# KILLING WILD RAMIFICATION

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Abstract. We compute the inertia group of the compositum of wildly ramified Galois covers. It is used to show that even the $p$-part of the inertia group of a Galois cover of $\mathbb{P}^1$ branched only at infinity can be reduced if there is a jump in the lower ramification filtration at two and a certain linear disjointness statement holds.

## 1. Introduction

Let $k$ be a field of characteristic $p > 0$. Let $X \to Y$ be a finite $G$-Galois cover of regular irreducible $k$-curves branched at $\tau \in Y$. Let $I$ be the inertia subgroup of $G$ at a point of $X$ above $\tau$. It is well known that $I = P \rtimes \mu_n$ where $P$ is a $p$-group, $\mu_n$ is a cyclic group of order $n$ and $(n, p) = 1$. Abhyankar's lemma can be viewed as a tool to modify the tame part of the inertia group. For instance, suppose $k$ contains the $n^{th}$-roots of unity. Let $y$ be a regular local parameter of $Y$ at $\tau$. Let $Z \to Y$ be the Kummer cover of regular curves given by the field extension $k(Y)[y^{1/n}]/k(Y)$ and $\tau' \in Z$ be the unique point lying above $\tau$. Then the pullback of the cover $X \to Y$ to $Z$ is a Galois cover of $Z$ branched at $\tau'$. But the inertia group at any point above $\tau'$ is $P$. Our main result, Theorem 3.6, is a wild analogue of this phenomenon.

Assume $k$ is also algebraically closed field and let $X \to \mathbb{P}^1$ be a $G$-Galois cover of $k$-curves branched only at $\infty$. Let $I$ be the inertia subgroup at some point above $\infty$ and $P$ be the sylow-$p$ subgroup of $I$. Then noting that the tame fundamental group of $\mathbb{A}^1$ is trivial, it can be seen that the conjugates of $P$ in $G$ generate the whole of $G$. Abhyankar's inertia conjecture states that the converse should also be true. More precisely, any subgroup of a quasi-$p$ group $G$ of the form $P \rtimes \mu_n$, where $P$ is a $p$-group and $(n, p) = 1$, such that conjugates of $P$ generate $G$ should be the inertia group of a $G$-cover of $\mathbb{P}^1$ branched only at $\infty$.

An immediate consequence of a result of Harbater ([Ha2, Theorem 2]) shows that the inertia conjecture is true for every sylow-$p$ subgroup of $G$. In fact Harbater's result shows that if a $p$-subgroup $P$ of $G$ occurs as the inertia group of a $G$-cover of $\mathbb{P}^1$ branched only at $\infty$ and $Q$ is a $p$-subgroup of $G$ containing $P$ then there exists a $G$-cover of $\mathbb{P}^1$ branched only at $\infty$ so that the inertia group is $Q$. In Theorem 3.7, it is shown that if there is a jump in the lower ramification filtration at two and there is no epimorphism from $G$ to any nontrivial quotient of $P$ then the given $G$-cover of $\mathbb{P}^1$ can be modified to obtain a $G$-cover of $\mathbb{P}^1$ branched only at $\infty$ so that the inertia group of this new cover is smaller than $P$. The proof is via Theorem 3.6 and a study of higher ramification filtration (Proposition 2.7).

So far the inertia conjecture is only known for some explicit groups. See for instance [BP, Theorem 5] and [MP, Theorem 1.1].

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## 2. Filtration on ramification group

For a complete discrete valuation ring (DVR) $R$, $v_R$ will denote the valuation associated to $R$ with the value group $\mathbb{Z}$. Let $S/R$ be a finite extension of complete DVRs such that $\mathrm{QF}(S)/\mathrm{QF}(R)$ is a Galois extension with Galois group $G$. The lower ramification filtration on $G$ is a decreasing filtration defined for $i \geq -1$ in the following way:

$$G_i = \{\sigma \in G : v_S(\sigma x - x) \geq i+1, \, \forall x \in S\}$$

Note that $G_{-1} = G$ and $G_0$ is the inertia subgroup. For every $i$, $G_i$ is a normal subgroup of $G$. One extends this filtration to the real line as follows: for $u \in \mathbb{R}, u \geq -1$, define $G_u = G_m$ where $m$ is the smallest integer greater than or equal to $u$. The Herbrand function $\phi$ is defined as

$$\phi(v) = \int_0^v \frac{du}{[G_0 : G_u]}$$

Note that $\phi$ is a bijective piece-wise linear function. The upper ramification filtration on $G$ is defined as $G^v = G_{\psi(v)}$ for $v \geq 0$ where $\psi$ is the inverse of $\phi$. Let $G^{v+} = \cup_{\epsilon > 0} G^{v+\epsilon}$. The following are some well-known results.

**Proposition 2.1.** [Ser, IV, 1, Proposition 2 and 3] *Let $S/R$ be a finite extension of complete DVRs such that $\mathrm{Gal}(\mathrm{QF}(S)/\mathrm{QF}(R)) = G$. Let $H$ be a subgroup $G$. Let $K$ be the fixed subfield of $\mathrm{QF}(S)$ under the action of $H$. Let $T$ be the normalization of $R$ in $K$. Then $T$ is a complete DVR, $\mathrm{Gal}(\mathrm{QF}(S)/K) = H$ and the lower ramification filtration on $H$ is induced from that of $G$, i.e. $H_i = G_i \cap H$. Moreover, if $H = G_j$ for some $j \geq 0$ then $(G/H)_i = G_i/H$ for $i \leq j$ and $(G/H)_i = \{e\}$ for $i \geq j$.*

The upper ramification filtration behaves well with quotients of $G$ rather than subgroups.

**Proposition 2.2.** [Ser, IV, 1, Proposition 14] *In the above setup if we assume $H$ is a normal subgroup of $G$ then the upper filtration on $G/H$ is induced from $G$, i.e. $(G/H)^v = G^v H/H$.*

**Remark 2.3.** Note that for any $i \geq 0$, $G_i = G$ iff $G^i = G$. Since $\phi(v) \leq v$, $G_v = G^{\phi(v)} \supset G^v$. This explains the "if part". For the "only if" part, one notes that for $v \leq i$, by definition, $\phi(v) = v$ and hence $\psi(v) = v$.

**Proposition 2.4.** [Ser, IV, 2, Corollary 2 and 3] *The quotient group $G_0/G_1$ is a prime-to-$p$ cyclic group and if the residue field has characteristic $p > 0$ then for $i \geq 1$, $G_i/G_{i+1}$ is an elementary abelian group of exponent $p$. In particular $G_1$ is a $p$-group.*

**Lemma 2.5.** [FV, Chapter III, Section 2, 2.5] *Let $R$ be a complete DVR over a field $k$ of characteristic $p > 0$ with perfect residue field. Let $L/QF(R)$ be an Artin-Schrier extension given by the polynomial $Z^p - Z - x$, for some $x \in \mathrm{QF}(R)$ such that $v_R(x) < 0$ and $(v_R(x), p) = 1$. Then $L/\mathrm{QF}(R)$ is a $\mathbb{Z}/p\mathbb{Z}$-Galois extension with the upper (and lower) ramification jump at $-v_R(x)$.*

**Lemma 2.6.** *Let $S/R$ be a totally ramified extension of complete DVRs over a field $k$ of characteristic $p > 0$. Suppose $\mathrm{QF}(S)$ is generated over $\mathrm{QF}(R)$ by $\alpha_1, \ldots, \alpha_r \in \mathrm{QF}(S)$ with $\alpha_i^p - \alpha_i \in \mathrm{QF}(R)$ and $v_R(\alpha_i^p - \alpha_i) = -1$ for $1 \le i \le r$. Then the degree of the different $d_{S/R} = 2|G| - 2$.*

*Proof.* Let $L_i = \mathrm{QF}(R)(\alpha_i) \subset \mathrm{QF}(S)$ for $1 \le i \le r$. Then for all $i$, $H_i = \mathrm{Gal}(L_i/QF(R)) = \mathbb{Z}/p\mathbb{Z}$. It follows from Lemma 2.5 that the upper ramification filtration on $H_i$ is given by $H_i^v = \mathbb{Z}/p\mathbb{Z}$ for $0 \le v \le 1$ and $H_i^v = \{e\}$ if $v > 1$. From Proposition 2.2, it is easy to see that the highest upper jump of the compositum of Galois extensions is the maximum of the upper jumps of these extensions. So if we let $G = \mathrm{Gal}(\mathrm{QF}(S)/\mathrm{QF}(R))$ then $G^v = \{e\}$ for $v > 1$. Since $G$ is a $p$-group and $S/R$ is totally ramified $G^v = G$ for $v \le 1$. Hence $G_1 = G$ and $G_2 = \{e\}$. Now $d_{S/R} = 2|G| - 2$ follows from Hilbert's different formula [Sti, Theorem 3.8.7]. □

For a finite Galois extension $M/K$ of local fields, let $M_{<i}$, $M_{\le i}$ and $M_{=i}$ denote the compositum of subfields $L$ of $M$ such that the largest upper numbering jump in the ramification filtration of $L/K$ is "$< i$", "$\le i$" and "$= i$" respectively.

**Proposition 2.7.** *Let $i \ge 1$ and $S/R$ be a finite extension of complete DVRs over a perfect field $k$ of characteristic $p$ such that $\mathrm{Gal}(\mathrm{QF}(S)/\mathrm{QF}(R)) = G = G_i$. Let $L$ be the subfield of $\mathrm{QF}(S)$ generated over $\mathrm{QF}(R)$ by $\{\alpha \in \mathrm{QF}(S) | \alpha^p - \alpha \in R, v_R(\alpha^p - \alpha) = -i\}$. Then $G_{i+1} = \mathrm{Gal}(\mathrm{QF}(S)/L)$.*

*Proof.* Let $L' = \mathrm{QF}(S)^{G_{i+1}}$ and $H = \mathrm{Gal}(\mathrm{QF}(S)/L) \le G$. Since $G_{i+1}$ is a normal subgroup of $G$, the extension $L'/\mathrm{QF}(R)$ is Galois and $\mathrm{Gal}(L'/\mathrm{QF}(R)) = G/G_{i+1}(= \bar{G}$ say). Moreover, by Proposition 2.1, the lower ramification filtration on $\bar{G}$ is given by $\bar{G}_i = \bar{G}$ and $\bar{G}_{i+1} = \{e\}$. So we have $\bar{G}^i = \bar{G}$ and $\bar{G}^{i+} = \{e\}$. Hence $L' \subset \mathrm{QF}(S)_{=i}$. By Lemma 2.5, $L \subset \mathrm{QF}(S)_{=i}$. Since $G^i = G$, we have that $\mathrm{QF}(S)_{<i} = \mathrm{QF}(R)$. Hence $\mathrm{QF}(S)_{\le i} = \mathrm{QF}(S)_{=i}$. From Proposition 2.2 and $G^i = G$, it also follows that $i$ is the only jump in the upper ramification filtration of the Galois extension $\mathrm{QF}(S)_{=i}/\mathrm{QF}(R)$. So by Remark 2.3 and Proposition 2.4, the Galois group $\mathrm{Gal}(\mathrm{QF}(S)_{=i}/\mathrm{QF}(R))$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^r$ for some $r \ge 1$. So by Artin-Schrier theory we obtain that $\mathrm{QF}(S)_{=i}$ is generated over $\mathrm{QF}(R)$ by $\alpha_1, \ldots, \alpha_r \in \mathrm{QF}(S) \setminus \mathrm{QF}(R)$ such that $\beta_j = \alpha_j^p - \alpha_j \in R$ for $1 \le j \le r$.

Let $x$ be a uniformizing parameter of $R$ then $R = k[[x]]$. If $v_R(\beta_j) > 0$ then $\alpha_j = c - \beta_j - \beta_j^p - \beta_j^{p^2} - \ldots \in R$ for some $c \in \mathbb{F}_p$. So $v_R(\beta_j) \le 0$. Moreover since $G_0 = G$, $S/R$ is totally ramified. So $v_R(\beta_j) \ne 0$ and hence $v_R(\beta_j) < 0$. If $v_R(\beta_j)$ is a multiple of $p$ then $\beta_j = c_0 x^{pl} + c_1 x^{pl+1} + \ldots$, for some integer $l < 0$. Let $c \in k$ be such that $c^p = c_0$ and let $\alpha'_j = \alpha_j - cx^l$. Then $\beta'_j := \alpha_j'^p - \alpha'_j = \beta_j - c_0 x^{pl} + cx^l$, $v_R(\beta'_j) > v_R(\beta_j)$ and $\mathrm{QF}(R)(\alpha_j) = \mathrm{QF}(R)(\alpha'_j)$. Hence by such modifications we may assume the negative integer $v_R(\alpha_j^p - \alpha_j)$ is coprime to $p$. So by Lemma 2.5 the upper jump of the ramification filtration of $\mathrm{QF}(R)(\alpha_j)/\mathrm{QF}(R)$ is $-v_R(\alpha_j^p - \alpha_j)$. Since $\mathrm{QF}(R)(\alpha_j) \subset \mathrm{QF}(S)_{=i}$ we must have $v_R(\alpha_j^p - \alpha_j) = -i$ (by Proposition 2.2). Hence we have that $L = \mathrm{QF}(S)_{=i}$. By [Yos, Proposition 5.4] $\mathrm{Gal}(\mathrm{QF}(S)/\mathrm{QF}(S)_{\le i}) = G^{i+} = G_{i+1}$, which means $\mathrm{QF}(S)_{\le i} = L'$. Hence $L = L'$, i.e., $G_{i+1} = \mathrm{Gal}(\mathrm{QF}(S)/L)$). □

**Corollary 2.8.** *Let $L/K$ be a purely wildly ramified Galois extension of local fields of characteristic $p$ and perfect residue field with Galois group $G$. Then $L^{G_{i+1}} = L^{G_i}(\alpha | v_i(\alpha^p - \alpha) = -i)$ for $i \ge 1$ where $v_i$ is the valuation associated to the local field $L^{G_i}$.*

*Proof.* Note that the extension $L/L^{G_i}$ is Galois with Galois group $H = G_i \leq G$. Moreover, by Proposition 2.1 the ramification filtration on $H$ is induced from that of $G$. So $H_j = H \cap G_j = G_i \cap G_j$ and hence, $H_j = G_i = H$ for $j \leq i$ and $H_j = G_j$ for $j \geq i$. Since $H_i = H$, we can apply the above result to conclude that

$$G_{i+1} = H_{i+1} = \mathrm{Gal}(L/L^{G_i}(\alpha | v_i(\alpha^p - \alpha) = -i)$$

Hence $L^{G_{i+1}} = L^{G_i}(\alpha | v_i(\alpha^p - \alpha) = -i)$. □

## 3. Reducing Inertia

For a local ring $R$, let $m_R$ denote the maximal ideal of $R$. In this section we shall show that the wild part of the inertia subgroup of a Galois cover can be reduced. We begin with the following lemma.

**Lemma 3.1.** *Let $R$ be a DVR and $K$ be the quotient field of $R$. Let $L$ and $M$ be finite separable extensions of $K$ and $\Omega = LM$ their compositum. Let $A$ be a DVR dominating $R$ with quotient field $\Omega$. Note that $S = A \cap L$ and $T = A \cap M$ are DVRs. Let $\hat{K}$, $\hat{L}$, $\hat{M}$ and $\hat{\Omega}$ be the quotient field of the complete DVRs $\hat{R}$, $\hat{S}$, $\hat{T}$ and $\hat{A}$ respectively. If $A/m_A = S/m_S$ then $\hat{\Omega} = \hat{L}\hat{M}$. Here all fields are viewed as subfields of an algebraic closure of $\hat{K}$.*

*Proof.* Note that $\hat{L}$ and $\hat{M}$ are contained in $\hat{\Omega}$. So $\hat{L}\hat{M} \subset \hat{\Omega}$. Let $\pi_A$ denote a uniformizing parameter of $A$. Then $\pi_A \in LM \subset \hat{L}\hat{M}$. So it is enough to show that $\hat{\Omega} = \hat{L}[\pi_A]$. Note that $\hat{S}[\pi_A]$ is a finite $\hat{S}$-module, hence it is a complete DVR [Coh]. Also $\hat{S} \subset \hat{S}[\pi_A] \subset \hat{A}$ and $\pi_A$ generate the maximal ideal of $\hat{A}$, hence $\pi_A S$ is the maximal ideal of $\hat{S}[\pi_A]$. Moreover, the residue field of $\hat{S}$ is equal to $S/m_S = A/m_A$ which is same as the residue field of $\hat{A}$. Hence the residue field of $\hat{S}[\pi_A]$ is also same as the residue field of $\hat{A}$. So $\hat{S}[\pi_A] = \hat{A}$ (by [Coh, Lemma 4]). Hence the quotient field of $\hat{S}[\pi_A]$ is $\hat{\Omega}$. But that means $\hat{L}[\pi_A] = \hat{\Omega}$. □

**Corollary 3.2.** *Let the notation be as in the above lemma. If $\hat{L} \subset \hat{M}$ then $A/T$ is an unramified extension.*

*Proof.* Since $\Omega/M$ is a finite extension, so is $\hat{\Omega}/\hat{M}$. Hence $\hat{A}$ is a finite $\hat{T}$-module. The above lemma and the hypothesis implies that $\hat{\Omega} = \hat{M}$. So $\hat{A} = \hat{T}$, i.e. $A/T$ is unramified. □

**Remark 3.3.** The above corollary can be viewed as a field theoretic generalization of Abhyankar's lemma which applies to wildly ramified extensions as well. In fact in the above corollary if it is also assumed that $L/K$ is tamely ramified then $\hat{L} \subset \hat{M}$ is equivalent to saying that the ramification index of $L/K$ divides the ramification index of $M/K$.

Let $k$ be any field.

**Theorem 3.4.** *Let $X \to Y$ and $Z \to Y$ be Galois covers of regular $k$-curves branched at a closed point $\tau$ in $Y$. Let $\tau_x$ and $\tau_z$ be closed points of $X$ and $Z$ respectively, lying above $\tau$. Suppose $k(\tau_z) = k(\tau)$. Let $W$ be an irreducible dominating component of the normalization of $X \times_Y Z$ containing the closed point $(\tau_x, \tau_z)$. Then $W \to Y$ is a Galois cover ramified at $\tau$ and the decomposition subgroup of the cover at $\tau$ is the Galois group of the field extension $\mathrm{QF}(\hat{\mathcal{O}}_{X,\tau_x})\,\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z})/\,\mathrm{QF}(\hat{\mathcal{O}}_{Y,\tau_y})$. Note that all fields here can be viewed as subfields of $\mathrm{QF}(\hat{\mathcal{O}}_{W,(\tau_x,\tau_w)})$.*

*Proof.* Let $R = \mathcal{O}_{Y,\tau}$. Note that $R$ is a DVR and let $K$ be the quotient field of $R$. Let $L$ and $M$ be the function field of $X$ and $Z$ respectively and $\Omega = LM$ be their compositum. By definition $W$ is an irreducible regular curve with function field $\Omega$ and the two projections give the covering morphisms to $X$ and $Y$. Let $\tau_w$ denote the closed point $(\tau_x, \tau_z) \in W$ and $A = \mathcal{O}_{W,\tau_w}$. Since $\tau_w$ lies above $\tau_x$ under the morphism $W \to X$ and above $\tau_z$ under the morphism $W \to Z$, we have that $A \cap L = \mathcal{O}_{X,\tau_x}(= S \text{ say})$ and $A \cap M = \mathcal{O}_{Z,\tau_z}(= T \text{ say})$. Since $k(\tau_z) = k(\tau)$ and $k(W) = k(X)k(Z)$ we get that $k(\tau_w) = k(\tau_z)k(\tau_x) = k(\tau_x)$. But this is same as $A/m_A = S/m_S$. So using Lemma 3.1, we conclude that $\hat{L}\hat{M} = \hat{\Omega}$.

The decomposition group of the cover $W \to Y$ at $\tau_w$ is given by the Galois group of the field extension $\hat{\Omega}/\hat{K}$ ([Bou, Chapter 6, Section 8.5, Corollary 4]). This completes the proof because $\hat{\Omega} = \hat{L}\hat{M} = QF(\hat{\mathcal{O}}_{X,\tau_x})QF(\hat{\mathcal{O}}_{Z,\tau_z})$ and $\hat{K} = QF(\hat{\mathcal{O}}_{Y,\tau})$. □

**Proposition 3.5.** *Let $\Phi : X \to Y$ be a $G$-cover of regular $k$-curves ramified at $\tau_x \in X$ and let $\tau = \Phi(\tau_x)$. Let $G_\tau$ and $I_\tau$ be the decomposition subgroup and the inertia subgroup respectively at $\tau_x$. Let $N \le I_\tau$ be a normal subgroup of $G_\tau$. Suppose there exist a Galois cover $\Psi : Z \to Y$ of regular $k$-curves ramified at $\tau_z \in Z$ with $\Psi(\tau_z) = \tau$ and $k(\tau_z) = k(\tau)$ such that the fixed field $\mathrm{QF}(\hat{\mathcal{O}}_{X,\tau_x})^N$ is same as the compositum $\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z})k(\tau_x)$. Let $W$ be an irreducible dominating component of the normalization of $X \times_Y Z$ containing $(\tau_x, \tau_z)$. Then the natural morphism $W \to Z$ is a Galois cover. The inertia group and the decomposition group at the point $(\tau_x, \tau_z)$ are $N$ and an extension of $N$ by $\mathrm{Gal}(k(\tau_x)/k(\tau))$ respectively.*

*Proof.* Let $\tau_w \in W$ be the point $(\tau_x, \tau_z)$. Applying Theorem 3.4, we obtain that the decomposition group of the Galois cover $W \to Y$ at $\tau_w$ is isomorphic to $G_{\tau_w} = \mathrm{Gal}(\mathrm{QF}(\hat{\mathcal{O}}_{X,\tau_x})\,\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z})/\,\mathrm{QF}(\hat{\mathcal{O}}_{Y,\tau}))$. Since $\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z}) \subset \mathrm{QF}(\hat{\mathcal{O}}_{X,\tau_x})$, we have $G_{\tau_w} = G_\tau = \mathrm{Gal}(\mathrm{QF}(\hat{\mathcal{O}}_{X,\tau_x})/\,\mathrm{QF}(\hat{\mathcal{O}}_{Y,\tau}))$. Since $k(\tau_z) = k(\tau)$, the inertia group and the decomposition group of the cover $Z \to Y$ at $\tau_z$ are both $\mathrm{Gal}(\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z})/\,\mathrm{QF}(\hat{\mathcal{O}}_{Y,\tau}))$. Since $\mathrm{QF}(\hat{\mathcal{O}}_{X,\tau_x})^N = \mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z})k(\tau_x)$, we also obtain that $\mathrm{Gal}(\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z})k(\tau_x)/\,\mathrm{QF}(\hat{\mathcal{O}}_{Y,\tau})) = G_\tau/N$. Moreover, we have $G_\tau/I_\tau = \mathrm{Gal}(k(\tau_x)/k(\tau)) = \mathrm{Gal}(k(\tau_x)\,\mathrm{QF}(\hat{\mathcal{O}}_{Y,\tau})/\,\mathrm{QF}(\hat{\mathcal{O}}_{Y,\tau}))$. Since $\hat{\mathcal{O}}_{Z,\tau_z}/\hat{\mathcal{O}}_{Y,\tau}$ is totally ramified, $\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z})$ and $k(\tau_x)\,\mathrm{QF}(\hat{\mathcal{O}}_{Y,\tau})$ are linearly disjoint over $\mathrm{QF}(\hat{\mathcal{O}}_{Y,\tau})$.

$$\begin{tikzcd}
 & \mathrm{QF}(\hat{\mathcal{O}}_{X,\tau_x}) & \\
 & \mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z})k(\tau_x) \arrow[u, "N"] & \\
\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z}) \arrow[ur, "G_\tau/I_\tau"] & & \mathrm{QF}(\hat{\mathcal{O}}_{Y,\tau})k(\tau_x) \arrow[ul] \\
 & \mathrm{QF}(\hat{\mathcal{O}}_{Y,\tau}) \arrow[ul] \arrow[uu, "G_\tau/N"] \arrow[ur, "G_\tau/I_\tau"'] &
\end{tikzcd}$$

So $\mathrm{Gal}(\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z})k(\tau_x)/\,\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z})) = \mathrm{Gal}(k(\tau_x)/k(\tau))$. Hence the decomposition group of $W \to Z$ at the point $\tau_w$ is $\mathrm{Gal}(\mathrm{QF}(\hat{\mathcal{O}}_{X,\tau})/\,\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z}))$. But this group is an extension of $N$ by $\mathrm{Gal}(k(\tau_x)/k(\tau))$. Finally, the inertia group is $\mathrm{Gal}(\mathrm{QF}(\hat{\mathcal{O}}_{X,\tau})/\,\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z})k(\tau_x)) = N$. □

From here onwards, let $k$ be an algebraically closed field of characteristic $p > 0$.

**Theorem 3.6.** *Let $\Phi : X \to Y$ be a $G$-Galois cover of regular $k$-curves. Let $\tau_x \in X$ be a ramification point and $\tau = \Phi(\tau_x)$. Let $I$ be the inertia group of $\Phi$ at $\tau_x$. There exists a cover $\Psi : Z \to Y$ of deg $|I|$, such that the cover $W \to Z$ is étale over $\tau_z$ where $W$ is the normalization of $X \times_Y Z$ and $\tau_z \in Z$ is such that $\Psi(\tau_z) = \tau$. Moreover if there is no epimorphism from $G$ to any nontrivial quotient of $P$ where $P$ is the $p$-sylow subgroup of $I$ then $W \to Z$ is a $G$-cover of irreducible regular $k$-curves.*

*Proof.* Since $I$ is the inertia group, it is isomorphic to $P \rtimes \mu_n$ where $(p, n) = 1$ and $\mu_n$ is a cyclic group of order $n$. First we shall reduce to the case $I = P$. Let $y$ be a local coordinate of $Y$ at $\tau$ such that $k(Y)[y^{1/n}] \cap k(X) = k(Y)$. Let $Z_1$ be the normalization of $Y$ in $k(Y)[y^{1/n}]$. Then $Z_1 \to Y$ is a $\mu_n$-cover branched at $\tau$ such that $k(Z_1)$ and $k(X)$ are linearly disjoint over $k(Y)$. Let $\tau_{z1} \in Z_1$ be a point lying above $\tau$. Let $X_1$ be the normalization of $X \times_Y Z_1$. Then by the above proposition, $\Phi_1 : X_1 \to Z_1$ is a $G$-cover of irreducible regular $k$-curves and the inertia group at $(\tau_x, \tau_{z1})$ is $P$.

Let $Y_1 = Z_1$, $\tau_{x1} = (\tau_x, \tau_{z1})$ and $\tau_1 = \tau_{z1}$. Then $\Phi_1 : X_1 \to Y_1$ is a $G$-cover with $\Phi_1(\tau_{x1}) = \tau_1$ and the inertia group of this cover at $\tau_{x1}$ is $P$. Once we obtain a cover $Z \to Y_1$ satisfying the conclusions of the theorem for the cover $\Phi_1 : X_1 \to Y_1$ and closed points $\tau_{x1}$ and $\tau_1$, then the cover $Z \to Y$, obtained by the composition $Z \to Y_1 \to Y$, satisfies the conclusions of the theorem. This is because the morphism $X \times_Y Z \to Z$ is same as $X_1 \times_{Y_1} Z \to Z$ and the degree of the morphism $Z \to Y$ is $|P|n = |I|$. So replacing $\Phi : X \to Y$, $\tau_x$ and $\tau$ by $\Phi_1 : X_1 \to Y_1$, $\tau_{x1}$ and $\tau_1$ respectively, we may assume that $I = P$.

Let $y$ be a regular parameter of $Y$ at $\tau$. Then $k(Y)/k(y)$ is a finite extension. Since $Y$ is a regular curve, we get a finite morphism $\alpha : Y \to \mathbb{P}^1_y$ such that $\alpha(\tau)$ is the point $y = 0$ and $\alpha$ is étale at $\tau$ (as $\hat{\mathcal{O}}_{Y_1,\tau} = k[[y]]$).

Note that $\mathrm{QF}(\hat{\mathcal{O}}_{X,\tau_x})/k((y))$ is a $P$-extension. By [Ha1, Cor 2.4], there exist a $P$-cover $V \to \mathbb{P}^1_y$ branched only at $y = 0$ (where it is totally ramified) such that $\mathrm{QF}(\hat{\mathcal{O}}_{V,\theta}) = \mathrm{QF}(\hat{\mathcal{O}}_{X,\tau_x})$ as extensions of $k((y))$. Here $\theta$ is the unique point in $V$ lying above $y = 0$. Since $V \to \mathbb{P}^1_y$ is totally ramified over $y = 0$ and $Y \to \mathbb{P}^1_y$ is étale over $y = 0$, the two covers are linearly disjoint. Let $Z$ be the normalization of $V \times_{\mathbb{P}^1_y} Y$. Then the projection map $Z \to Y$ is a $P$-cover. Let $\tau_z \in Z$ be the closed point $(\theta, \tau)$. By Lemma 3.1, $\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z}) = \mathrm{QF}(\hat{\mathcal{O}}_{V,\theta})\,\mathrm{QF}(\hat{\mathcal{O}}_{Y,\tau}) = \mathrm{QF}(\hat{\mathcal{O}}_{X,\tau_x})$. Applying Proposition 3.5 with $N = \{e\}$, we get that an irreducible dominating component $W$ of the normalization of $X \times_Y Z$ is a Galois cover of $Z$ such that the inertia group over $\tau_z$ is $\{e\}$. Hence the normalization of $X \times_Y Z$ is a cover of $Z$ étale over $\tau_z$.

Moreover, there is no epimorphism from $G$ to any nontrivial quotient of $P$ implies that $k(Z)$ and $k(X)$ are linearly disjoint over $k(Y)$. Hence $W \to Z$ is a $G$-cover. □

**Theorem 3.7.** *Let $\Phi : X \to \mathbb{P}^1$ be a $G$-Galois cover of regular $k$-curves. Suppose $\Phi$ is branched only at one point $\infty \in \mathbb{P}^1$ and the inertia group of $\Phi$ over $\infty$ is $I$. Let $P$ be a subgroup of $I$ such that $I_1 \supset P \supset I_2$. Suppose there is no epimorphism from*

*$G$ to any nontrivial quotient of $P$. Then there exists a $G$-cover $W \to \mathbb{P}^1$ ramified only at $\infty$ and the inertia group at $\infty$ is $P$.*

*Proof.* Let $n = [I : I_1]$ be the tame ramification index of $\Phi$ at $\infty$. Let $x$ be a local coordinate on $\mathbb{P}^1$ such that the point $\infty$ is $x = \infty$. Let $\mathbb{P}^1_y \to \mathbb{P}^1_x$ be the Kummer cover obtained by sending $y^n$ to $x$. Since $\Phi$ is étale at $x = 0$ and the cover $\mathbb{P}^1_y \to \mathbb{P}^1_x$ is totally ramified at $x = 0$ the two covers are linearly disjoint. So letting $W$ to be the normalization of $X \times_{\mathbb{P}^1_x} \mathbb{P}^1_y$, we obtain a $G$-cover $\Phi_1 : W \to \mathbb{P}^1_y$ of regular $k$-curves. Moreover, by Abhyankar's lemma, $\Phi_1$ is ramified only at $y = \infty$ and the inertia group of $\Phi_1$ at $y = \infty$ is the subgroup $I_1$ of $I$. So replacing $\Phi$ by $\Phi_1$, we may assume $I = I_1$. Also since $I_1/I_2$ is abelian, $P$ is a normal subgroup of $I$.

Let $\tau \in X$ be a point above $x = \infty$. Let $S = \hat{\mathcal{O}}_{X,\tau}$ and $R = \hat{\mathcal{O}}_{\mathbb{P}^1,\infty}$ then $R = k[[x^{-1}]]$ and $\mathrm{Gal}(QF(S)/\mathrm{QF}(R)) = I$. Let $L = \mathrm{QF}(S)^P$. Then by Proposition 2.7, $L = \mathrm{QF}(R)(\alpha_1, \ldots, \alpha_l)$ where $\alpha_i \in \mathrm{QF}(S)$ is such that $v_R(\alpha_i^p - \alpha_i) = -1$ for $1 \le i \le l$. Let $T$ be the normalization of $R$ in $L$. Then $\mathrm{Spec}(T)$ is a principal $P$-cover of $\mathrm{Spec}(R)$. By [Ha1, Corollary 2.4], this extends to a $P$-cover $\Psi : Z \to \mathbb{P}^1_x$ ramified only at $x = \infty$, where it is totally ramified. Let $\tau_z \in Z$ be the point lying above $x = \infty$, then $\mathrm{QF}(\hat{\mathcal{O}}_{Z,\tau_z}) = L = \mathrm{QF}(S)^P$. By Lemma 2.6, $d_{T/R} = 2|P| - 2$. So by Riemann-Hurwitz formula, the genus of $Z$ is given by

$$2g_Z - 2 = |P|(0 - 2) + d_{T/R}$$

Hence $g_Z = 0$. So $Z$ is isomorphic to $\mathbb{P}^1$.

Since there is no epimorphism from $G$ to any nontrivial quotient of $P$, $\Phi$ and $\Psi$ are linearly disjoint covers of $\mathbb{P}^1_x$. Let $W$ be the normalization of $X \times_{\mathbb{P}^1_x} Z$. Now we are in the situation of Proposition 3.5. Hence the $G$-cover $W \to Z$ is branched only at $\tau_z$ and the inertia group at $\tau_z$ is $P$. This completes the proof, as $Z$ is isomorphic to $\mathbb{P}^1$. □

**Remark 3.8.** Note that if $G$ is a simple group different from $\mathbb{Z}/p\mathbb{Z}$ then the group theoretic hypothesis of the above results are satisfied.

**Corollary 3.9.** *Let $\Phi : X \to \mathbb{P}^1$ be a $G$-Galois cover of regular $k$-curves branched only at one point $\infty \in \mathbb{P}^1$ and the inertia group of $\Phi$ over $\infty$ is $I$. Suppose there is no epimorphism from $G$ to any nontrivial quotient of $I_2$. Then the conjugates of $I_2$ generate $G$.*

*Proof.* Applying the above theorem with $P = I_2$, we get an étale $G$-cover of $\mathbb{A}^1$ with the inertia group $I_2$ at $\infty$. Let $N$ be the normal subgroup of $G$ generated by the conjugates of $I_2$. By Galois theory, $N$ leads to a $G/N$-Galois cover of $\mathbb{P}^1$ which is étale over $\mathbb{A}^1$ and has trivial inertia group at $\infty$. So $N = G$, as $\mathbb{P}^1$ has no nontrivial étale cover. □

## References

[Bou] Nicholas Bourbaki *Commutative algebra. Chapters 1–7* Translated from the French. Reprint of the 1989 English translation. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. xxiv+625 pp.

[BP] I. I. Bouw and R. J. Pries *Rigidity, reduction, and ramification*, Math. Ann. 326:4 (2003), 803824.

[MP] Jeremy Muskat and Rachel Pries *Title: Alternating group covers of the affine line*, http://arxiv.org/abs/0908.2140v2 (arxiv preprint).

[Coh] I. S. Cohen *On the structure and ideal theory of complete local rings*, Trans. Amer. Math. Soc. 59, (1946). 54106.

[FV] I. B. Fesenko and S. V. Vostokov *Local fields and their extensions*, With a foreword by I. R. Shafarevich. Second edition. Translations of Mathematical Monographs, 121. American Mathematical Society, Providence, RI, 2002. xii+345 pp.

[Ha1] David Harbater *Moduli of p-covers of curves*, Comm. Algebra 8 (1980), no. 12, 10951122.

[Ha2] David Harbater *Embedding problems and adding branch points*, in "Aspects of Galois Theory", London Mathematical Society Lecture Note series, **256** Cambridge University Press, pages 119-143, 1999.

[Ser] Jean-Pierre Serre *Local fields*, Translated from the French by Marvin Jay Greenberg. Graduate Texts in Mathematics, 67. Springer-Verlag, New York-Berlin, 1979. viii+241 pp.

[Sti] Henning Stichtenoth *Algebraic function fields and codes*, Second edition. Graduate Texts in Mathematics, 254. Springer-Verlag, Berlin, 2009. xiv+355 pp.

[Yos] Manabu Yoshida *Ramification of local fields and Fontaine's property ($P_m$)* J. Math. Sci. Univ. Tokyo 17 (2010), no. 3, 247265 (2011).

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