MATHEMATICAL MORPHOLOGY: FUNDAMENTALS

FOUNDING FATHERS OF

MATHEMATICAL MORPHOLOGY

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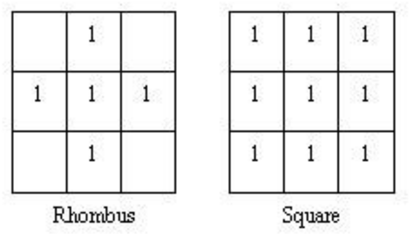
Mathematical Morphology

Binary Mathematical Morphology

Greyscale Morphology

Mathematical Morphology (MM)

- Mathematical morphologic transformations (Matheron, 1975; Serra, 1982) have shown its speciality and strength in the context of geomorphology such as significant geomorphologic features extraction, basic measures of water bodies estimation, geomorphic processes modelling and simulation, fractal landscapes generation, etc.
- In the entire investigation, both $f(x,y)$s are analysed as greyscale image (3-D) and the extracted networks as thresholded sets (binary form).
- In order to process the binary sets such as channel networks, binary morphological transformations are employed.
- Grey-scale mathematical morphological transformations are used to process the three-dimensional images such as $f(x,y)$ (e.g. DEMs).
- The geometrical and topological structures of $f(x,y)$ are examined by matching it with structuring elements of various shapes and sizes at different locations in the $f(x,y)$.
- Figure below provides two examples of structuring elements (B), which are in the shape of rhombus and square of size 3X3. (1's and 0's stand for foreground and background regions, respectively).



Basic Transformations

- Mathematical Morphology
  - Dilation
  - Erosion
  - Opening
  - Closing

Mathematical Morphology (cont)

<u>Binary MM</u>

- Binary erosion transformation of *X* by structuring element, *B*
  - the set of points *s* such that the translated B_x is contained in the original set *X*, and is equivalent to intersection of all the translates.
  - $X \ominus B = \{x: B_x \subseteq X\} = \bigcap_{b \in B} X_{-b}$
- Binary dilation transformation of *X* by *B*
  - the set of all those points *s* such that the translated B_x intersects *X*, and is equivalent to the union of all translates.
  - $X \oplus B = \{x: B_x \cap X \neq \varnothing\} = \bigcup_{b \in B} X_b$
- The dilation with an elementary structuring template expands the set with a uniform layer of elements, while the erosion operator eliminates a layer from the set.
- Multiscale erosions and dilations are

  - $(X \ominus B) \ominus B \ominus \ldots \ominus B = (X \ominus nB)$,
  - $(X \oplus B) \oplus B \oplus \ldots \oplus B = (X \oplus nB)$,

  where $nB = B \oplus B \oplus \ldots \oplus B$ and *n* is the number of transformation cycles.

Mathematical Morphology (cont)

<u>Binary MM (cont)</u>

- By employing erosion and dilation of *X* by *B*, opening and closing transformations are further represented as:
  - $X \circ B = ((X \ominus B) \oplus B))$
  - $X \bullet B = ((X \oplus B) \ominus B))$
- After eroding *X* by *B*, the resultant eroded version is dilated to achieve the opened version of *X* by *B.*
- Similarly, closed version of *X* by *B* is obtained by first performing dilation on *X* by *B* and followed by erosion on the resultant dilated version.
- Multiscale opening and closing transformations are implemented by performing erosions and dilations recursively as shown below.

  - $(X \circ nB) = [(X \ominus nB) \oplus nB)]$,

  - $(X \bullet nB) = [(X \oplus nB) \ominus nB)]$,

  where *n* is the number of transformations cycles.

Mathematical Morphology (cont)

<u>Greyscale MM</u>

- Greyscale dilation and erosion operations - expansion and contractions respectively
- Let $f(x,y)$ be a function on Z^2, and B be a fixed structuring element of size one. The erosion of $f(x)$ by B replaces the value of f at a pixel (x, y) by the minima values of the image in the window defined by the structuring template B
  - $(f \ominus B)(x,y) = \min_{(i,j)\in B}\{f(x+i, y+j)\}$,
- The dilation of $f(x)$ by B replaces the value of f at a pixel (x, y) by the maxima values of the image in the window defined by the structuring template B

$$(f \oplus B)(x,y) = \max_{(i,j)\in B}\{f(x-i, y-j)\}$$

- In other words, $(f \ominus B)$ and $(f \oplus B)$ can be obtained by computing *minima* and *maxima* over a moving template B, respectively.
- Erosion is the dual of dilation :
  - Eroding foreground pixels is equivalent to dilating the background pixels.

Mathematical Morphology (cont)

<u>Grey-scale MM (cont)</u>

- Opening and closing are both based on the dilation and erosion transformations.
- Opening of *f* by *B* is achieved by eroding *f* and followed by dilating with respect to *B*, $(f \circ B) = [(f \ominus B) \oplus B]$,
- Closing of *f* by *B* is defined as the dilation of *f* by *B* followed by erosion with respect to B, $(f \bullet B) = [(f \oplus B) \ominus B]$,
- Opening eliminates specific image details smaller than *B*, removes noise and smoothens the boundaries from the inside, whereas closing fills holes in objects, connects close objects or small breaks and smoothens the boundaries from the outside.
- Multiscale opening and closing can be performed by increasing the size (scale) of the structuring template *nB*, where *n* = 0, 1, 2,…, N. These multiscale opening and closing of *f* by *B* are mathematically represented as:

$$(f \circ nB) = \{[(f \ominus B) \ominus B \ominus \ldots \ominus B] \oplus B \oplus B \oplus \ldots \oplus B\} = [(f \ominus nB) \oplus nB],$$

$$(f \bullet nB) = \{[(f \oplus B) \oplus B \oplus \ldots \oplus B] \ominus B \ominus B \ominus \ldots \ominus B\} = [(f \oplus nB) \ominus nB],$$

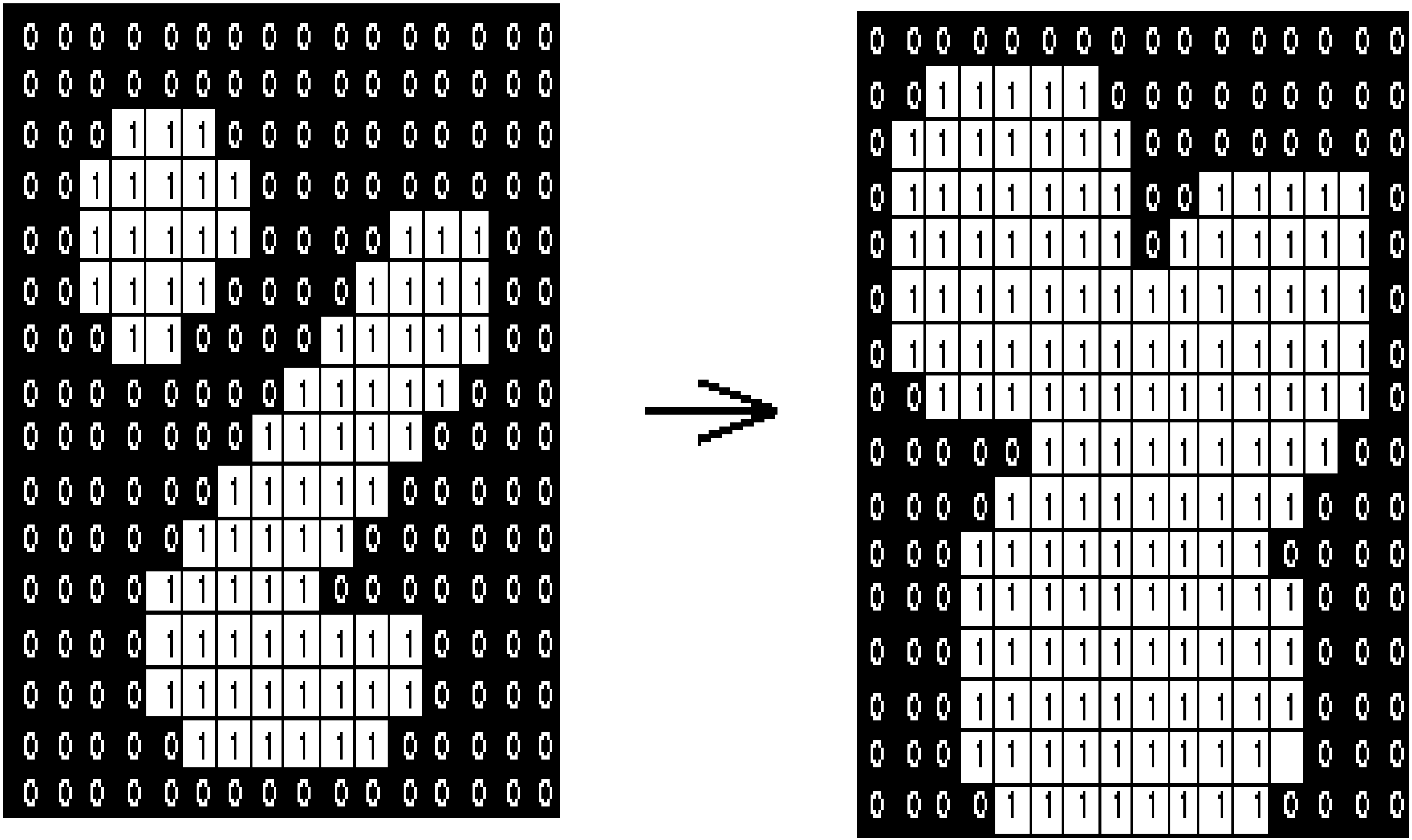
at scale *n* = 0, 1, 2,…, N.

- Performing opening and closing iteratively by increasing the size of *B* transforms the function *f*(x,y) into lower resolutions correspondingly.

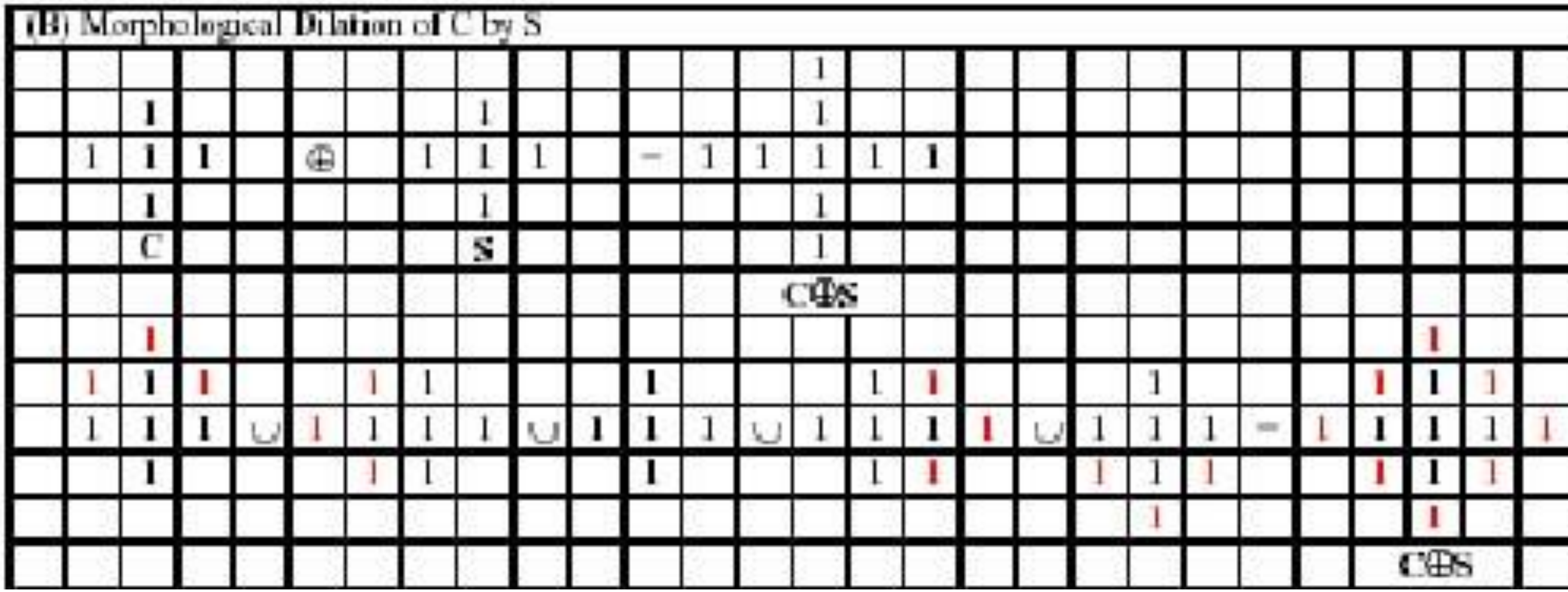
Mathematical Morphology (cont)

- Multiscale opening and closing of *f* by *nB* effect spatially distributed greyscale regions in the form of smoothing of contours to various degrees. The shape and size of *B* control the shape of smoothing and the scale respectively.
- Important problems like feature detection and characterisation often require analysing greyscale functions at multiple spatial resolutions. Recently, non-linear filters have been used to obtain images at multi-resolution due to their robustness in preserving the fine details.
- Advantages of mathematical morphology transformations
  - popular in object recognition and representation studies.
  - The non-linearity property in preserving the fine details.

Effect of Dilation using 3X3 structuring element



Steps in Dilation of *X* by *B*



Effect of Erosion using 3X3 structuring element

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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Steps in Erosion of *X* by *B*

(A) Morphological Erosion of C by S																											
	1						1							0													
1	1	1		⊖		1	1	1		=			0	1	0												
	1						1							0													
	C						S							C⊖S													
	0				1				1				1				1				0						
1	1	1	∩	0	1	1	∩	1	1	1	∩	1	1	0	∩	1	1	1	−	0	1	0					
	1				1				1				1				0				0						
																					C⊕S						

Effect of Opening using 3X3 structuring element

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0
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0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

→

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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Steps in Opening of *X* by *B*

(C) Morphological Opening of C by S

1	1	1					1						0	0	0					1						1	
1	1	1		$\ominus$		1	1	1		=			0	1	0		$\oplus$		1	1	1		=		1	1	1
1	1	1					1						0	0	0					1						1	
	C						S							$C \ominus S$						S						$C \ominus S \oplus S$	

Effect of Closing using 3X3 structuring element

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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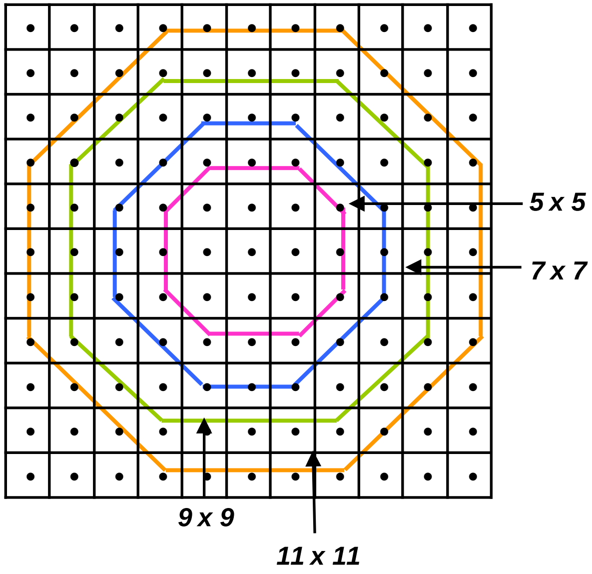
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Steps in Closing of *X* by *B*

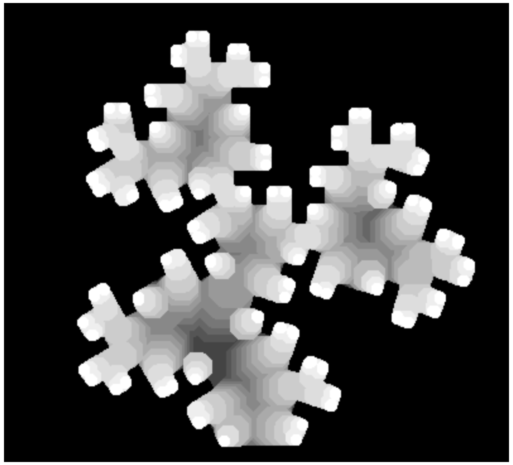
(D) Morphological Closing of C by S

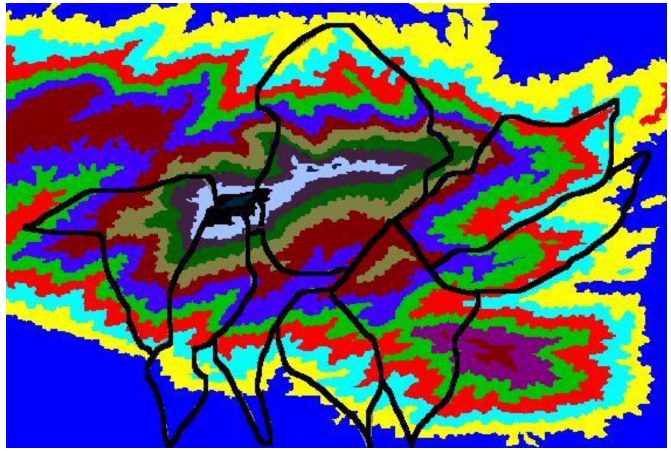
											1	1	1														
1	1	1				1				1	1	1	1	1					1				1	1	1		
1	0	1	⊕		1	1	1		=	1	1	1	1	1		⊖		1	1	1	=		1	1	1		
1	1	1				1				1	1	1	1	1					1				1	1	1		
	C					S					1	1	1						S					C			
												C⊕S												C⊕S⊖S			

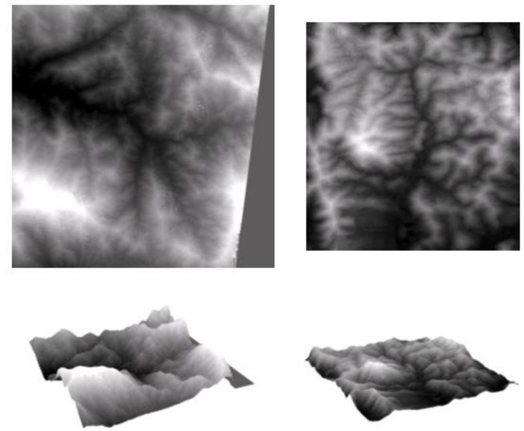


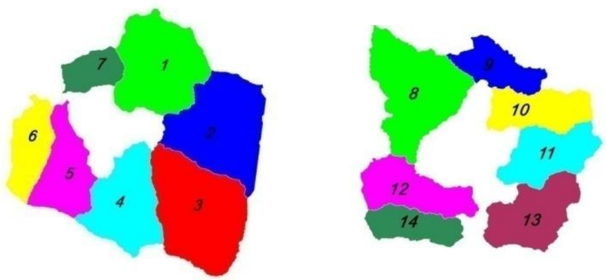
Octagonal symmetric structuring elements of various primitive sizes ranging from 5 × 5 to 11 × 11. These primitive sizes can be considered as *B*.

Digital Elevation Models

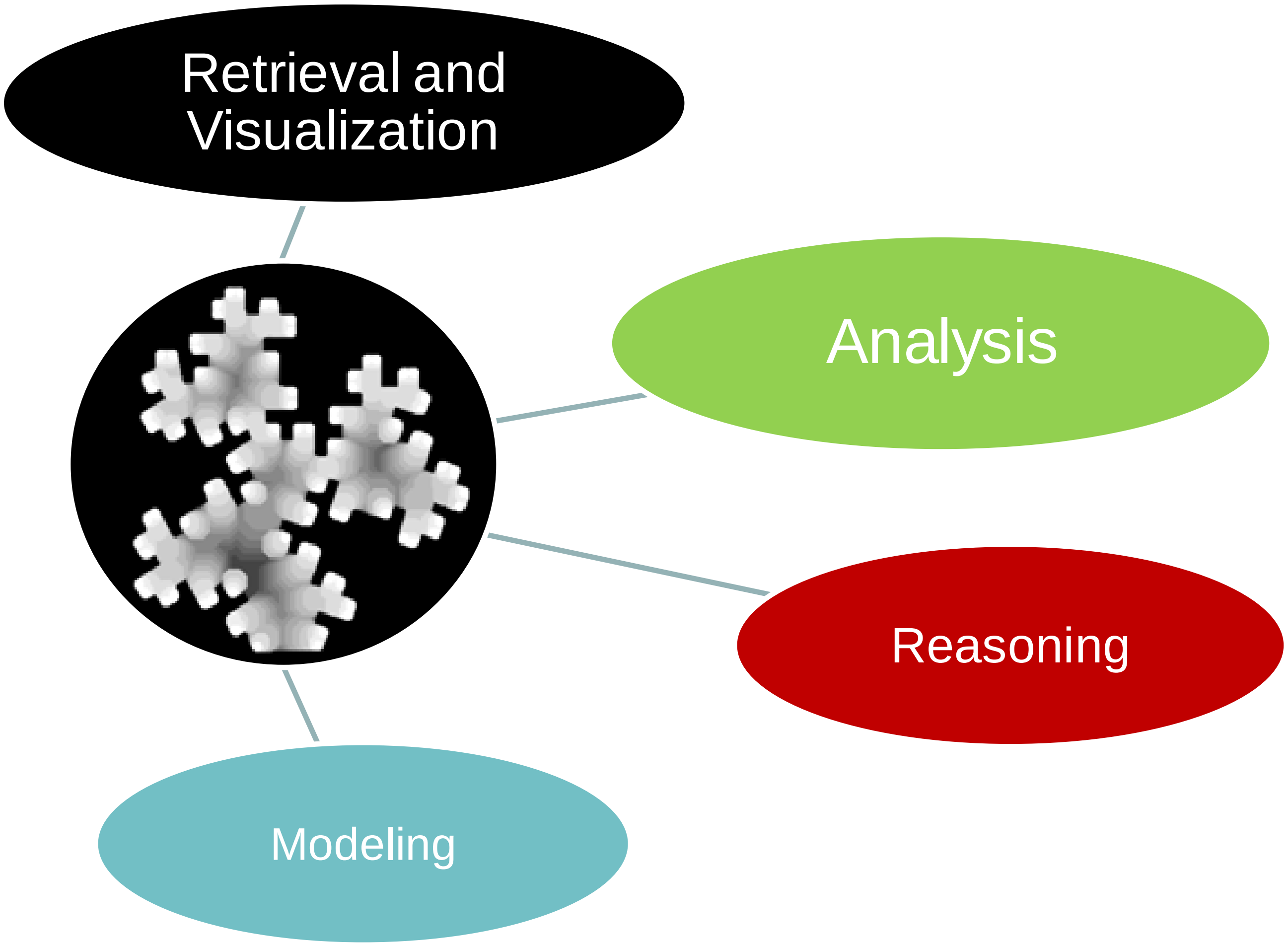




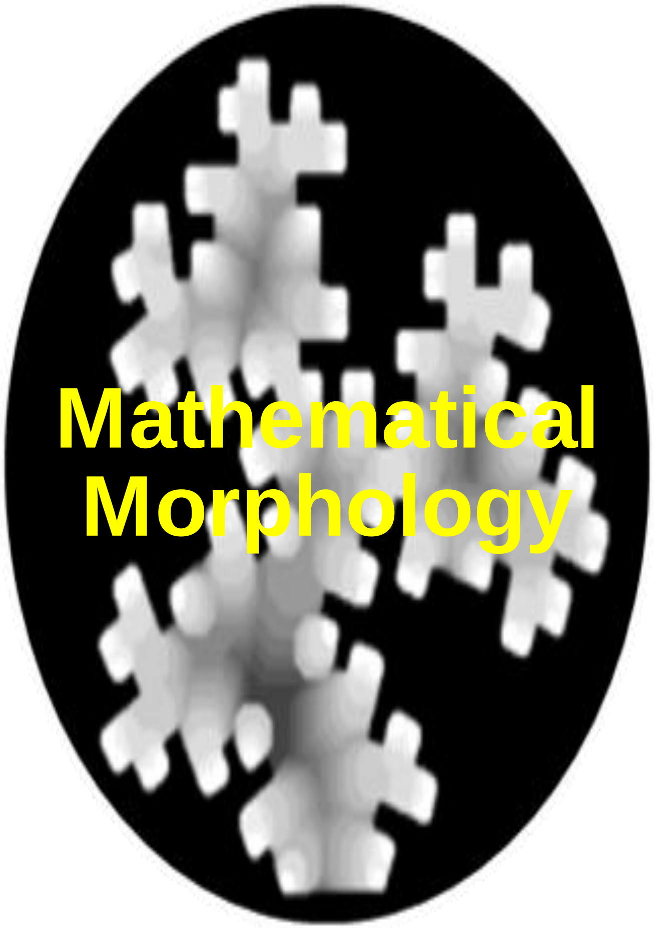




Mathematical Morphology in Retrieval, Analysis, Reasoning, Modeling and Visualization



Concepts, Techniques & Tools



- Morphological Skeletonization
- Multiscale operations, Hierarchical segmentation
- Recursive Morphological Pruning
- Hit-or-Miss Transformation
- Morphological Thinning
- Morphological Reconstruction
- Watersheds
- Morphological shape decomposition
- Granulometries
- Hausdorff dilation (erosion) distance
- Morphological interpolation
- Directional Distances
- SKIZ and WSKIZ

Outline

Basic description of Terrestrial Data

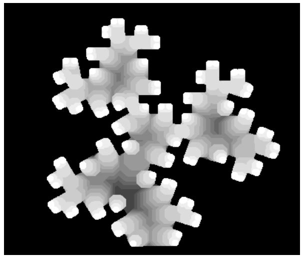
Mathematical Morphology in Geomorphology and GISci

Retrieval of unique phenomena (e.g. Networks), Analysis and quantitative characterization of Geomorphological phenomena and processes via various metrics

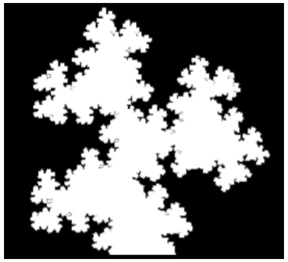
Spatial interpolation, Spatio-temporal modeling, spatial reasoning, spatial information visualization

Terrestrial Data : Various Representations

Functions (DEMs, Satellite Images, Microscopic Images etc)



Sets (Thresholded Elevation regions, Binary images decomposed from images)



Skeletons (Unique topological networks)

Thank You

Q & A

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