb{Q}$$

is an isomorphism. Similarly, when $q-2a=1$, the map

$$R_{\mathcal{D}} \otimes \mathbb{Q} : CH^{a+1}(X,1) \otimes \mathbb{Q} \oplus B^a(X) \otimes \mathbb{Q} \longrightarrow PCH^a(Y_v) \otimes \mathbb{Q}$$

is an isomorphism. However, the maps are defined *integrally*, so the kernel and cokernel are torsion. We conjecture that they are actually finite. Let $b(X,q,a)$ denote the order of the kernel and $c(X,q,a)$ denote

$$c(X,q,a) := \begin{cases} \left|\frac{\oplus_v CH^{q-a-1}(Y_v, q-a-1)/\{tors\}}{Im(R_{\mathcal{D}})}\right| \\ \left|\frac{\oplus_v PCH^a(Y_v)/\{tors\}}{Im(R_{\mathcal{D}})}\right| \end{cases}$$

9.2 A special value conjecture

We then have the following conjecture:

CONJECTURE CFF:

- C1: If $q-2a>1$ then

$$\operatorname{ord}_{s=a} \Lambda(X,s) = \dim_{\mathbb{Q}} CH^{q-a}(X, q-2a) \otimes \mathbb{Q}$$

- C2: If $q-2a=1$, the space $CH^{q-a}(X,1) \otimes \mathbb{Q}$ is **not** finite dimensional, but

$$\operatorname{ord}_{s=a} \Lambda(X,s) = -\dim_{\mathbb{Q}} B^a(X) \otimes \mathbb{Q}$$

Further,

$$\Lambda^*(X,a) = \pm \frac{c(X,q,a)}{b(X,q,a)} \log(q)^{\operatorname{ord}_{s=a} \Lambda(X,s)} \tag{1}$$

A few remarks are in order.

Remark 9.1. *Let $\mathcal{C}$ denote the projective curve over $\mathbb{F}_\ell$ whose generic point is K and $\mathcal{X}$ a model of X over $\mathcal{C}$. The numbers $c(X,q,a)$ and $b(X,q,a)$ have the following interpretation.*

There is a localization sequence which relates the higher chow groups' $CH^{q-a}(\mathcal{X}, q-2a)$ to the usual higher Chow groups as follows:

$$\cdots \to CH^{q-a}(\mathcal{X}, q-2a) \to CH^{q-a}(X, q-2a) \xrightarrow{R_{\mathcal{D}}} \bigoplus_v CH^{q-a-1}(Y_v, q-2a-1) \to \cdots$$

$$\cdots \to CH^{q-a}(\mathcal{X}, q-2a-1) \to CH^{q-a}(X, q-2a-1) \to \cdots$$

The number $b(X, q, a)$ can be understood as the order of the image of $CH^{q-a}(\mathcal{X}, q-2a)$ in the localization sequence. Similarly, the number $c(X, q, a)$ can be understood as the order of the group $Ker : CH^{q-a}(\mathcal{X}, q-2a-1) \to CH^{q-a}(X, q-2a-1)$.

With this observation, one sees that the special value can be viewed as the obstruction of the regulator map being an isomorphism integrally. If you assume the Parshin-Soulé conjecture for $\mathcal{X}$ the fact that the map is rationally an isomorphism follows.

Remark 9.2. *When $q-2a > 1$, the formula (1) does not seem to hold in general. When X is $Spec(\mathbb{F}_\ell(T)))$ and $q = 1$ and $a = -1$, so $q-2a = 3$ and $q-a = 2$, the groups $CH^1(Y_v, 2)$ are 0 for all v, so $b(X, 1.-1) = |ker(R_{\mathcal{D}})| = |CH^2(X, 3)|$ and $c(X, 1, -1) = 1$. The special value of the zeta function is*

$$\Lambda_K(-1) = \frac{1}{(1-\ell)(1-\ell^2)}$$

while

$$\frac{c(X, 1, -1)}{b(X, 1, -1)} = \frac{1}{\ell^2 - 1}$$

so there is a factor missing. One has a similar problem for other values of q and a as well.

Remark 9.3. *One can also formulate an exact value conjecture in the S-integral case and this can be viewed as the conjecture when S is the empty set.*

There are other integral structures which could be used to explain the special value. For example, Beilinson's motivic cohomology groups. When tensored with $\mathbb{Q}$, they are isomorphic to the rational higher Chow groups. Further, they satisfy a localization sequence. The same formulation can then be used to obtain a special value conjecture. However, in all the examples we have these two groups are isomorphic integrally, so we have chosen to formulate things in terms of higher Chow groups. Perhaps the correct question to ask is what is the integral structure on the motivic cohomology such that the above conjecture holds.

10 Examples

In this section we exhibit some evidence for the conjectures. Many of these results are just the rephrasing of well known results in our terms.

10.1 Points

10.1.1 Local Case – Conjecture A – The (non Archimedean) Hodge-$\mathcal{D}$-conjecture

- $X = Spec(K_v)$ – a non-Archimedean local field, $q = 1, a = 0$, so $q - 2a = 1$.

  Here one is considering the map

  $$CH^1(Speck(K_v), 1) \otimes \mathbb{Q} \xrightarrow{\log|\cdot|_v} CH^0(Spec(\mathbb{F}_v)) \otimes \mathbb{Q}$$

  Its well known (for example from [Blo86]) that $CH^1(Speck(K_v), 1) = K_v^*$. The local L-factor is

  $$L_v(H^0(Speck(K_v)), s) = \frac{1}{1 - N(v)^{-s}}$$

  which has a pole of order 1 at 0 which shows that $CH^0(Spec(\mathbb{F}_v))$ is of rank 1. Hence the map is clearly surjective.

- When $X = Spec(\mathbb{R})$ then the regulator map is $\log|\cdot| : \mathbb{R}^* \to \mathbb{R}$ which is clearly surjective. A similar result holds when $X = Spec(\mathbb{C})$.

- $X = Spec(K_v)$, $q - 2a > 1$, K_v non-Archimedean.

  Here one knows by the work of Quillen [Qui73] that $K_{q-2a}(Spec(\mathbb{F}_v)) \otimes \mathbb{Q} = 0$. Hence the surjectivity of the regulator map is trivial. This agrees with the fact that the local L-factor has no poles.

- $X = Spec(\mathbb{R})$ or $X = Spec(\mathbb{C})$.

  Here the local L-factor has poles or not depending on whether $q - 2a$ is odd or even. The surjectivity of the regulator map is an easy consequence of the celebrated theorem of Borel [Bor77].

10.1.2 Global Case – Conjectures B and C

- $X = Spec(K)$, K a number field. Let

$$\zeta_{K,S}(s) = \prod_{v \notin S} L_v(Spec(K), s)$$

  Conjecture B2 is the usual S-unit theorem of Dirichlet. The fact that the regulator map is surjective follows from the finiteness of the class number.

  In this case one also has an expression for the special value along the lines of **Conjecture CFF** which follows from the class number formula and the functional equation.

  We describe this in the case when S is the set of Archimedean primes. Here

$$L_\infty(Spec(K), s) = 2^{-r_2 s}\pi^{-\frac{ns}{2}}\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}$$

  and

$$\Lambda_K(s) = |D(K)|^{s/2} L_\infty(Spec(K), s)\zeta_K^\infty(s)$$

  where $D(K)$ is the discriminant. $\Lambda_K(s)$ satisfies the functional equation

$$\Lambda_K(s) = \Lambda_K(1 - s)$$

  and has a simple pole at $s = 0$ with residue

$$\Lambda_K^*(0) = Res_{s=0}\Lambda_K(s) = -\frac{R_K h_K}{w_K}$$

  Here R_K is the classical regulator, h_K is the class number and w_K is the number of roots of unity. h_K is the order of the cokernel of the non-archimedean regulator map and w_K is the order of the kernel, so the formula is very similar to that expected in **Conjecture CFF**. In this case there is a way of defining a second integral structure on the Deligne cohomology and R_K is the cokernel of that map.

  The simple pole at $s = 0$ shows that **Conjecture C2** holds in this case as well.

- Function field case

In this case the precise analogue of all the conjectures is known and can be found in the book of Rosen [Ros02]. If K is a function field of transcendence degree 1 over $\mathbb{F}_\ell$, Rosen (pg 244) shows, for example, that the completed zeta function $\Lambda_K(s)$ has a simple pole at $s = 0$ and

$$\Lambda_K^*(0) = -\frac{h_K}{(\ell - 1)\log(\ell)}$$

which is precisely what is expected by **Conjecture C2** as h_K is the class number and is the order of the cokernel, while the kernel is the number of elements of finite order in K^* which is $\ell - 1$, the cardinality of $\mathbb{F}_\ell^*$.

10.2 Curves

10.2.1 Local Case – Conjecture A – The (non Archimedean) Hodge-$\mathcal{D}$-conjecture

- $X = E$, an elliptic curve, $q = 2, a = 0$ so $q - 2a = 2$ over a local field K_v with finite residue field $\mathbb{F}_v$. We are looking at the map

  $$CH^2(E, 2) \otimes \mathbb{Q} \longrightarrow CH^1(\mathcal{E}_v, 1) \otimes \mathbb{Q}$$

  Here when v is a prime of good reduction, the group $CH^1(\mathcal{E}_v, 1) = \mathbb{F}_v^*$ so is finite and as a result the conjecture is trivial. When v is a prime of semi-stable (split multiplicative) reduction then $CH^1(\mathcal{E}_v, 1)$ is one dimensional. Bloch and Grayson [BG86] showed that the regulator map is surjective if the elliptic curve is defined over $\mathbb{Q}$.

- $X = X_0(N)$, a modular curve, $q = 2, a = 0$ so $q - 2a = 2$

  A precise computation of the boundary of an element of $CH^2(X, 2)$ can be found in the work Shappacher and Scholl [SS91]. This result is also referred to as the 'boundary of the Eisenstein symbol'.

10.2.2 Global Case – Conjectures B and C

- $X = E$, an elliptic curve over $\mathbb{Q}$, $q = 2, a = 0$ so $q - 2a = 2$.

  The first non-classical case of the relations between K-theory and special values of L-functions was Bloch's [Blo74] proof of the relation between K_2 of a CM elliptic curve E and the value $L(E, 2)$ of the L-function of the elliptic curve. Since all finite primes have potentially good reduction, this proves the conjecture for CM elliptic curves over $\mathbb{Q}$ - one does not have to consider an integral model.

  Further, Beilinson [Beĭ84] showed a similar result for modular curves - and as a consequence of the Shimura-Taniyama theorem - for all elliptic curves over $\mathbb{Q}$. This is the S-integral conjecture for S being the set of Archimedean primes.

  The full conjecture, however, asserts that the rank of K_2 of an elliptic curve E over $\mathbb{Q}$ is $s + 1$, where s is the number of semi-stable primes of E. While Bloch and Grayson [BG86] gave example of elements in K_2 with boundaries at semi-stable primes as far as I am aware there are no known cases of an elliptic curve without CM for which it is known that there are at least that many elements. There is some description of this situation in the survey by Ramakrishan [Ram89]and some further progress in the work of Schappacher and Rolschausen[RS98].

- Function fields

  In the function field case the analogue of Beilinson's theorem for Drinfeld modular curves is due independently to Kondo [Kon02] and Pál [P0́7].

10.3 Surfaces

10.3.1 Local Case – Conjecture A – The (non Archimedean) Hodge-$\mathcal{D}$-conjecture

- $X = E_1 \times E_2$, E_1 and E_2 elliptic curves, v a prime of good reduction for both E_1 and E_2, $q = 3, a = 1$, so $q - 2a = 1$.

  Here one is asking for the surjectivity of the map

  $$CH^2(E_1 \times E_2, 1) \otimes \mathbb{Q} \xrightarrow{\partial} PCH^1(\mathcal{E}_{1,v} \times \mathcal{E}_{2,v}) \otimes \mathbb{Q}$$

  This case is due to Spiess [Spi99]. The idea is to use genus 2 curves whose Jacobian are isogenous to $E_1 \times E_2$ to construct elements of the group $CH^2(E_1 \times E_2, 1)$.

- X as above, but E_1 and E_2 non-isogenous v a prime of split multiplicative reduction for both.

  This case is due to me [Sre08]. Here one had to understand the group PCH^1 in the bad reduction case and then generalize Spiess' results to work in this context.

  If E_1 and E_2 are isogenous, then the target space is 0 dimensional, so there is nothing to be proved.

- X an Abelian surface or a $K3$ surface over $\mathbb{R}$.

  The Hodge-$\mathcal{D}$-conjecture for abelian surfaces or $K3$ surfaces is due to Chen and Lewis [CL05]. Once again, , if $A = E_1 \times E_2$ with E_1 and E_2 isogenous there is nothing to be proved.

- $X = C \times C$, C a curve with semi-stable reduction at v. $q = 3$, $a = 0$.

  Consani [Con99] , answering a question of Kato, showed that the nilpotent monodromy operator determines a cycle in the special fibre of $C \times C$ which lies in the higher Chow group $CH^2(\mathcal{X}_v, 2)$, where $\mathcal{X}$ is a semi-stable model of $C \times C$. This cycle *a priori* depends on the semi-stable model but should be intrinsic to $C \times C$.

  The surjectivity conjecture implies that there should exist a higher Chow cycle in $CH^3(C \times C, 3)$ whose non-Archimedean regulator is the cycle determined by the nilpotent monodromy. In [RS] we show the existence such a cycle in some special cases.

  In general, for X a variety of dimension d, the i^{th} power of the nilpotent monodromy operator determines a cycle in $CH^{d+i}(\mathcal{X} \times \mathcal{X}, 2i)$, where $\mathcal{X}$ is a semi-stable model of X and **Conjecture A** predicts the existence of a cycle in the higher Chow group $CH^{d+i+1}(X \times X, 2i + 1)$ which bounds it. Such a cycle would give a motivic interpretation of the nilpotent monodromy.

10.3.2 Global Case – Conjectures B and C

- $X = E \times E$, E an elliptic curve over $\mathbb{Q}$. $q = 3, a = 1$ as before.

  The surjectivity of the regulator map at primes of good reduction of a (modular) elliptic curve over $\mathbb{Q}$ was shown by Mildenhall [Mil92]. This makes essential use of the modularity as the elements he constructs are in $CH^2(X_0(N) \times X_0(N), 1)$, where $X_0(N)$ is a modular curve covering E, and uses the covering map to push the elements down to $E \times E$. At primes of bad reduction there is nothing to be proved as the target space is trivial. When E is an elliptic curve with CM he shows further that the co-kernel of the regulator map is finite.

- $X = X_0(N) \times X_0(N)$, $X_0(N)$ a modular curve.

  Here the Beilinson conjecture was actually shown by Beilinson [Beĭ84].

- Function Field Case

  $X = X_0(N) \times X_0(N)$, $X_0(N)$ a Drinfel'd modular curve over the function field $\mathbb{F}_\ell[T]$, $q = 3$ and $a = 1$.

  Some evidence for this conjecture is due to me [Sre]. We show that the regulator at the prime $\infty = 1/T$, is related to the special value of the L-function. We construct elements in the higher chow group mimicking the construction of Beilinson. We first show that the special value has a representation as an integral over the Drinfeld modular curve. To show that this is the regulator of an element of the higher Chow group we use the fact that the associated special fibre at ∞ of a Drinfeld modular curve is the dual graph of the Bruhat-Tits graph of Drinfeld modular curve - so an integral of an integer valued function over the vertices of the graph is nothing but a sum of components and hence is a 1 cycle.

10.4 Final Remarks

There are several cases in which these conjectures are known for trivial reasons - for example, when every element of the higher chow groups are decomposable. However, there are many seemingly simple cases when the conjecture is not known. One case I am currently studying is the case of a simple abelian surface over a local field. While there is evidently only one decomposable element in the higher chow group $CH^2(A, 1)$, Tate shows that the Picard number of the special fibre is at least of rank 2. So the conjecture asserts that there must exist an indecomposable element of $CH^2(A, 1)$ but it is far from clear what that element is.

There are a few isolated results in higher dimensions. One consequence of the work of Schappacher and Scholl [SS91] is the conjecture for some symmetric powers of an elliptic curve. They show the map

$$CH^{m+1}(Sym^m(E), m+1) \otimes \longrightarrow CH^m(Sym^m(\mathcal{E})_v, m) \otimes \mathbb{Q}$$

is surjective by constructing a fibre of this map - the so called 'Eisenstein Symbol'.

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