Defn Let V be a vector space and $v_1, \ldots, v_n \in V$. A linear combination of $v_1, ..., v_n$ is any vector of the type $r_1 v_1 + r_2 v_2 + \cdots + r_n v_n$, where $r_1, ..., r_n$ is are real nos.

Thm The set of all l.c. of $v_1, ..., v_n$ is a subspace of V.

This subspace is denoted by $< v_1, ..., v_n >$ and is called the subspace or vector space spanned by $v_1, ..., v_n$.

Ex 1 Let $v_1 = (1, 2, 0), v_2 = (1, 0, 1), v_3 = (2, 2, 1)$ and $v_4 = (1, 1, 1)$.

Is v_4 a linear combination of v_1, v_2, v_3?

Is v_3 a l.c. of v_1, v_2?

Ex 2 Consider the v.s. $F(\mathbb{R})$.
Is sin(x) a l.c. of cos(x) and cos(2x)?

Ex 3 In Ex 1, show that the v.s. spanned by $< v_1, v_2, v_4 >$ is $\mathbb{R}^3$.

Proof Let $(a, b, c) \in V$. Want to find r, s, t so that

$$rv_1 + sv_2 + tv_4 = (a, b, c)$$

i.e., want to find r, s, t so that

$$(r + s + t, 2r + t, s + t) = (a, b, c)$$

Treating r, s, t as variables we get the following equations:

Use gauss elimination to solve the equation

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 2 & 0 & 1 & b \\ 0 & 1 & 1 & c \end{bmatrix}$$

R_2 replaced by $R_2 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & -2 & -1 & b-2a \\ 0 & 1 & 1 & c \end{bmatrix}$$

interchange R_2, R_3

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 1 & c \\ 0 & -2 & -1 & b-2a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 1 & c \\ 0 & 0 & 1 & b-2a+2c \end{bmatrix}$$

$$t = b - 2a + 2c, s = c - t, r = a - s - t$$

$$t = b - 2a + 2c,$$

$$s = c - b + 2a - 2c,$$

$$r = a - c + b - 2a + 2c - b + 2a - 2c$$

[[End of proof]]

Is the v.s. spanned by $< v_1, v_2, v_3 >$ the whole $\mathbb{R}^3$?

No.

What is the subspace $< v_1, v_2, v_3 >$?

Ex 4

Show that vector space P_3 polynomials of degree atmost three is spanned by $1, x, x^2, x^3$.

Thm Let v, w span the vector space V. Let $x, y \in V$ be such that v and w are l.c. of x, y then $V \subset < x, y >$

proof

Note that since v and w are l.c. of x, y

$$v = ax + by$$

$$w = cx + dy$$

for some real numbers a, b, c, d.

Let $u \in V$. Since v, w span V.

$$u = rv + sw$$

for some real numbers r, s.

substituing for v, w we get

$$u = r(ax + by) + s(cx + dy)$$

$$u = (ra)x + (rb)y + (sc)x + (sd)y$$

$$u = (ra + sc)x + (rb + sd)y$$

So every vector of V is spanned by x, y

Linear Independence

Defn Let V be a vector space. $v_1, ..., v_n \in V$ is said to be linearly dependent if there exist real numbers $r_1, ..., r_n$, not all zero, so that $r_1 v_1 + ... + r_n v_n = 0$.

Otherwise $v_1, ..., v_n$ are said to be linearly independent.

Let $v_1 = (1,2,0), v_2 = (1,0,1), v_3 = (2,2,1)$ and $v_4 = (1,1,1)$.

Ex 1 v_1, v_2, v_3 are linearly dependent.

Ex 2 Let V be a v.s. and $v \in V$. v is dependent if and only if $v = 0$.

Thm Let V be a v.s. and $v, w \in V$ be nonzero vectors. v, w are dependent if and only if $v = rw$ for real number r.

Ex 3 Is $\{v_1, v_2, v_4\}$ linearly dependent set?

Let $r, s, t \in \mathbb{R}$ be such that

$$rv_1 + sv_2 + tv_3 = 0$$

then we get the following homogenous equation in r, s, t.

$$r + s + t = 0$$

$$2r + \quad + t = 0$$

$$s + t = 0$$

The coeffcient matrix is the following:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

R_2 replaced by $R_2 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

interchange R_2, R_3

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{bmatrix}$$

Replace R_3 by $R_3 + 2R_2$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

So $t = 0$, $s + t = 0$ and $r + s + t = 0$, i.e.

$r = s = t = 0$. So v_1, v_2, v_4 are linearly independent.

Ex 4 Does the following matrices form a linearly independent set?

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

What if we add the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ to the set?

Thm Let V be a v.s. and $v_1, ..., v_n \in V$. The set $\{v_1, ..., v_n\}$ is lin. depend. if and only if one of the vector in this set can be written as a linear combination of the other vectors in the set.

proof If $\{v_1, ...v_n\}$ is linearly dependent set then

there exist $r_1, ..., r_n \in \mathbb{R}$ not all zero so that $r_1 v_1 + ... + r_n v_n = 0$. Let $r_i \neq 0$.

$$-r_i v_i = r_1 v_1 + ... + r_{i-1} v_{i-1} + r_{i+1} v_{i+1} + ... + r_n v_n$$

Since $r_i \neq 0$, we can multiply the above by $-\frac{1}{r_i}$ to get

$$v_i = \frac{-r_1}{r_i} v_1 + ... + \frac{-r_{i-1}}{r_i} v_{i-1} + \frac{-r_{i+1}}{r_i} v_{i+1} + ... + \frac{-r_n}{r_i} v_n$$

So v_i is a l.c. of the other vectors.

Conversely, suppose one of the vector is a l.c. of other vectors. i.e.,

$v_i = s_1 v_1 + ... + s_{i-1} v_{i-1} + s_{i+1} v_{i+1} + ... + s_n v_n$

for some real numbers $s_1, ..., s_n$.

Then we get,

$s_1 v_1 + ... + s_{i-1} v_{i-1} + (-1) v_i + s_{i+1} v_{i+1} + ... + s_n v_n = 0$

Since the coefficient of v_i is nonzero, $\{v_1, ..., v_n\}$ is a lin. depend. set.

Defn Let V be a vector space. A subset $\{v_1, v_2, ..., v_n\}$ is said to be a basis of V if $\{v_1, ..., v_n\}$ is a l.i. set and it spans V.

Ex 1 $\{(1, 0), (0, 1)\}$ is a basis of $\mathbb{R}^2$. More basis of $\mathbb{R}^2$?

Ex 2 $\{1, x, x^2, x^3, x^4, ..., x^n\}$ is a basis on $\mathbb{P}^n$, polynomials of degree at most n.

Ex 3 Give a basis of $M(2, 2)$.

Shortly we will prove the following theorem:

Thm Let V be a v.s. If $\{v_1, ..., v_n\}$ is a basis of V and $\{w_1, ..., w_m\}$ is also a basis of V then $n = m$.

We will prove the following result first:

Thm Let V be a v.s. If $\{v_1, ..., v_n\}$ is a l.i. subset of V and $\{w_1, ..., w_m\}$ spans V then $m \geq n$.

proof

Since $\{w_1, ...w_m\}$ spans V, every v_i is a l.c. of $\{w_1, ..., w_m\}$.

So there are real numbers c_{ij} such that

$$v_1 = c_{11}w_1 + c_{12}w_2 + ... + c_{1m}w_m$$

$$v_2 = c_{21}w_1 + c_{22}w_2 + ... + c_{2m}w_m$$

$$.....$$

$$v_n = c_{n1}w_1 + c_{n2}w_2 + ... + c_{nm}w_m$$

Suppose $m < n$ then the following system of homogeous equations has a non trivial solution:

$$c_{11}x_1 + c_{21}x_2 + ... + c_{n1}x_n = 0$$

$$c_{12}x_1 + c_{22}x_2 + ... + c_{n2}x_n = 0$$

$$......$$

$$c_{1m}x_1 + c_{2m}x_2 + ... + c_{nm}x_n = 0$$

since there are more variables and less equations.

Let $x_1 = r_1, ..., x_n = r_n$ be a nontrivial solution. Then

$$r_1 v_1 + ... r_n v_n =$$

$$r_1 c_{11} w_1 + r_1 c_{12} w_2 + ... + r_1 c_{1m} w_m +$$

$$r_2 c_{21} w_1 + r_2 c_{22} w_2 + ... + r_2 c_{2m} w_m +$$

$$.....$$

$$r_n c_{n1} w_1 + r_n c_{n2} w_2 + ... + r_n c_{nm} w_m =$$

$$(c_{11} r_1 + c_{21} r_2 + ... + c_{n1} r_n) w_1 +$$

$$(c_{12} r_1 + c_{22} r_2 + ... + c_{n2} r_n) w_2 +$$

$$......$$

$$(c_{1m} r_1 + c_{2m} r_2 + ... + c_{nm} r_n) w_m$$

$$= 0$$

This contradicts the linear independence of the set $\{v_1, ..., v_n\}$.

[[End of proof]]

Thm Let V be a v.s. If $\{v_1, ..., v_n\}$ is a basis of V and $\{w_1, ..., w_m\}$ is also a basis of V then $n = m$.

Proof

Since $\{v_1, ..., v_n\}$ is a l.i. set and $\{w_1, ..., w_m\}$ generates V, by previous theorem $m \geq n$.

Also Since $\{w_1, ..., w_m\}$ is a l.i. set and $\{v_1, ..., v_n\}$ generates V, by previous theorem $n \geq m$.

Hence $n = m$.

[[End of proof]]

Defn Let V be a v.s. the dimension of V is the number of elements in a basis of V.

Remark By above theorem dimension of V is well defined.

Ex What is the dimension of $\mathbb{R}^n$?

Ex What is the dimension of $\mathbb{P}^n$?

Ex What is the dimension of $M(2,2)$?

Defn A v.s. is said to be finite dimensional if it has a basis consisting of finite number of elements. Otherwise it is called infinite dimensional v.s.

Ex $\mathbb{P}$, the set of polynomials is an infinite dimensional v.s.

Thm Suppose $\{v_1, ..., v_n\}$ spans a v.s. V. Then some subset of $\{v_1, ..., v_n\}$ is a basis of V.

proof If $\{v_1, ..., v_n\}$ is a l.i. set then $\{v_1, ..., v_n\}$ is a basis and we are done.

Otherwise, $\{v_1, ..., v_n\}$ is linearly dependent. So by Thm 3.5, one of the vector is a l.c. of the others. Let

$$v_i = s_1v_1 + ... + s_{i-1}v_{i-1} + s_{i+1}v_{i+1} + ... + s_nv_n$$

Let $u \in V$, then

$$u = r_1v_1 + ... + r_iv_i + ... + r_nv_n$$

$$= r_1v_1 + ... + r_{i-1}v_{i-1} +$$

$$r_i(s_1v_1 + ... + s_{i-1}v_{i-1} + s_{i+1}v_{i+1} + ... + s_nv_n)$$

$$+ r_{i+1}v_{i+1} + ... + r_nv_n$$

So $\{v_1, ..., v_{i-1}, v_{i+1}, ..., v_n\}$ spans V.

If $\{v_1, ..., v_{i-1}, v_{i+1}, ..., v_n\}$ is l.i. then it is a basis of V, it is a subset of $\{v_1, ..., v_n\}$ and we are done.

Otherwise keep continuing the above steps to get smaller subsets which spans V.

Eventually we will stop with a l.i. set or an empty set. If we get an empty set means $V = \{0\}$ and is spanned by the empty set. In this case the basis of V is the empty set.

[[End of proof]]

Corollary If a v.s. V is spanned by finite number of vectors then V is finite dimensional.

Proof Let $\{v_1, ..., v_n\}$ spans V.

By the above theorem, V has a basis consisting of at most n elements.

So V is finite dimensional.

Ex $\{x^2+x+2, x^2+1, x+1, x^2+2x, 3x^2+2x+1\}$ spans $\mathbb{P}^2$. Find a subset which is a basis.

Lemma Let $\{v_1, ..., v_n\}$ be a l.i. subset of a v.s. V. If $v \in V$ is not a l.c. of $\{v_1, ..., v_n\}$ then $\{v_1, ..., v_n, v\}$ is a l.i. set.

Proof

Let $r_1, ..., r_n, r$ be real numbers such that

$$r_1 v_1 + r_2 v_2 + ... + r_n v_n + rv = 0$$

If $r \neq 0$ then

$$v = \frac{-r_1}{r} v_1 + \frac{-r_2}{r} v_2 + ... + \frac{-r_n}{r} v_n$$

which contradicts the hypothesis.

So $r = 0$ and

$$r_1 v_1 + r_2 v_2 + ... + r_n v_n = 0$$

But $\{v_1, ..., v_n\}$ is a l.i. set.

So $r_i = 0$ for all i. And we know that $r = 0$.

Hence $\{v_1, ..., v_n, v\}$ is a l.i. set.

[[End of Proof]]

Thm Let $\{v_1, ..., v_n\}$ be a l.i. subset of a finite dimensional v.s. V. There exist $v_{n+1}, v_{n+2}, ..., v_m \in V$, where m is the dimension of V such that $\{v_1, ..., v_m\}$ is a basis of V.

proof If $\{v_1, ..., v_n\}$ spans V then $\{v_1, ..., v_n\}$ is a basis.

$m = n$ and we are done.

Otherwise let $v_{n+1} \in V$ but not in $< v_1, ..., v_n >$.

Then by the above theorem, $\{v_1, ..., v_n, v_{n+1}\}$ is a l.i. set.

Continue the above process to obtain $v_{n+1}, ..., v_m \in V$ so that

$\{v_1, ..., v_n, v_{n+1}, ..., v_m\}$ is a l.i. set.

If $\{v_1, ..., v_n, v_{n+1}, ..., v_m\}$ does not span V then

by previous theorem there exist $v \in V$ so that

$\{v_1, ..., v_n, v_{n+1}, ..., v_m, v\}$ is l.i. set.

But dimension V is m. So V can be spanned by m elements.

So by thm 3.9, V cannot have a l.i. set contain more than m elements.

A contradiction. So $\{v_1, ..., v_n, v_{n+1}, ..., v_m\}$ spans V and hence is a basis of V.

Ex Extend the following to a basis of $M(2, 2)$.
$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Cor Let S be a subspace of a v.s. V then $\dim(S) \leq \dim(V)$.

proof If $\{v_1, ..., v_n\}$ is a basis of S then it is a l.i. subset of V.

By above theorem, $\dim(V) \geq n = \dim(S)$.

[[End of Proof]]

Cor If $\dim(V) = n$ and $\{v_1, ..., v_n\}$ is l.i. then it is a basis of V. If $\{w_1, ..., w_n\}$ spans V then $\{w_1, ..., w_n\}$ is a basis of W.

Consider the following vectors in $\mathbb{R}^2$,

$(1, 1), (1, 2), (1, 0)$

They span $\mathbb{R}^2$ but they are not a basis.

A vector $(5, 21)$ can be expressed in many ways using these vectors.

$$(5, 21) = 5(1, 1) + 8(1, 2) - 8(1, 0)$$

$$= 21(1, 1) - 16(1, 0)$$

$$= (1, 1) + 10(1, 2) - 6(1, 0)$$

But $(1, 1), (1, 2)$ is a basis.

$$(5, 21) = -11(1, 1) + 16(1, 2)$$

there is no other l.c. of $(1, 1)$ and $(1, 2)$ which will give $(5, 21)$.

Let $B = \{(1, 1), (1, 2)\}$ be an ordered set. i.e. we remember the order in which the elements are listed. Then *the* co-ordinates of $(5, 21)$ with respect to B is $(-11, 16)_B$.

We shall switch notation for $\mathbb{R}^n$.

A $n \times 1$ coloumn matrix will denote an element of $\mathbb{R}^n$.

Defn Let V be a v.s. and $B = \{v_1, ..., v_n\}$ be an ordered basis of V. Let $u \in V$. $\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$ is the <u>coordinate</u> of u with respect to B

if $u = r_1 v_1 + r_2 v_2 + ... + r_n v_n$.

We write $u = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}_B$

Ex 1 Let $B = \{e_1, ..., e_n\}$ be the ordered standard basis of $\mathbb{R}^n$.

Then $\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}_B$

Ex 2 Note $B = \{x+2, x-2, x(x-2), x^2(x-2)\}$ is a basis of $\mathbb{P}^3$. Write the coordinates of x^3-2 w.r.t. B.

Thm Let $\{v_1, ..., v_n\}$ be a subset of a v.s. V. $\{v_1, v_2, ..., v_n\}$ is a basis of V if and only if for every $u \in V$, there exist unique $r_1, ..., r_n \in \mathbb{R}$ so that $u = r_1 v_1 + r_2 v_2 + ... + r_n v_n$.

proof

($\Rightarrow$)

Let $u \in V$. Since $\{v_1, v_2, ..., v_n\}$ is a basis there exist $r_1, ..., r_n \in \mathbb{R}$ such that

$$u = r_1 v_1 + r_2 v_2 + ... + r_n v_n$$

We only have to show uniqueness. Suppose there exist $s_1, ..., s_n \in \mathbb{R}$ so that

$$u = s_1 v_1 + s_2 v_2 + ... + s_n v_n$$

substracting the two expression, we get

$$0 = (r_1 - s_1)v_1 + (r_2 - s_2)v_2 + ... + (r_n - s_n)v_n$$

Since $\{v_1, v_2, ..., v_n\}$ is l.i., $r_i - s_i = 0$ for all i.

So $r_i = s_i$ for all i proving uniqeness of $r_1, ..., r_n$.

($\Leftarrow$)

By hypothesis, $\{v_1, v_2, ..., v_n\}$ spans V.

$$r_1 v_1 + r_2 v_2 + ... + r_n v_n = 0$$

and

$$0 v_1 + 0 v_2 + ... + 0 v_n = 0$$

So by uniqueness, $r_i = 0$ for all i. Hence $\{v_1, v_2, ..., v_n\}$ is l.i.

Hence $\{v_1, v_2, ..., v_n\}$ is a basis.

Chapter 4: Inner products

We shall define various operations on vector spaces which is behaves like multiplication. These operations are called inner products.

Inner product is a function from $V \times V$ to $\mathbb{R}$, where V is a vector space. So given a pair of vectors, you can "multiply" them to get a real number.

Defn Let V be a vector space. A function which assigns a real number $\langle v, w \rangle$ to vectors $v, w \in V$ is said to be an *inner product* if the function satisfies the following properties. Let u, v, w be any vectors in V

1. $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if and only if $v = \vec{0}$.

2. $\langle v, w \rangle = \langle w, v \rangle$.

3. $\langle rv, w \rangle = r\langle v, w \rangle = \langle v, rw \rangle$ for any $r \in \mathbb{R}$.

4. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$.

V together with an inner product is called an *inner product space*.

Example Let $V = \mathbb{R}^3$, define $\langle v, w \rangle := v \cdot w$, the dot product. i.e. $\langle (a_1, a_2, a_3), (b_1, b_2, b_3) \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$.

This is an inner product on $\mathbb{R}^3$.

More generally, on $\mathbb{R}^n$ the following is an inner product $\langle (a_1, a_2, ..., a_n), (b_1, b_2, ..., b_n) \rangle = a_1 b_1 + a_2 b_2 + ... + a_n b_n$.

Thm Let V be a vector space and $\langle ., . \rangle$ be an inner product on V. Then
$\langle v, 0 \rangle = \langle 0, v \rangle = 0$ and
$\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$.

proof $\langle v, 0\rangle = \langle 0, v\rangle$ by axiom 2 of inner product.

Shall show $\langle 0, v\rangle = 0$.
$\langle 0 + 0, v\rangle = \langle 0, v\rangle + \langle 0, v\rangle$ by axiom 4
$\Rightarrow \langle 0, v\rangle = \langle 0, v\rangle + \langle 0, v\rangle$
$\Rightarrow \langle 0, v\rangle - \langle 0, v\rangle = \langle 0, v\rangle + \langle 0, v\rangle - \langle 0, v\rangle$
$\Rightarrow \langle 0, v\rangle = 0$

$$
\begin{aligned}
\langle u, v + w\rangle &= \langle v + w, u\rangle \text{ (by axiom 2)} \\
&= \langle v, u\rangle + \langle w, u\rangle \text{ (by axiom 4)} \\
&= \langle u, v\rangle + \langle u, w\rangle \text{ (by axiom 2)}
\end{aligned}
$$

[[End of proof]]

Examples (1) For $a < b$ real numbers, let $V = C([a, b])$ be the vector space of continuous real valued function on the interval $[a, b]$. The following is an inner product on V:

For $f, g \in V$ let

$$\langle f, g \rangle = \int_a^b f(x) g(x) dx$$

(2) On $\mathbb{R}^n$ the following is an inner product

$\langle (a_1, a_2, , a_3), (b_1, b_2, b_3) \rangle = 2a_1b_1 + \sqrt{3}a_2b_2 + \pi a_n b_n$

(3) On $\mathbb{P}_3$ there is an inner product given by

$$\langle f, g \rangle_1 = \int_0^1 f(x) g(x) dx$$

and

$$\langle f, g \rangle_2 = a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3$$

where $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and $g(x) = b_0 + b_1x + b_2x^2 + b_3x^3$

Defn A *norm* is a function from a an inner product space $(V, \langle ., . \rangle)$ to $\mathbb{R}$ given by $||v|| = \sqrt{\langle v, v \rangle}$

Thm Let V be an inner product space and $v \in V$.

1. $||v|| \geq 0$ and $||v|| = 0$ if and only if $v = \vec{0}$.

2. $||rv|| = |r| \, ||v||$ for any $r \in \mathbb{R}$.

$\mathbb{R}^n$ will be viewed as an inner product space where the inner product $\langle v, w \rangle$ is the dot product $v \cdot w$, unless otherwise mentioned.

Thm Let θ be the angle between two nonzero vectors v, w in $\mathbb{R}^n$, then

$$v \cdot w = ||v|| \, ||w|| \cos(\theta)$$

Remarks

(1) So given two nonzero vectors $v, w \in \mathbb{R}^n$, the angle between v and w is given by

$$\theta = \arccos(\frac{v \cdot w}{||v|| \, ||w||}); \qquad \text{where } \theta \in [0, \pi]$$

(2) Note that $\theta = \pi/2$ if and only if $v \cdot w = 0$

Defn Two vectors v, w in $\mathbb{R}^n$ are said to be *orthogonal* if $v \cdot w = 0$.

Remark $\vec{0}$ is orthogonal to very vector in $\mathbb{R}^n$.

Example
Let $V = \{v \in \mathbb{R}^3 | v$ is orthogonal to $(1, 2, 3)\}$. Show V is a subspace of $\mathbb{R}^3$. Find a basis of V.

Orthogonal projection

Thm (Cauchy-Schwarz Inequality) Let V be an inner product space and $v, w \in V$.
Then $\langle v, w \rangle \leq ||v|| \, ||w||$.

Proof If $v = 0$ or $w = 0$ then both sides are zero and the inequality holds.

So assume v, w are nonzero vectors. So $a = 1/||v||$ and $b = 1/||w||$ are real numbers.

$$\langle av - bw, av - bw \rangle \geq 0$$

$$\Rightarrow \langle av, av - bw \rangle - \langle bw, av - bw \rangle \geq 0$$

$$\Rightarrow \langle av, av \rangle - \langle av, bw \rangle - \langle bw, av \rangle + \langle bw, bw \rangle \geq 0$$

$$\Rightarrow a^2 ||v||^2 - 2ab\langle v, w \rangle + b^2 ||w||^2 \geq 0$$

$$\Rightarrow 1 - \frac{2\langle v, w \rangle}{||v|| \, ||w||} + 1 \geq 0$$

$\Rightarrow 2 \geq 2\frac{\langle v,w\rangle}{||v||\,||w||}$

$\Rightarrow ||v||\,||w|| \geq \langle v, w\rangle$

[[End of Proof]]

Thm (Triangle inequality) Let V be a inner product space and $u, v \in V$ then $||u+v|| \leq ||u|| + ||v||$

Proof $||u+v||^2 = ||u||^2 + 2\langle u, v\rangle + ||v||^2$
$\leq ||u||^2 + 2||u||\,||v|| + ||v||^2$ by Cauchy's Schwarz inequality
$= (||u|| + ||v||)^2$

$\Rightarrow ||u+v|| \leq ||u|| + ||v||$

[[End of proof]]

Defn Let V be an innerproduct space and $u \in V$ be nonzero vector. For a vector $v \in V$ the *orthogonal projection* of v on u is denoted by $proj_u(v)$ and defined as

$$proj_u(v) = \frac{\langle u, v \rangle}{\langle u, u \rangle} u$$

Remark $v - proj_u(v)$ is orthogonal to u. Hence

$$v = proj_u(v) + (v - proj_u(v)$$

is a decomposition of v into two orthogonal components.

Defn Let V be an inner product space. A vector $u \in V$ is said to be a *unit vector* if $||u|| = 1$.

Example Let $(1, 2, 3) \in \mathbb{R}^3$. Find a unit vector which has the same direction as $(1, 2, 3)$.

Defn Let V be an inner product space. A subset S of V is said to be *orthogonal set* if given any two distinct vectors $u, v \in S$ u is orthogonal to v. Moreover, if each vector in S is a unit vector then S is called an *orthonormal set*.

Example In $\mathbb{R}^3$, $\{(1,0,0),(0,2,0),(0,0,3)\}$ is an orthogonal set.

In general in $\mathbb{R}^n$, $\{e_1, ..., e_n\}$ is an orthonormal set.

Thm Let $\{v_1, ...v_n\}$ be orthogonal set of nonzero vectors in an inner product space V then $\{v_1, ..., v_n\}$ is a linearly independent set.

proof Suppose for some real number $r_1, r_2, ..., r_n$,

$$r_1 v_1 + r_2 v_2 + ... + r_n v_n = 0$$

Then for each i,

$$\Rightarrow \langle v_i, r_1 v_1 + r_2 v_2 + ... + r_n v_n \rangle = \langle v_i, 0 \rangle$$

$$\Rightarrow \langle v_i, r_1 v_1 \rangle + \langle v_i, r_2 v_2 \rangle + ... + \langle v_i, r_n v_n \rangle = 0$$

$$\Rightarrow r_1 \langle v_i, v_1 \rangle + r_2 \langle v_i, v_2 \rangle + ... + r_n \langle v_i, v_n \rangle = 0$$

Since $\{v_1, ... v_n\}$ is an orthogonal set, $\langle v_i, v_j \rangle = 0$ for $j \neq i$. So

$$r_i \langle v_i, v_i \rangle = 0$$

$\Rightarrow r_i = 0$ (Since $v_i \neq 0$.)

Hence $\{v_1, ..., v_n\}$ are l.i.

Defn A subset $\{v_1, ..., v_n\}$ of an inner product space V is said to be an *orthogonal basis* if $\{v_1, ..., v_n\}$ is a basis and an orthogonal set.

$\{v_1, ..., v_n\}$ is said to be an *orthonormal basis* if $\{v_1, ..., v_n\}$ is a basis and an orthonormal set.

Examples

Next we shall see that from any given basis of an IP space V be can obtain an orthonormal basis by an algorithm called Gram-Schmidt orthonormalization process.