FUNDAMENTAL GROUP IN NONZERO CHARACTERISTIC

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Abstract. A proof of freeness of the commutator subgroup of the fundamental group of a smooth irreducible affine curve over a countable algebraically closed field of nonzero characteristic. A description of the abelianizations of the fundamental groups of affine curves over an algebraically closed field of nonzero characteristic is also given.

1. Introduction

The algebraic fundamental group of smooth curves over an algebraically closed field of characteristic zero is a well understood object, thanks to Grothendieck's Riemann existence theorem [SGAI, XIII, Corollary 2.12, page 392] . But if the characteristic of the base field is $p > 0$ and the curve is affine then there may be wild ramification over the points at infinity. So computing the algebraic fundamental group in this scenario is not as simple. Though Grothendieck gave a description of the prime-to-p part of the fundamental group. The prime-to-p part is analogous to the characteristic zero case. But the structure of the whole group is still elusive in spite of the fact that all the finite quotients of this group are now known. A necessary and sufficient condition for a finite group to be a quotient of the fundamental group of a smooth affine curve was conjectured by Abhyankar (see the theorem below) and was proved by Raynaud [Ra1] (in the case of the affine line) and Harbater [Ha2] (for arbitrary smooth affine curves). For a finite group G and a prime number p, let $p(G)$ denote the subgroup of G generated by all the p-Sylow subgroups. $p(G)$ is called the *quasi-p* part of G.

Theorem 1.1. (**Raynaud, Harbater**) *Let C be a smooth projective curve of genus g over an algebraically closed field of characteristic $p > 0$ and for some $n \geq 0$, let $x_0, \cdots, x_n$ be some points on C. Then a finite group G is a quotient of the fundamental group $\pi_1(C \setminus \{x_0, \cdots, x_n\})$ if and only if $G/p(G)$ is generated by $2g + n$ elements. In particular a finite group G is a quotient of $\pi_1(\mathbb{A}^1)$ if and only if $G = p(G)$, i.e., G is a quasi-p group.*

The "if part" of the above theorem is the nontrivial part, the "only if part" was proved long back by Grothendieck. Serre made some significant advancement towards solving this conjecture in [Se2]. More precisely, Serre proved the "if part" for the affine line under the assumption that the group G is solvable. Then Raynaud, using the induction on the cardinality of G and dealing with other cases, completed the proof of the conjecture for the affine line. Once the affine line case was done,

I would like to thank my advisor, Prof. Donu Arapura, for his guidance and numerous suggestions in developing the theory and verification of the proofs. I am also grateful to Prof David Harbater for his useful suggestions and comments which helped me obtain the result on the commutator subgroup in this generality, i.e., for any smooth affine irreducible curve and improve the presentation of the paper. Department of Mathematics, Purdue University.

Harbater used the technique of formal patching to combine the quasi-p covers of the affine line and prime-to-p covers of the given curve to construct covers with desired group, solving the conjecture in the general case. A flavour of formal patching technique will be seen in this manuscript as well.

From now on we shall assume that the characteristic of the base field is $p > 0$. Consider the following exact sequence for the fundamental group of a smooth affine curve C.

$$1 \to \pi_1^c(C) \to \pi_1(C) \to \pi_1^{ab}(C) \to 1$$

where $\pi_1^c(C)$ and $\pi_1^{ab}(C)$ are the commutator subgroup and the abelianization of the fundamental group $\pi_1(C)$ of C respectively. In this paper we give a description of the abelianization (Corollary 3.5). This is a simple consequence of the theory of Witt vectors and some calculation involving étale cohomology. But the main result of this paper is the following (see Theorem 4.8, Theorem 5.3, Theorem 5.5 and Theorem 6.12):

Theorem 1.2. *Let k be a countable algebraically closed field of characteristic p. Let C be any irreducible smooth affine curve over k, then $\pi_1^c(C)$ is free.*

It is worth noting here that the tame part or the prime-to-p part of the fundamental group of an affine curve may be very small. For instance, the prime-to-p part of the fundamental group of the affine line is trivial. But obviously the prime-to-p part of the commutator subgroup is different from the commutator subgroup of the prime-to-p part of the fundamental group.

The result on the commutator subgroup can be interpreted as some analogue of the so called Shafarevich's conjecture for global fields. Recall that the Shafarevich conjecture says that the commutator subgroup of the absolute Galois group of the rational numbers $\mathbb{Q}$ is free. David Harbater ([Ha6]), Florian Pop ([Pop]) and later Dan Haran and Moshe Jarden ([HJ]) have shown, using different patching methods, that the absolute Galois group of the function field of a curve over an algebraically closed field is free. See [Ha7] for more details on these kind of problems.

The Section 2 of this thesis mainly consists of definitions and notations. In Section 3, "Abelianization", we give a description of the p-part of the abelianization of the algebraic fundamental group of any normal affine algebraic variety over an algebraically closed field in terms of Witt vectors. We deduce the fact that the abelianization of the algebraic fundamental group determines $W_n(A)/P(W_n(A))$ as a group (see Remark 3.6) where $W_n(A)$ is the ring of finite Witt vectors over the coordinate ring A of the affine variety under consideration and P is the additive group endomorphism of $W_n(A)$ which sends $(a_1, \cdots, a_n)$ to $(a_1^p, \cdots, a_n^p) - (a_1, \cdots, a_n)$ (here $''-''$ is subtraction in the Witt ring). It is conjectured by Harbater that the algebraic fundamental group of an affine curve should determine its coordinate ring.

The rest of the manuscript is devoted to proving that the commutator subgroup of the algebraic fundamental group of a smooth irreducible affine curve over a countable algebraically closed field k is a free profinite group of countable rank. Section 4 consist of some group theory results, some results on embedding problems and it is also shown that the commutator subgroup of the algebraic fundamental group is projective. These results are then used to reduce Theorem 1.2 to finding proper solution for all split embedding problem with perfect quasi-p group as the kernel, abelian p-group as kernel and prime-to-p group as kernel (for definitions and terminology see Section 2). The embedding problems with quasi-p kernels are easy to handle thanks to the results of Florian Pop (Theorem 5.1). These are dealt in Section 5. The nontrivial part of this document is solving the embedding problems with prime-to-p kernel.

The section on "prime-to-p embedding problems" (Section 6) is the longest one and is devoted to finding proper solutions for prime-to-p embedding problems. The first subsection of this section is on formal patching methods which were developed by Harbater (see [Ha1], [Ha2] and [Ha3]), Ferrand, Raynaud, Artin, etc. These patching results have been moulded for the situation at hand. The next subsection contains the proof of the main theorem (Theorem 6.12). It starts with some technical lemmas and propositions. First the prime-to-p embedding problems are solved for the commutator subgroup of the fundamental group of the affine line. The modifications needed for the general case appears next. The last section consists of a few more short exact sequences involving the fundamental group of affine curves. These are fallouts of the proof. A brief outline of how to modify the proof to get these results have also been mentioned.

2. Definitions and notations

Let p be a fixed prime number. For a ring A of characteristic p, $(W_n(A), +, .)$ will denote the ring of Witt vectors of length n over R. This ring as a set consists of n-tuples of elements of R. But the group operations are entirely different. The multiplicative identity is $(1, 0, 0, \cdots, 0)$ and the additive identity is the zero vector $(0, 0, \cdots, 0)$.

Let F denote the Frobenius endomorphism on $W_n(A)$ which sends $(a_0, \cdots, a_{n-1})$ to $(a_0^p, \cdots, a_{n-1}^p)$ and $P\colon W_n(A) \to W_n(A)$ be the homomorphism of abelian groups sending $(a_0, \cdots, a_{n-1})$ to $(a_0^p, \cdots, a_{n-1}^p) - (a_0, \cdots, a_{n-1})$, i.e., $F - Id$. Note that $(a_0^p, \cdots, a_{n-1}^p) - (a_0, \cdots, a_{n-1})$ is not the same as $(a_0^p - a_0, \cdots, a_{n-1}^p - a_{n-1})$ since the "$-$" in the Witt ring is different from component-wise subtraction.

For a field K of characteristic p, these different Witt rings characterize the abelian p-group field extensions of K. For a detailed treatment of Witt vectors, readers are advised to look at [Jac].

An *embedding problem* consists of surjections $\phi\colon \pi \twoheadrightarrow G$ and $\alpha\colon \Gamma \twoheadrightarrow G$

$$\begin{tikzcd}
 & & & \pi \arrow[dl, dashed, "\psi"'] \arrow[d, "\phi"] & \\
1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r, "\alpha"'] & G \arrow[r] \arrow[d] & 1 \\
 & & & 1 &
\end{tikzcd}$$

where G, Γ and π are groups. Let $H = \ker(\alpha)$. It is also sometimes called embedding problem for π. It is said to have a *weak solution* if there exists a group homomorphism ψ which makes the diagram commutative, i.e., $\alpha \circ \psi = \phi$. Moreover, if ψ is an epimorphism then it is said to have a *proper* solution. It is said to be a finite embedding problem if Γ is finite. All the embedding problems considered here is assumed to be finite. An embedding problem is said to be a *split embedding problem* if there exists a group homomorphism from G to Γ which is a right inverse of α. It is called a *quasi-p* embedding problem or embedding problem with quasi-p kernel if H is a quasi-p group, i.e., H is generated by its Sylow p-subgroups and similarly it is called a prime-to-p embedding problem if H is a prime-to-p group, i.e., the order of H is prime to p. H will sometimes be referred to as the kernel of the embedding problem.

A *profinite group* is a compact Hausdorff totally disconnected topological group. In the context of this thesis, a more useful definition (which is equivalent to the previous one) is that a profinite group is the inverse limit of an inverse system of finite groups with discrete topology.

A profinite group is called *free* if it is a profinite completion of a free group. A *generating set* of a profinite group π is a subset I so that the closure of the group generated by I is the whole group π. The *rank* of a profinite group is the minimum of the cardinalities of its generating sets.

For a reduced ring R, $\mathrm{frac}(R)$ will denote the total ring of quotients of R. A ring extension $R \subset S$ is said to be *generically separable* if R is a domain, S is reduced, $\mathrm{frac}(S)$ is separable extension of $\mathrm{frac}(R)$ and no nonzero element of R becomes a zero divisor in S. A morphism $\phi \colon Y \to X$ is said to be generically separable if X can be covered by affine open subsets $U = \mathrm{Spec}(R)$ such that the ring extension $R \subset \mathcal{O}(\Phi^{-1}(U))$ is generically separable.

For an affine variety X over a field k, $k[X]$ will denote the coordinate ring of X and $k(X)$ will denote its function field. For a scheme X and a point $x \in X$, let $\hat{\mathcal{K}}_{X,x}$ denote the fraction field of complete local ring $\hat{\mathcal{O}}_{X,x}$ whenever the latter is a domain. For domains $A \subset B$, $\overline{A}^B$ will denote the integral closure of A in B.

As in [Ha2], a morphism of schemes, $\Phi \colon Y \to X$, is said to be a *cover* if Φ is finite, surjective and generically separable. For a finite group G, Φ is said to be a G-*cover* (or a G-Galois cover) if in addition there exists a group homomorphism $G \to \mathrm{Aut}_X(Y)$ which acts transitively on the geometric generic fibers of Φ.

For a scheme X, let $\mathcal{M}(X)$ denote the category of coherent sheaves of $\mathcal{O}_X$-modules, $\mathcal{AM}(X)$ denote the category of coherent sheaves of $\mathcal{O}_X$-algebras and $\mathcal{SM}(X)$ denote the subcategory of $\mathcal{AM}(X)$ for which the sheaves of algebras are generically separable and locally free. For a finite group G, let $G\mathcal{M}(X)$ denote the category of generically separable coherent locally free sheaves of $\mathcal{O}_X$-algebras S together with a G-action which is transitive on the geometric generic fibers of $\mathrm{Spec}_{\mathcal{O}_X}(S) \to X$. For a ring R, we may use $\mathcal{M}(R)$ instead of $\mathcal{M}(\mathrm{Spec}(R))$, etc. Given categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ and functors $\mathcal{F} \colon \mathcal{A} \to \mathcal{C}$ and $\mathcal{G} \colon \mathcal{B} \to \mathcal{C}$, $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ will denote the fiber product category. The objects of this category are triples (A, B, C), where A, B and C are objects of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ respectively together with isomorphisms of C with $\mathcal{F}(A)$ and with $\mathcal{G}(B)$ in $\mathcal{C}$. The morphisms are triples (a, b, c), where a, b and c are morphisms in $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ respectively, so that $\mathcal{F}(a)$ and $\mathcal{G}(b)$ under the functors $\mathcal{F}$ and $\mathcal{G}$ are morphism in $\mathcal{C}$ which agrees with c in the natural way. That is, the following two squares commute.

$$\begin{array}{ccc} C & \xrightarrow{c} & C' \\ \downarrow & & \downarrow \\ \mathcal{F}(A) & \xrightarrow[\mathcal{F}(a)]{} & \mathcal{F}(A') \end{array} \qquad\qquad \begin{array}{ccc} C & \xrightarrow{c} & C' \\ \downarrow & & \downarrow \\ \mathcal{G}(B) & \xrightarrow[\mathcal{G}(b)]{} & \mathcal{G}(B') \end{array}$$

For a connected scheme X, let $Cov(X)$ denote the category whose objects are finite étale covers of X, i.e., finite surjective étale morphisms $f \colon Y \to X$. Morphisms between two objects $f \colon Y \to X$ and $g \colon Z \to X$ of this category are morphisms of

X-schemes. A geometric point x of X is a morphism $\mathrm{Spec}(K) \to X$, with K a separably algebraically closed field containing the residue field of the point $x_0 \in X$, where x_0 is the image of the given morphism $\mathrm{Spec}(K) \to X$. Note that x_0 need not be a closed point.

For each geometric point b of X we shall define a functor F_b from the category $Cov(X)$ to the category of sets (denoted by $Sets$). This functor sends an object $f\colon Y \to X$ of $Cov(X)$ to the set $\mathrm{Hom}_X(b, Y)$, the set of X-morphisms from b to Y. Moreover, a morphism $h\colon Y \to Z$ in $Cov(X)$ is sent to the set map $F_b(h)$ which sends $\alpha \in \mathrm{Hom}_X(b, Y)$ to $h \circ \alpha \in \mathrm{Hom}_X(b, Z)$.

Two functors F and G from $Cov(X)$ to $Sets$ are said to be isomorphic if for every object $f\colon Y \to X$ there is a bijection $I_Y\colon F(Y) \to G(Y)$ and these bijections are compatible with the morphisms in the two categories. In terms of commutative diagram this means that for any morphism $h\colon Y \to Z$ in the category $Cov(X)$ we have the following commutative diagram:

$$\begin{tikzcd} F(Y) \arrow[r, "I_Y"] \arrow[d, "F(h)"'] & G(Y) \arrow[d, "G(h)"] \\ F(Z) \arrow[r, "I_Z"'] & G(Z) \end{tikzcd}$$

I is said to be an isomorphism from the functor F to G. The *étale fundamental group* (or the *algebraic fundamental group*) of X with a base point b, a geometric point of X, is the group of automorphisms of the fiber functor F_b defined above. This group is denoted by $\pi_1(X, b)$.

Note that for each object Y of $Cov(X)$, $\pi_1(X, b)$ induces a group of automorphisms of $F_b(Y)$. Moreover these groups form an inverse system of groups and $\pi_1(X, b)$ is the inverse limit of this inverse system. Hence it is a profinite group. As in the classical (topological) fundamental group case, here as well $\pi_1(X, b)$ and $\pi_1(X, c)$ are (non canonically) isomorphic for any two geometric points b and c of X.

If X is a smooth variety over $\mathbb{C}$ (or any algebraically closed field of characteristic zero), then the étale fundamental group of X, $\pi_1(X, b)$, for a closed point $b \in X$ is isomorphic to the profinite completion of the topological fundamental group $\pi_1^{top}(X, b)$. This is a consequence of the famous (Grothendieck) Riemann Existence Theorem. Let k be an algebraically closed field. It is easy to see from the Riemann-Hurwitz formula that $\pi_1(\mathbb{P}^1_k, b)$ is trivial and that if the characteristic of k is 0 then $\pi_1(\mathbb{A}^1_k, b)$ is also trivial.

In the subsequent sections when we deal with an integral scheme X , we shall drop the base point from the notation of the fundamental group. In those situations, the base point b is assumed to be the generic geometric point of X corresponding to the ring monomorphism $k(X) \hookrightarrow \overline{k(X)}^s$, where the function field $k(X)$ is viewed as the residue field of the generic point and $\overline{k(X)}^s$ is the separable closure of $k(X)$. Hence, if X is integral $\pi_1(X) := \pi_1(X, b)$ where b is the above generic geometric point of X. With this notation, we see that $\pi_1(X) = \varprojlim \mathrm{Aut}(k(Y)/k(X))$ where Y varies over finite étale covers of X and $\mathrm{Aut}(k(Y)/k(X))$ is the group of field automorphisms of $k(Y)$ fixing $k(X)$. Moreover, the Galois étale covers of X form a cofinal system in this inverse system, hence $\pi_1(X) = \varprojlim \mathrm{Gal}(k(Y)/k(X))$ where

Y varies over finite Galois étale covers of X. This Galois theoretic aspect of $\pi_1(X)$ has been used throughout in this thesis.

3. Abelianization

In this section, the theory of Witt vectors and their intimate relationship with abelian p-group field extensions are used to understand the abelianizations of the fundamental groups of affine varieties. The field extensions defined by Witt vectors can be thought of as generalizations of Artin-Schrier field extensions.

Let $X = \mathrm{Spec}(A)$ be a normal affine algebraic variety over an algebraically closed field k of characteristic $p > 0$. Let $G = \pi_1^{ab}(X)$ and let $G^{(p)}$ be the maximal p-quotient of G, i.e., $G^{(p)}$ is the quotient group of G such that every finite p-group quotient of G factors through $G^{(p)}$. Note that $G^{(p)}$ is also a subgroup of G. Let $G_W = \mathrm{Hom}(\varinjlim W_n(A)/P(W_n(A)), S^1)$, where the group homomorphism from $W_n(A)/P(W_n(A))$ to $W_{n+1}(A)/P(W_{n+1}(A))$ is given by sending $[(a_1, \cdots, a_n)]$ to $[(0, a_1, \cdots, a_n)]$. Here S^1 is the unit circle in the complex plane viewed as a topological group. The main result of this section is the following:

Theorem 3.1. *$G^{(p)}$, the p-part of the abelianization of the fundamental group of X, is isomorphic to G_W.*

The proof requires a few technical lemmas which we shall see now. For a sheaf of rings $\mathcal{F}$ of characteristic p on a topological space X, let $W_n(\mathcal{F})$ denote the sheaf which assigns to an open set U, the ring $W_n(\mathcal{F}(U))$. A version of the following lemma can be found in [Se1].

Lemma 3.2. (Serre) *Let A be a noetherian ring of characteristic p. Let B be a ring extension of A and a finite A-module. Then for every $n \geq 1$ and $k > 0$, $H^k_{et}(\mathrm{Spec}(A), W_n(\mathcal{B})) = 0$, where $\mathcal{B} = \theta_* \mathcal{O}_{\mathrm{Spec}(B)}$ and $\theta \colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is the morphism induced from $A \hookrightarrow B$.*

Proof. Since B is a finite module over A, $\mathcal{B}$ is a coherent sheaf over $\mathrm{Spec}(A)$. We shall use induction on n to prove the lemma. Note that $W_1(\mathcal{B}) = \mathcal{B}$. By Serre's Vanishing Theorem and the fact that the étale cohomology of coherent sheaves agrees with the Zariski cohomology (see [Mil, 3.7, 3.8, page 114]), the lemma holds for $n = 1$. For the induction step, consider the following exact sequence.

$$0 \longrightarrow \mathcal{B} \longrightarrow W_{n+1}(\mathcal{B}) \longrightarrow W_n(\mathcal{B}) \longrightarrow 0$$

Here, for a fixed open set U, the surjection (on the level of rings) is given by sending $(b_1, \cdots, b_n, b_{n+1})$ to $(b_1, \cdots, b_n)$, and clearly the kernel is the subgroup $\{(0, \cdots, 0, b) \in W_{n+1}(\mathcal{B}(U)) : b \in \mathcal{B}(U)\}$ which is isomorphic $\mathcal{B}(U)$ as a group. This induces a long exact sequence

$$\cdots \to H^k_{et}(\mathrm{Spec}(A), \mathcal{B}) \to H^k_{et}(\mathrm{Spec}(A), W_{n+1}(\mathcal{B})) \to H^k_{et}(\mathrm{Spec}(A), W_n(\mathcal{B})) \to \cdots$$

By the induction hypothesis on n, $H^k_{et}(\mathrm{Spec}(A), W_n(\mathcal{B})) = 0$ for $k > 0$ and hence $H^k_{et}(\mathrm{Spec}(A), W_{n+1}(\mathcal{B})) = 0$, for all $k > 0$. □

Extending the Artin-Schrier theory to Witt vectors we get the following result.

Lemma 3.3. *Let A be a finitely generated normal domain over an algebraically closed field k of characteristic $p > 0$. Let $\pi_1(X)$ be the fundamental group of $X = \mathrm{Spec}(A)$ and $(W_n(A), +, .)$ the ring of Witt vectors of length n. Let P be the additive group endomorphism $F - Id$ of $W_n(A)$. Then for every $n \geq 1$, we have* *a natural isomorphism* $W_n(A)/P(W_n(A)) \xrightarrow{\Phi} \mathrm{Hom}(\pi_1(X), W_n(\mathbb{Z}/p\mathbb{Z}))$ *so that the following diagram commutes.*

$$\begin{tikzcd} W_n(A)/P(W_n(A)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\#"] & \mathrm{Hom}(\pi_1(X), W_n(\mathbb{Z}/p\mathbb{Z})) \arrow[d] \\ W_{n+1}(A)/P(W_{n+1}(A)) \arrow[r] & \mathrm{Hom}(\pi_1(X), W_{n+1}(\mathbb{Z}/p\mathbb{Z})) \end{tikzcd}$$

Here the first vertical map sends $[(a_1, \cdots, a_n)]$ *to* $[(0, a_1, \cdots, a_n)]$ *and the second vertical map is induced by the inclusion of* $W_n(\mathbb{Z}/p\mathbb{Z})$ *in* $W_{n+1}(\mathbb{Z}/p\mathbb{Z})$*.*

Proof. Let K be an algebraic closure of the fraction field $\mathrm{frac}(A)$ of A. Let K^{un} be the compositum of all subfields L of K with the property that $L/\mathrm{frac}(A)$ is a finite field extension and the integral closure of A in L, $\overline{A}^L$, is an étale ring extension of A. We shall define a map ϕ from $W_n(A) \to \mathrm{Hom}(\pi_1(X), W_n(\mathbb{Z}/p\mathbb{Z}))$ as follows. Given $(a_1, \cdots, a_n) \in W_n(A)$, let $(r_1, \cdots, r_n) \in W_n(K)$ be such that $P(r_1, \cdots, r_n) = (a_1, \cdots, a_n)$. Note that $r_1, \cdots, r_n \in K^{un}$; to see this first observe that $r_i^p - r_i \in A[r_1, \cdots, r_{i-1}]$ for each $i \geq 1$. We know that the ring extension given by a polynomial of the form $Z^p - Z - a$, for $a \in A$ is an unramified extension. $\pi_1(X) = \mathrm{Gal}(K^{un}/\mathrm{frac}(A))$, so $\pi_1(X)$ acts on K^{un} fixing $\mathrm{frac}(A)$. For $g \in \pi_1(X)$, let $\phi(a_1, \cdots, a_n)(g) = (gr_1, \cdots, gr_n) - (r_1, \cdots, r_n)$. Note that

$$\begin{aligned} P((gr_1, \cdots, gr_n) - (r_1, \cdots, r_n)) &= P((gr_1, \cdots, gr_n)) - P(r_1, \cdots, r_n) \\ &= gP(r_1, \cdots, r_n) - (a_1, \cdots, a_n) = 0. \end{aligned}$$

The second equality holds because $(r_1^p, \cdots, r_n^p) - (r_1, \cdots, r_n)$ is given by a polynomial in $r_1, \cdots, r_n$ with integer coefficients. Hence $F((gr_1, \cdots, gr_n) - (r_1, \cdots, r_n)) = (gr_1, \cdots, gr_n) - (r_1, \cdots, r_n)$, which yields $(gr_1, \cdots, gr_n) - (r_1, \cdots, r_n) \in W_n(\mathbb{Z}/p\mathbb{Z})$. To see that $\phi(a_1, \cdots, a_n)$ is independent of the choice of $(r_1, \cdots, r_n)$, let $(s_1, \cdots, s_n)$ be such that $P(s_1, \cdots, s_n) = (a_1, \cdots, a_n)$. Then the difference $(r_1, \cdots, r_n) - (s_1, \cdots, s_n) \in W_n(\mathbb{Z}/p\mathbb{Z})$, hence is fixed by g. So $g((r_1, \cdots, r_n) - (s_1, \cdots, s_n)) = (r_1, \cdots, r_n) - (s_1, \cdots, s_n)$ which yields $(gr_1, \cdots, gr_n) - (r_1, \cdots, r_n)$ is same as $(gs_1, \cdots, gs_n) - (s_1, \cdots, s_n)$. Next we shall see that $\phi(a_1, \cdots, a_n)$ is a homomorphism from $\pi_1(X)$ to $W_n(\mathbb{Z}/p\mathbb{Z})$. Let $g, h \in \pi_1(X)$ then

$$\begin{aligned} \phi(a_1, \cdots, a_n)(gh) &= (ghr_1, \cdots, ghr_n) - (r_1, \cdots, r_n) \\ &= (ghr_1, \cdots, ghr_n) - (hr_1, \cdots, hr_n) + (hr_1, \cdots, hr_n) \\ &\quad - (r_1, \cdots, r_n) \\ &= \phi(a_1, \cdots, a_n)(g) + \phi(a_1, \cdots, a_n)(h) \end{aligned}$$

since $P(hr_1, \cdots, hr_n) = (a_1, \cdots, a_n)$. Now we shall see that ϕ is a homomorphism. To simplify notation, $\underline{a}$ may be used for $(a_1, \cdots, a_n)$. Let $\underline{a}, \underline{b} \in W_n(A)$ and $\underline{r}, \underline{s} \in W_n(K)$ be such that $P(\underline{r}) = \underline{a}, P(\underline{s}) = \underline{b}$ then $P(\underline{r} + \underline{s}) = \underline{a} + \underline{b}$. Hence $\phi(\underline{a} + \underline{b}) = \phi(\underline{a}) + \phi(\underline{b})$. To determine the kernel of ϕ, note that $\phi(\underline{a}) = 0$ iff $g\underline{r} = \underline{r}$ for all $g \in G$, i.e. $\underline{r} \in W_n(\mathrm{frac}(A))$. But A is normal, hence this happens iff $\underline{r} \in W_n(A)$. Hence the kernel of ϕ is $P(W_n(A))$. Let Φ be the induced map on $W_n(A)/P(W_n(A))$. The fact that the diagram commutes is obvious by the construction. So to complete the proof it suffices to show that ϕ is surjective.

Let $\alpha \in \mathrm{Hom}(\pi_1(X), W_n(\mathbb{Z}/p\mathbb{Z}))$, we shall find a Witt vector $(a_1, \cdots, a_n)$ so that $\alpha = \phi(a_1, \cdots, a_n)$. Note that α corresponds to a Galois étale extension B of A with the Galois group of frac(B) over frac(A) being im$(\alpha)(= H$ say). Let $SW_n(B) = \{(r_1, \cdots, r_n) \in W_n(B) : P(r_1, \cdots, r_n) \in W_n(A)\}$. Clearly $W_n(A) \hookrightarrow SW_n(B)$. Let $\hat{H} = \mathrm{Hom}(H, S^1)$ be the character group of H. For $\underline{r} \in SW_n(B)$ and $h \in H$ define $\chi_{\underline{r}}(h) = h\underline{r} - \underline{r} := g\underline{r} - \underline{r}$ where g is any element of $\alpha^{-1}(h)$. As noted earlier $\chi_{\underline{r}}$ is a character (after identifying $W_n(\mathbb{Z}/p\mathbb{Z})$ with the unique cyclic subgroup of S^1). We know that $\hat{H} \cong H$. Also as seen earlier $\lambda \colon \underline{r} \mapsto \chi_{\underline{r}}$ is a group homomorphism from $SW_n(B)$ to $\hat{H}$ whose kernel is precisely $W_n(A)$. So we have an exact sequence

$$0 \longrightarrow W_n(A) \longrightarrow SW_n(B) \longrightarrow \hat{H}$$

where the last homomorphism is λ. Next we shall show that λ is surjective. Since $\hat{H} \cong H$ is a quotient of $\pi_1(X)$, $H^1(\hat{H}, W_n(B)) \hookrightarrow H^1(\pi_1(X), W_n(B))$ ([Wei, 6.8.3]). By the Hochschild-Serre spectral sequence, $H^1(\pi_1(X), H^0_{et}(X, W_n(\mathcal{B})))$ embeds into $H^1_{et}(X, W_n(\mathcal{B}))$ (see [Mil, 2.21(b), page 106]). Also $H^0_{et}(X, W_n(\mathcal{B}))$ is simply $W_n(B)$ and by Lemma 3.2, $H^1_{et}(X, W_n(\mathcal{B})) = 0$. So $H^1(\hat{H}, W_n(B)) = 0$, i.e., every cocycle is a coboundary. Viewing $\hat{H}$ as $\mathrm{Hom}(H, W_n(\mathbb{Z}/p\mathbb{Z})$, if $\chi \in \hat{H}$ then $\chi(hh') = \chi(h) + \chi(h') = h\chi(h') + \chi(h)$, hence χ is a cocycle and therefore a coboundary. So there exists an $\underline{r} \in W_n(B)$ such that $\chi(h) = h\underline{r} - \underline{r}, \forall h \in H$. Since $\chi(h) \in W_n(\mathbb{Z}/p\mathbb{Z})$, $P(\chi(h)) = 0$, i.e., $hP(\underline{r}) = P(\underline{r}), \forall h \in H$. Since A is normal and B is integral over A, $P(\underline{r}) \in W_n(A)$, hence $\underline{r} \in SW_n(B)$. This proves the surjectivity of λ. So we have, $SW_n(B)/W_n(A) \cong \hat{H} \cong H$. Since H is a subgroup of $W_n(\mathbb{Z}/p\mathbb{Z})$, H has a generator of the type $h = (0, \cdots 0, 1, 0, \cdots, 0)$. Let the coset $(r_1, \cdots, r_n) + W_n(A)$ be a generator of $SW_n(B)/W_n(A)$. It follows that $\chi_{\underline{r}}$ is a generator of $\hat{H}$ and hence $h_1 := \chi_{\underline{r}}(h)$ is also a generator of H. So there is an $h_2 \in W_n(\mathbb{Z}/p\mathbb{Z})$ such that $h_1 \cdot h_2 = h$ (Witt product). Let $g \in \alpha^{-1}(h)$ then $g\underline{r} - \underline{r} = \chi_{\underline{r}}(h) = h_1$. Let $(a_1, \cdots, a_n) = P(h_2 \cdot \underline{r})$. This Witt vector is our candidate for preimage of α, we shall show $\alpha = \phi(a_1, \cdots, a_n)$. By assumption $\alpha(g) = h$ and

$$\begin{aligned}
\phi(a_1, \cdots, a_n)(g) &= g(h_2 \cdot \underline{r}) - h_2 \cdot \underline{r} \\
&= gh_2 \cdot g\underline{r} - h_2 \cdot \underline{r} \\
&= h_2 \cdot g\underline{r} - h_2.\underline{r} \ (h_2 \in W_n(A)) \\
&= h_2 \cdot (g\underline{r} - \underline{r}) \\
&= h_2 \cdot h_1 = h
\end{aligned}$$

For arbitrary $g_1 \in G$, since H is cyclic $\alpha(g_1) = h + \cdots + h$, l times, for some l. Then

$$\begin{aligned}
\phi(a_1, \cdots, a_n)(g_1) &= h_2 \cdot (g_1\underline{r} - \underline{r}) \\
&= h_2 \cdot (\chi_{\underline{r}}(h + \cdots + h)) \\
&= h_2 \cdot (h_1 + \cdots + h_1) = h + \cdots + h
\end{aligned}$$

So $\phi(a_1, \cdots, a_n)$ agrees with α on whole of $\pi_1(X)$. □

Now we are ready to prove Theorem 3.1.

Proof. **(Theorem 3.1)** We know that $G^{(p)} \cong \mathrm{Hom}(\mathrm{Hom}_{cont}(G^{(p)}, S^1), S^1)$ by Pontriagin duality. Let K_n be the compositum of all the fraction fields frac(B) where each $A \hookrightarrow B$ is a finite Galois étale extension with Galois group a subgroup of $(\mathbb{Z}/p^n\mathbb{Z})^m$ for some m. And let G_n be the Galois group of K_n over frac(A). The natural group homomorphism from G_{n+1} to G_n corresponding to the Galois extension $K_{n+1} \supset K_n \supset \mathrm{frac}(A)$ makes $(G_n)_{n\geq 1}$ into an inverse system and $G^{(p)} = \varprojlim G_n$. So we have $\mathrm{Hom}_{cont}(G^{(p)}, S^1) \cong \mathrm{Hom}_{cont}(\varprojlim G_n, S^1)$. Since $\mathrm{Hom}(-, S^1)$ is a contravariant functor and the dual of inverse limit is direct limit, $\mathrm{Hom}_{cont}(G^{(p)}, S^1) \cong \varinjlim \underset{cont}{\mathrm{Hom}}(G_n, S^1)$. Since G_n is a p^n torsion group and $W_n(\mathbb{Z}/p\mathbb{Z})$ and can be identified with the unique cyclic subgroup of S^1 of order p^n, $\mathrm{Hom}_{cont}(G_n, S^1) \cong \mathrm{Hom}_{cont}(G_n, W_n(\mathbb{Z}/p\mathbb{Z}))$. Also $\mathrm{Hom}_{cont}(G_n, W_n(\mathbb{Z}/p\mathbb{Z})) \cong \mathrm{Hom}_{cont}(G^{(p)}, W_n(\mathbb{Z}/p\mathbb{Z}))$, since all the homomorphisms from $G^{(p)}$ to $W_n(\mathbb{Z}/p\mathbb{Z})$ factor through G_n. Similarly, all the homomorphisms from $\pi_1(X)$ to $W_n(\mathbb{Z}/p\mathbb{Z})$ have to factor through $G^{(p)}$. Therefore, $\mathrm{Hom}_{cont}(G^{(p)}, W_n(\mathbb{Z}/p\mathbb{Z}))$ is isomorphic to $\mathrm{Hom}_{cont}(\pi_1(X), W_n(\mathbb{Z}/p\mathbb{Z}))$. But, by Lemma 3.3 $\varinjlim \mathrm{Hom}_{cont}(\pi_1(X), W_n(\mathbb{Z}/p\mathbb{Z}))$ is isomorphic to $\varinjlim W_n(A)/P(W_n(A))$. □

The following result is a corollary of a classical result of Grothendieck [SGAI, XIII, Corollary 2.12, page 392].

Theorem 3.4. (Grothendieck) *The prime to p part of the abelianization of the fundamental group of an affine curve $C = \mathrm{Spec}(A)$ over an algebraically closed field k of characteristic $p > 0$ is given by $\bigoplus_{i=1}^{2g+r-1} (\prod_{l\neq p\ prime} \mathbb{Z}_l)$ where g is the genus of the smooth compactification curve and r is the number of points in the compactification which are not in C.*

Corollary 3.5. *Under the assumption of the previous theorem, abelianization of the fundamental group of C, $\pi_1^{ab}(C)$, is given by*

$$\mathrm{Hom}(\varinjlim W_n(A)/P(W_n(A)), S^1) \bigoplus \bigoplus_{i=1}^{2g+r-1} (\prod_{l\neq p\ prime} \mathbb{Z}_l)$$

Proof. This follows directly from Theorem 3.1 and Theorem 3.4. □

Remark 3.6. *Since the rank of $\pi_1^{ab}(C)$ is the same as the cardinality of k, as a consequence of Lemma 3.3, we obtain another proof of a known result that $\pi_1^{ab}(C)$ determines the cardinality of the base field. In fact just the p-part of $\pi_1^{ab}(C)$ determines $W_n(A)/P(W_n(A))$ for all n.*

4. Embedding problems

4.1. **Group theory and embedding problems.** In this subsection we will see some well known group theoretic results. Some of these results connects the "freeness" of a profinite group to solving certain embedding problems.

Theorem 4.1. (Iwasawa [Iwa, page 567], [FJ, Corollary 24.2]) *A profinite group π of countably infinite rank is free if and only if every finite embedding problem for π has a proper solution.*

This was generalized by Melnikov and Chatzidakis for any cardinality (cf [Jar, Theorem 2.1]). The Melnikov-Chatzidakis result says that for an infinite cardinal m, a profinite group π is free of rank m if and only if every finite nontrivial embedding problem for π has exactly m solutions. The following is a variant of this result which has been proved in [HS] and is useful if we know that π is a projective group.

Theorem 4.2. **([HS, Theorem 2.1])** *Let π be a profinite group and let m be an infinite cardinal. Then π is a free profinite group of rank m if and only if the following conditions are satisfied:*
(i) π is projective.
(ii) Every split embedding problem for π has exactly m solutions.

If π is a projective profinite group then there is a standard argument which allows us to reduce the problem of finding proper solutions to an embedding problem for π to finding proper solutions to a *split* embedding problem with the same kernel. To see this let us consider the following embedding problem:

$$\begin{tikzcd} & & & \pi \arrow[dl, dashed, "\theta"'] \arrow[d, "\phi"] & \\ 1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r, "\alpha"'] & G \arrow[r] \arrow[d] & 1 \\ & & & 1 & \end{tikzcd}$$

Since π is projective there exists a weak solution θ to this embedding problem. Let $G' = \operatorname{Im}\theta$. G' acts on H by conjugation, since H is a normal subgroup of Γ. Let $\Gamma' = H \rtimes G'$ so that we have a natural surjection $\beta\colon \Gamma' \to \Gamma$ given by $(h, g) \mapsto hg$. So if we have a proper solution θ' for the split embedding problem,

$$\begin{tikzcd} & & & \pi \arrow[dl, dashed, "\theta'"'] \arrow[d, "\tilde{\psi}"] & \\ 1 \arrow[r] & H \arrow[r] & \Gamma' \arrow[r] & G' \arrow[r] \arrow[d] & 1 \\ & & & 1 & \end{tikzcd}$$

then $\psi = \beta \circ \theta'$ provides a proper solution to the original embedding problem. This trick is used again below to reduce the problem further to certain special cases.

Theorem 4.3. *Let π be any projective profinite group of rank at most m. Then π is free of rank m if and only if for any finite group Γ and any minimal normal subgroup H of Γ, the split embedding problem*

$$\begin{tikzcd} & & & \pi \arrow[dl, dashed, "\psi"'] \arrow[d, "\phi"] & \\ 1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r, "\alpha"'] & G \arrow[r] \arrow[d] & 1 \\ & & & 1 & \end{tikzcd}$$

has m distinct solutions (and at least one solution if m is the countable cardinal) in the following three cases:
(1) *H is a quasi-p perfect group, i.e. $H = [H, H]$.*
(2) *H is an abelian p-group.*
(3) *H is a prime-to-p group.*

Proof. In view of Theorem 4.2 (and Theorem 4.1 if m is the countable cardinal), the “only if part” is trivial and for the “if part” it is enough to show that the split embedding problem for π has m distinct proper solutions (and at least one solution if m is a countable cardinal) for any finite group H, this will, in particular, force the rank of π to be m. We induct on the cardinality of H. Suppose H is not a minimal normal subgroup. Let H_1 be a proper nontrivial subgroup of H and a normal subgroup of Γ. Then we have the following two proper nontrivial embeddding problems.

$$\begin{tikzcd} & & & \pi \arrow[dl, dashed] \arrow[d] & \\ 1 \arrow[r] & H/H_1 \arrow[r] & \Gamma/H_1 \arrow[r] & G \arrow[r] \arrow[d] & 1 \\ & & & 1 & \end{tikzcd}$$

and

$$\begin{tikzcd} & & & \pi \arrow[dl, dashed] \arrow[d] & \\ 1 \arrow[r] & H_1 \arrow[r] & \Gamma \arrow[r] & \Gamma/H_1 \arrow[r] \arrow[d] & 1 \\ & & & 1 & \end{tikzcd}$$

The cardinalities of H_1 and H/H_1 are strictly smaller than the cardinality of H. By induction hypothesis, after replacing the above two embedding problems by the corresponding split ones if necessary, we obtain m distinct proper solutions to both of them (respectively at least one in countable case). Hence we have m distinct proper solutions (respectively at least one) to the original embedding problem. Therefore we may assume H is a nontrivial minimal normal subgroup of Γ. So $H \cong \mathbb{S} \times .. \times \mathbb{S}$ for some finite simple group $\mathbb{S}$ **provide ref**. If $\mathbb{S}$ is prime-to-p then H is prime-to-p, hence we are done by case (3). If $\mathbb{S}$ is quasi-p nonabelian group then H being the product of perfect groups is perfect. So we are done by case (1). And finally if $\mathbb{S}$ a is quasi-p abelian then $\mathbb{S} \cong \mathbb{Z}/p\mathbb{Z}$. Hence H is an abelian p-group and we are done by case (2). □

We will need the following group theory result later.

Lemma 4.4. *Given any finite abelian p-group A there exists a finite p-group B such that the commutator $[B, B]$ of B is isomorphic to A.*

Proof. Since A is an abelian p-group, A is a direct sum of cyclic p-groups. Observe that the commutator of the group $B_1 \times B_2$ is isomorphic to $[B_1, B_1] \times [B_2, B_2]$ for any two groups B_1 and B_2. So we may assume A is a cyclic p-group, say $\mathbb{Z}/p^m\mathbb{Z}$. Consider the Heisenberg group over $\mathbb{Z}/p^m\mathbb{Z}$, i.e., the group of 3×3 upper triangular matrices with diagonal entries 1. It is a group of order p^{3m} generated by the matrices

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and one could easily check that the commutator of this group is the subgroup generated by

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is clearly isomorphic to $(\mathbb{Z}/p^m\mathbb{Z}, +)$. □

The construction of such a group using Heisenberg matrices was pointed out to me by a friend Sandeep Varma and also by Prof. Donu Arapura.

4.2. **Reduction to solving embedding problems.** In this subsection we shall use the results above to reduce the main theorem (Theorem 4.8) to solve some special kinds of embedding problems.

Let k be an algebraically closed field of characteristic $p > 0$. Let $\pi_1(C)$ be the algebraic fundamental group of a smooth affine curve C over k and $\pi_1^c(C) = [\pi_1(C), \pi_1(C)]$ be the commutator subgroup. Assume that $\mathrm{card}(k) = m$.

Lemma 4.5. *With the above notation the rank of $\pi_1^c(C)$ is at most m.*

Proof. Let C be a smooth affine curve over an algebraically closed field k of cardinality m. Since $k(C)$, the function field of C, is also of cardinality m, there are only m polynomials over $k(C)$. Hence the absolute Galois group of $k(C)$ is the inverse limit of finite groups over a set of cardinality m and hence has generating set of cardinality m (generating set in the topological sense). So $\pi_1(C)$, being a quotient of the absolute Galois group of $k(C)$, is m-generated and hence the commutator subgroup $\pi_1^c(C)$ is m-generated. Hence $\pi_1^c(C)$ has rank at most m. □

Proposition 4.6. *For an irreducible smooth affine curve C over k, the commutator subgroup $\pi_1^c(C)$ of the fundamental group $\pi_1(C)$ is a projective group. More explicitly, given*

$$\begin{tikzcd} & \pi_1^c(C) \arrow[dl, dashed, "\exists\psi"'] \arrow[d, two heads, "\phi"] \\ \Gamma \arrow[r, two heads, "\alpha"'] & G \end{tikzcd}$$

with surjections ϕ and α to a finite group G from $\pi_1^c(C)$ and another finite group Γ respectively, there exists a group homomorphism ψ from $\pi_1^c(C)$ to Γ so that the above diagram commutes, i.e., $\alpha \circ \psi = \phi$

Proof. Let K^{ab} be the compositum of the function fields of abelian étale covers of C, i.e., the compositum of all L, $k(C) \subset L \subset \overline{k(C)}$ with $L/k(C)$ finite, the integral closure $\overline{k[C]}^L$ of $k[C]$ in L an étale extension of $k[C]$, and $\mathrm{Gal}(L/k(C))$ an abelian group.

A surjection $\phi\colon \pi_1^c(C) \to G$ corresponds to a Galois field extension M/K^{ab} with the Galois group $\mathrm{Gal}(M/K^{ab}) = G$ and $M \subset K^{un}$, where K^{un} is the compositum of the function fields of all the étale covers of C.

Since M/K^{ab} is a finite field extension, there exist, a finite Galois extension L of $k(C)$ with $k(C) \subset L \subset K^{ab}$, a Galois extension L' of L with $\mathrm{Gal}(L'/L) = G$, and $L'K^{ab} = M$. Let $\pi_1^L = \pi_1(\mathrm{Spec}(\overline{k[C]}^L))$. So we have the following tower of fields.

Moreover $\mathrm{Gal}(K^{un}/K^{ab}) = \pi_1^c(C)$, $\mathrm{Gal}(K^{un}/L) = \pi_1^L$ and $\pi_1^c(C)$ is a subgroup of π_1^L. The field extension L'/L gives a surjection $\tilde{\phi}\colon \pi_1^L \to G$. Since L'/L is a

$$\begin{tikzcd}
K^{un} & M = L'K^{ab} \arrow[l] \\
K^{ab} \arrow[u, "\pi_1^c(C)"] \arrow[ur, "G"'] & L' \arrow[u] \\
L \arrow[u] \arrow[ur, "G"'] \arrow[uu, bend left=60, "\pi_1^L"] & \\
k(C) \arrow[u] &
\end{tikzcd}$$

FIGURE 1

descent of the field extension M/K^{ab}, $\tilde{\phi}|_{\pi_1^c(C)} = \phi$. By [Se2, Proposition 1] and [Se3, I, 5.9, Proposition 45], which says that the fundamental group of any affine curve is projective, we have $\pi_1^L :\ = \pi_1(\mathrm{Spec}(\overline{k[C]}^L))$ is projective. So there exists a lift, $\tilde{\psi}$, to Γ of $\tilde{\phi}$. i.e.,

$$\begin{tikzcd}
 & \pi_1^L \arrow[dl, dashed, "\exists\tilde{\psi}"'] \arrow[d, "\tilde{\phi}"] \\
\Gamma \arrow[r, two heads, "\alpha"'] & G
\end{tikzcd}$$

with $\alpha \circ \tilde{\psi} = \tilde{\phi}$. So $\alpha \circ \tilde{\psi}|_{\pi_1^c(C)} = \tilde{\phi}|_{\pi_1^c(C)} = \phi$. So $\tilde{\psi}|_{\pi_1^c(C)}$ gives a lift of ϕ. $\square$

Proposition 4.7. *Consider the following split embedding problem for $\pi_1^c(C)$*

$$\begin{tikzcd}
 & & & \pi_1^c(C) \arrow[dl, dashed, "\psi"'] \arrow[d, "\phi"] & \\
1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r, "\alpha"'] & G \arrow[r] \arrow[d] & 1 \\
 & & & 1 &
\end{tikzcd}$$

Suppose this problem has m distinct solutions (and at least one solution if m is the countable cardinal) in the following three cases:
(1) H is a quasi-p perfect group, i.e. $H = [H, H]$.
(2) H is an abelian p-group.
(3) H is a prime-to-p group.
Then $\pi_1^c(C)$ is free of rank m.

Proof. In view of Theorem 4.3, this follows directly from Proposition 4.6 and Lemma 4.5. $\square$

In Section 5, it is shown that the embedding problem of the above theorem has a solution in cases (1) and (2) (Theorem 5.3, Theorem 5.5). This follows very easily from some well known results. In Section 6, a solution to the embedding problem with prime-to-p kernel has been exhibited (Theorem 6.12). Using these results and Proposition 4.7 we get the main theorem:

Theorem 4.8. *If m is the countable cardinal then $\pi_1^c(C)$ is free of countable rank.*

5. Quasi-p embedding problems

In this section it will be shown that every split embedding problem for $\pi_1^c(C)$ has a solution if H is a perfect quasi-p group or H is a p-group. This is an easy consequence of the following result of Florian Pop on quasi-p embedding problems.

Theorem 5.1. *(**Florian Pop, [Pop], [Ha3, Theorem 5.3.4]**) Let k be an algebraically closed field of characteristic $p > 0$, $\mathrm{card}(k) = m$, and let C be an an irreducible affine smooth curve over k. Then every quasi-p embedding problem for $\pi_1(C)$ has m distinct proper solutions.*

Theorem 5.2. *(**[Ha5, Theorem 1b]**) Let π be a profinite group so that $H^1(\pi, P)$ is infinite for every finite elementary abelian p-group P with continuous π-action. Then every p-embedding problem for π has a proper solution if and only if every p-embedding problem has a weak solution (equivalently, p cohomological dimension of π, $\mathrm{cd}_p(\pi) \leq 1$).*

Theorem 5.3. *The following split embedding problem has $\mathrm{card}(k) = m$ proper solutions*

$$\begin{tikzcd} & & & \pi_1^c(C) \arrow[d] \arrow[dl, dashed] & \\ 1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r] & G \arrow[r] \arrow[d] & 1 \\ & & & 1 & \end{tikzcd}$$

Here H is a quasi-p perfect group (i.e. $[H, H] = H$) and $\pi_1^c(C)$ is the commutator of the algebraic fundamental group of an irreducible smooth affine curve C over an algebraically closed field k of characteristic p.

Proof. As in Proposition 4.6 (also see Figure 1), let K^{un} denote the compositum (in some fixed algebraic closure of $k(C)$) of the function fields of all étale Galois covers of C. And let K^{ab} be the subfield of K^{un} obtained by considering only abelian étale covers of C. In terms of Galois theory, $\pi_1^c(C)$ is $\mathrm{Gal}(K^{un}/K^{ab})$. So giving a surjection from $\pi_1^c(C)$ to G is the same as giving a Galois extension $M \subset K^{un}$ of K^{ab} with Galois group G. Since K^{ab} is an algebraic extension of $k(C)$ and M is a finite extension of K^{ab}, we can find a finite abelian extension $L \subset K^{ab}$ of $k(C)$ and $L' \subset K^{un}$ a Galois extension of L with Galois group G so that $M = K^{ab}L'$. Let X be the normalization of C in L and Φ_X the normalization morphism. Then X is an étale abelian cover of C and the function field $k(X)$ of X is L. Let W_X be the normalization of X in L' and Ψ_X the corresponding normalization morphism. Then Ψ_X is étale and $k(W_X) = L'$.

By applying Theorem 5.1 to the affine curve X and translating the conclusion into Galois theory, we conclude that there exist m distinct smooth irreducible étale Γ-covers. Each one of these Γ-covers, Z, of X has the property that $Z/H = W_X$. Clearly $k(Z) \subset K^{un}$. We also have $\mathrm{Gal}(k(Z)K^{ab}/K^{ab}) \subset \Gamma$ and by assumption $\mathrm{Gal}(k(W_X)K^{ab}/K^{ab}) = G$. Moreover, the Galois group of $k(Z)/k(W_X)$ is H which is a perfect group and $k(W_X)K^{ab}/k(W_X)$ is a pro-abelian extension. Hence they are linearly disjoint, so $\mathrm{Gal}(k(Z)K^{ab}/k(W_X)K^{ab}) = H$. So $\mathrm{Gal}(k(Z)K^{ab}/K^{ab}) =$

Γ. Also if Z and Z' are two distinct solutions then $\mathrm{Gal}(k(Z)k(Z')/k(W_X))$ is a quotient of $H \times H$ and hence perfect. So $\mathrm{Gal}(k(Z)k(Z')K^{ab}/k(W_X)K^{ab}) = \mathrm{Gal}(k(Z)k(Z')/k(W_X))$ and consequently $k(Z)K^{ab}$ and $k(Z')K^{ab}$ are distinct fields. □

Now we shall consider the case when H is an abelian p-group.

Lemma 5.4. *Let P be any nonzero finite abelian p-group, then there exist infinitely many distinct epimorphisms from $\pi_1^c(C)$ to P.*

Proof. Let $n \geq 1$ be a natural number. By Lemma 4.4 there exists a p-group P_1 such that its commutator $[P_1, P_1] = P^n$. By Theorem 5.1 there exists a surjective homomorphism from $\pi_1(C)$ to P_1. And clearly this epimorphism when restricted to the commutator $\pi_1^c(C)$ surjects onto $[P_1, P_1] = P^n$. Call this restricted epimorphism ϕ. Now if we compose ϕ with any of the projection maps from P^n to P, we obtain n surjections from $\pi_1^c(C)$ to P. Since n was arbitrary, we are done. □

Theorem 5.5. *The following split embedding problem has a proper solution*

$$\begin{tikzcd}
 & & & \pi_1^c(C) \arrow[dl, dashed] \arrow[d] & \\
1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r] & G \arrow[r] \arrow[d] & 1 \\
 & & & 1 &
\end{tikzcd}$$

Here H is a minimal normal subgroup of Γ and an abelian p-group; and $\pi_1^c(C)$ is the commutator of the algebraic fundamental group of a smooth affine curve C over an algebraically closed field k of characteristic p.

Proof. Again, as in the proof of Theorem 5.3, let K^{un} and K^{ab} be the "function fields" of the maximal étale cover and maximal abelian étale cover of C respectively. Let M be the G-Galois field extension of K^{ab} corresponding to the epimorphism $\pi_1^c(C) \to G$ of the embedding problem. And again because of M being a finite field extension of K^{ab} we can descend it to an étale G-cover W_X of a finite abelian étale cover X of C. Since the center $Z(\Gamma)$ of Γ and H are both normal subgroups of Γ, so is $Z(\Gamma) \cap H$. Since H is a minimal normal subgroup of Γ, $Z(\Gamma) \cap H$ is trivial or $H \subset Z(\Gamma)$. If $H \subset Z(\Gamma)$ then Γ acts trivially on H, hence $\Gamma \cong G \times H$. By Lemma 5.4 there are infinitely many distinct surjections of $\pi_1^c(C)$ onto H. Hence there are infinitely many linearly disjoint field extensions of K^{ab} contained in K^{un} with Galois group H. M being a finite field extension of K^{ab}, all but finitely many of these H-extensions are linearly disjoint with M over K^{ab}. Hence there exists an H-extension of K^{ab} (in fact infinitely many of them) which is linearly disjoint with M over K^{ab} and the compositum of this H-extension with M leads to a Γ-extension of K^{ab}. Hence we have a solution to the embedding problem. Now suppose $Z(\Gamma) \cap H$ is trivial, i.e., Γ acts on H nontrivially. By Theorem 5.1 there exists a proper solution to the following embedding problem for $\pi_1(X)$:

$$\begin{tikzcd} & & & \pi_1(X) \arrow[dl, dashed] \arrow[d] & \\ 1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r] & G \arrow[r] \arrow[d] & 1 \\ & & & 1 & \end{tikzcd}$$

So there exists a smooth irreducible étale H-cover Z of W_X which is a Γ-cover of X. We shall show that $\mathrm{Gal}(k(Z)K^{ab}/K^{ab})$ is isomorphic to Γ. Suppose not, then $k(Z)$ is not linearly disjoint with $M = k(W_X)K^{ab}$ over $k(W_X)$. So there exists a nontrivial field extension $L''/k(W_X)$ with $L'' = k(Z) \cap k(W_X)K^{ab}$. So $L'' = Kk(W_X)$ for some finite field extension K of $k(X)$ such that $K \subset K^{ab}$. $K^{ab}/k(X)$ is a pro-abelian extension, so $K/k(X)$ is a Galois extension (in fact abelian). Hence $L''/k(X)$ is a Galois extension. So we conclude that $\mathrm{Gal}(k(Z)/L'')$ is a normal subgroup of $\Gamma = \mathrm{Gal}(k(Z)/k(X))$, but $\mathrm{Gal}(k(Z)/L'') \subset H = \mathrm{Gal}(k(Z)/k(W_X))$. H being a minimal normal subgroup of Γ and $L''/k(W_X)$ being a nontrivial extension force $L'' = k(Z)$ and hence $\mathrm{Gal}(K/k(X)) = H$. But this contradicts the fact that Γ acts on H nontrivially. □

Below we give an alternative approach to asserting the existence of a proper solution of the embedding problem for $\pi_1^c(C)$ when H is any p-group. This is a cohomological approach which needs the following cohomological result.

Proposition 5.6. *Let P be any nonzero finite elementary abelian p-group with a continuous action of $\pi_1^c(C)$, then the first group cohomology $H^1(\pi_1^c(C), P)$ is infinite.*

Proof. Let Φ be the kernel of the action of $\pi_1^c(C)$ on P. Then Φ is a normal subgroup of $\pi_1^c(C)$ of finite index. We know $\pi_1^c(C)$ acts on K^{un} and has fixed field K^{ab}. Let M be the fixed field of Φ, so $\mathrm{Gal}(M/K^{ab}) = \pi_1^c(C)/\Phi$. Since M is a finite extension of K^{ab}, there exist a finite abelian extension L of $k(C)$ and a finite extension L' of L such that $\mathrm{Gal}(L'/L) = \mathrm{Gal}(M/K^{ab})$ and $L'K^{ab} = M$. Let X be the normalization of C in L and Y the normalization of C in L'. If we translate all these relations between the above Galois extensions to their Galois groups, we get the following commutative diagram:

$$\begin{tikzcd} \Phi \arrow[r, hook] \arrow[d, hook] & \pi_1(Y) \arrow[d, hook] \\ \pi_1^c(C) \arrow[r, hook] \arrow[d, two heads] & \pi_1(X) \arrow[d, two heads] \\ \pi_1^c(C)/\Phi \arrow[r, "\sim"] & \pi_1(X)/\pi_1(Y) \end{tikzcd}$$

So we notice that $\pi_1(X) = \pi_1^c(C)\pi_1(Y)$. Now we define an action of $\pi_1(X)$ on P by defining it to be trivial on $\pi^1(Y)$ and the given action on $\pi_1^c(C)$. This is well defined because $\pi_1^c(C) \cap \pi_1(Y) = \Phi$, which is the kernel of the action of $\pi_1^c(C)$ on

P. Now consider the following short exact sequence of groups:

$$1 \to \pi_1^c(C) \to \pi_1(X) \to \Pi \to 1$$

Here Π is simply the quotient $\pi_1(X)/\pi_1^c(C)$. Applying the Hochschild-Serre spectral sequence for group cohomology [Wei 7.5.2] to this short exact sequence, we get the following long exact sequence:

$$0 \to H^1(\Pi, H^0(\pi_1^c(C), P)) \to H^1(\pi_1(X), P) \to H^0(\Pi, H^1(\pi_1^c(C), P)) \\ \to H^2(\Pi, H^0(\pi_1^c(C), P))$$

If the action of $\pi_1^c(C)$ on P is such that it fixes only 0, then $H^0(\pi_1^c(C), P) = 0$, hence the first and the fourth term in the above long exact sequence is 0. Also we know that $H^1(\pi_1(X), P)$ is infinite by [Ha4, Proposition 3.8]. So $H^1(\pi_1^c(C), P) \supset H^0(\Pi, H^1(\pi_1^c(C), P))$ is infinite. So we may assume that $P^{\pi_1^c(C)}$ is nonzero. In this case we have a short exact sequence of $\pi_1^c(C)$-modules:

$$0 \to P^{\pi_1^c(C)} \to P \to P/P^{\pi_1^c(C)} \to 0$$

Here $\pi_1^c(C)$ acts trivially on the first term and fixes nothing in the third term, i.e., $H^0(\pi_1^c(C), P/P^{\pi_1^c(C)}) = 0$, so we get the long exact sequence of group cohomology which looks like:

$$\cdots \to H^0(\pi_1^c(C), P/P^{\pi_1^c(C)}) \to H^1(\pi_1^c(C), P^{\pi_1^c(C)}) \to H^1(\pi_1^c(C), P) \to \cdots$$

Since $\pi_1^c(C)$ acts trivially on $P^{\pi_1^c(C)}$, $H^1(\pi_1^c(C), P^{\pi_1^c(C)}) = \mathrm{Hom}(\pi_1^c(C), P^{\pi_1^c(C)})$. And we know that $\mathrm{Hom}(\pi_1^c(C), P^{\pi_1^c(C)})$ is infinite by Lemma 5.4. So we conclude that $H^1(\pi_1^c(C), P)$ is infinite. □

Proof. Alternative approach to **Theorem 5.5**. The result follows trivially from Theorem 5.2 and Proposition 5.6. □

6. Prime-to-p embedding problems

In this section, certain results on formal patching will be proved and they will be used to solve the prime-to-p embedding problems for the commutator of the algebraic fundamental group of a smooth affine curve. We begin with some patching results (Theorem 6.1, Lemma 6.3, Proposition 6.4) which roughly mean the following: given a proper $k[[t]]$-scheme T whose special fiber is a collection of smooth irreducible curves intersecting at finitely many points, finding a cover of T is equivalent to finding a cover of these irreducible curves away from those finitely many intersection points and covers of formal neighbourhood of the intersection points so that they agree in the punctured formal neighbourhoods of the intersection points. In our situation, the special fiber of T is a connected sum of X and N copies of Y, for some $N \geq 1$. Let the i^{th} copy of Y intersect X at a point r_i of X and a point s of Y for $1 \leq i \leq n$. Now suppose we have an irreducible G-cover $\Psi_X \colon W_X \to X$ étale at $r_1, \cdots, r_n$ and an irreducible H-cover $\Psi_Y \colon W_Y \to Y$ étale at s. Let $\Gamma = G \rtimes H$. A Γ-cover of T is constructed by patching the Γ-cover of X, namely, $\mathrm{Ind}_G^\Gamma W_X = (\Gamma \times W_X)/\sim$, where $(\gamma, w) \sim (\gamma g^{-1}, gw)$ for $\gamma \in \Gamma$, $g \in G$ and w a point of W_X; and the Γ-cover $\mathrm{Ind}_H^\Gamma W_Y$ of Y. This is possible since both these covers restrict to Γ-covers induced from the trivial cover in the formal punctured neighbourhood of the intersection points, so we can pick trivial Γ-covers of the intersection points which obviously will restrict to the trivial Γ-cover on the punctured neighbourhood. Now we proceed to show how all this works. We start with some patching results.

6.1. **Formal Patching.**

Theorem 6.1. ([Ha3, Theorem 3.2.12]) *Let (A, p) be a complete local ring and let T be a proper A-scheme. Let $\{\tau_1, \cdots, \tau_N\}$ be a set of closed points of T and $T^o = T \setminus \{\tau_1, \cdots, \tau_N\}$. Let $\hat{T}_i = \operatorname{Spec}(\hat{\mathcal{O}}_{T,\tau_i})$, $\widetilde{T^o}$ be the p-adic completion of T^o and $\mathcal{K}_i$ be the p-adic completion of $T_i \setminus \{\tau_i\}$. Then the base change functor*

$$\mathcal{M}(T) \to \mathcal{M}(\widetilde{T^o}) \times_{\mathcal{M}(\cup_{i=1}^N \mathcal{K}_i)} \mathcal{M}(\cup_{i=1}^N \hat{T}_i)$$

is an equivalence of categories. And this remains true with $\mathcal{M}$ replaced by $\mathcal{AM}$, $\mathcal{SM}$ or $G\mathcal{M}$ for any finite group G.

In fact Theorem 3.2.12 of [Ha3] is even stronger and allows one to assert the equivalence of categories even if one replaces T, T^o, etc. with their pull-backs by proper morphisms. The proof of the above theorem uses Grothendieck's Existence Theorem and a result of Ferrand-Raynaud or rather the following generalization by M. Artin.

Theorem 6.2. (M. Artin [Ha3, Theorem 3.1.9]) *Let T be a noetherian scheme, W a finite set of closed points of T, $T^o = T \setminus W$, $\hat{W}$ the completion of T along W and $W^o = \hat{W} \times_T T^o$. Then the base change functor is an equivalence of categories between $\mathcal{M}(T)$ and $\mathcal{M}(T^o) \times_{\mathcal{M}(W^o)} \mathcal{M}(\hat{W})$.*

We shall specialize the above patching result (Theorem 6.1) to something we need. Let k be a field. Let X and Y be smooth projective k-curves with finite k-morphisms $\Phi_X : X \to \mathbb{P}^1_x$ and $\Phi_Y : Y \to \mathbb{P}^1_y$, where $\mathbb{P}^1_x$ and $\mathbb{P}^1_y$ are projective lines with local coordinates x, y respectively. Also assume that Φ_Y is totally ramified at $y = 0$. Let R and S be such that $\operatorname{Spec}(R) = X \setminus \Phi_X^{-1}(\{x = \infty\})$ and $\operatorname{Spec}(S) = Y \setminus \Phi_Y^{-1}(\{y = \infty\})$. So $k[x] \subset R$ and $k[y] \subset S$. Let $A = (R \otimes_k S \otimes_k k[[t]])/(t - xy)$ and $T^a = \operatorname{Spec}(A)$. Let T be the closure of T^a in $X \times_k Y \times_k \operatorname{Spec}(k[[t]])$. A more geometric way of describing T is the following. Let F be the graph of $t - xy = 0$ in $\mathbb{P}^1_x \times_k \mathbb{P}^1_y \times_k \operatorname{Spec}(k[[t]])$ then $T = (X \times_{\mathbb{P}^1_x} F) \times_F (F \times_{\mathbb{P}^1_y} Y)$ where the morphisms from F to $\mathbb{P}^1_x$ and $\mathbb{P}^1_y$ are restrictions of the projection morphisms from $\mathbb{P}^1_x \times_k \mathbb{P}^1_y \times_k \operatorname{Spec}(k[[t]])$.

$$\begin{tikzcd}
 & & T \arrow[dl] \arrow[dr] & & \\
 & X \times F \arrow[dl] \arrow[dr] & & F \times Y \arrow[dl] \arrow[dr] & \\
X \arrow[dr] & & F \arrow[dl] \arrow[dr] & & Y \arrow[dl] \\
 & \mathbb{P}^1_x & & \mathbb{P}^1_y &
\end{tikzcd}$$

Let L be the affine line $\operatorname{Spec}(k[z])$. The k-algebra homomorphism $k[[t]][z] \to A$ given by $z \mapsto x + y$ induces a $k[[t]]$-morphism ϕ from T^a to $L^* = L \times_k k[[t]]$. Let $\lambda \in L$ be the closed point $z = 0$. L is contained in L^* as the special fiber, so λ viewed as a closed point of L^* corresponds to the maximal ideal (z, t) in $k[[t]][z]$.

Let $\phi^{-1}(\lambda) = \{\tau_1, \cdots, \tau_N\} \subset T^a$. Note that the special fiber of T is a reducible curve consisting of X and N copies of Y, each copy of Y intersecting X at τ_i for some i, since the locus of $t = 0, x + y = 0$ is same as the locus of $x = 0$ and $y = 0$. Let r_i denote the point of X corresponding to τ_i, so $\Phi_X^{-1}(x = 0) = \{r_1, \cdots, r_N\}$, and let s denote the point on each copy of Y corresponding to τ_i, so s is the unique point of Y lying above $y = 0$. Borrowing the notation from Theorem 6.1, let $T^o = T \setminus \{\tau_1, \cdots, \tau_N\}$ and $X^o = X \setminus \{r_1, \cdots, r_N\}$. Let $\hat{T}_i = \mathrm{Spec}(\hat{\mathcal{O}}_{T,\tau_i})$ and let $T_X = T^o \setminus \{x = 0\}$ which is the same as the closure of $\mathrm{Spec}(A[1/x])$ in $X^o \times_k Y \times_k \mathrm{Spec}(k[[t]])$. Similarly, define $T_Y = T^o \setminus \{y = 0\}$.

Let $\hat{K}_{X,r_i}$ denote the quotient field of $\hat{\mathcal{O}}_{X,r_i}$. Define $\mathcal{K}_{X,r_i} = \mathrm{Spec}(\hat{K}_{X,r_i}[[t]] \otimes_{k[y]} \mathcal{O}_{Y,s})$ where we regard $\hat{K}_{X,r_i}[[t]]$ as a $k[y]$-module via the homomorphism which sends y to t/x. Similarly, define $\mathcal{K}_Y^i = \mathrm{Spec}(\hat{K}_{Y,s}[[t]] \otimes_{\mathrm{Spec}(k[x])} \mathcal{O}_{X,r_i})$, where we regard $\hat{K}_{Y,s}[[t]]$ as a $k[x]$-module via the homomorphism which sends x to t/y. Let x_i be a local coordinate of X at r_i and y_0 a local coordinate of Y at s. For any $k[[t]]$-scheme V, $\widetilde{V}$ will denote its (t)-adic completion.

With these notations we shall deduce the following result from Theorem 6.1. This result is analogous to [Ha2, Corollary 2.2].

Lemma 6.3. *The base change functor*

$$\mathcal{M}(T) \to \mathcal{M}(\widetilde{T_X} \cup \widetilde{T_Y}) \times_{\mathcal{M}(\cup_{i=1}^N (\mathcal{K}_{X,r_i} \cup \mathcal{K}_Y^i))} \mathcal{M}(\cup_{i=1}^N \hat{T}_i)$$

is an equivalence of categories. Moreover, the same assertion holds if one replaces $\mathcal{M}$ by $\mathcal{AM}$, $\mathcal{SM}$ and $G\mathcal{M}$ for a finite group G.

Proof. First of all we observe that the closed fiber of T^o, which is the subscheme defined by the ideal (t), is disconnected. This is because the closed fiber of $T_X \cup T_Y$ is the closed fiber of T^o and the closed fibers of T_X and T_Y, as subsets of the closed fiber of T^o, are open and disjoint. So considering their (t)-adic completions we deduce $\widetilde{T^o} = \widetilde{T_X} \cup \widetilde{T_Y}$. Similarly the punctured spectrum $\hat{T}_i \setminus \{\tau_i\}$ is the spectrum of the ring $k[[x_i, y_0]][(x+y)^{-1}])$. Since the only prime ideals of $k[[x_i, y_0]][(x+y)^{-1}]$ containing (t) are (x_i) and (y_0), we may first localize $k[[x_i, y_0]][(x + y)^{-1}]$ with respect to the complement of $(x_i) \cup (y_0)$ and then take the (t)-adic completion. Now using [Mat, 8.15], we get that the (t)-adic completion of $\hat{T}_i \setminus \{\tau_i\}$ is $\mathcal{K}_{X,r_i} \cup \mathcal{K}_Y^i$. So, with all this identification, we get the result from Theorem 6.1. □

Let G and H be subgroups of a finite group Γ, such that $\Gamma = G \rtimes H$.

Proposition 6.4. *Under the notation and the assumptions of Lemma 6.3, let $\Psi_X : W_X \to X$ be an irreducible normal G-cover étale over the points $r_1, \cdots, r_N$; and $\Psi_Y : W_Y \to Y$ an irreducible normal H-cover étale over s. Let W_{XT} be the normalization of an irreducible dominating component of $W_X \times_X T$ and similarly W_{YT} the normalization of an irreducible dominating component of $W_Y \times_Y T$. Then there exists an irreducible normal Γ-cover $W \to T$ such that*

(1) $W \times_T \widetilde{T_X} = \mathrm{Ind}_G^\Gamma \widetilde{W_{XT} \times_T T_X}$

(1') $W \times_T \widetilde{T_Y} = \mathrm{Ind}_H^\Gamma \widetilde{W_{YT} \times_T T_Y}$

(2) $W \times_T \hat{T}_i$ is a Γ-cover of $\hat{T}_i$ induced from the trivial cover.

(3) $W \times_T \mathcal{K}_{X,r_i}$ is a Γ-cover of $\mathcal{K}_{X,r_i}$ induced from the trivial cover.

(4) $W \times_T \mathcal{K}_Y^i$ is a Γ-cover of $\mathcal{K}_Y^i$ induced from the trivial cover.

(5) $W/H \cong W_{XT}$ as a cover of T.

Proof. Let $\widetilde{W_X} = \mathrm{Ind}_G^\Gamma \widetilde{W_{XT} \times_T T_X}$ and $\widetilde{W_Y} = \mathrm{Ind}_H^\Gamma \widetilde{W_{YT} \times_T T_Y}$. So $\widetilde{W_X}$ and $\widetilde{W_Y}$ are Γ-covers of $\widetilde{T_X}$ and $\widetilde{T_Y}$ respectively. Hence their union $\widetilde{W^o}$ is an object of $\Gamma\mathcal{M}(\widetilde{T_X} \cup \widetilde{T_Y})$. Now for each i, $\widetilde{W_X} \times_{\widetilde{T_X}} \mathcal{K}_{X,r_i} = \mathrm{Ind}_G^\Gamma W_X \times_X \mathcal{K}_{X,r_i}$. But $W_X \times_X \mathcal{K}_{X,r_i}$ is isomorphic to disjoint union of card(G) copies of $\mathcal{K}_{X,r_i}$, since $W_X \to X$ is étale over r_i. And similarly, $\widetilde{W_Y} \times_{\widetilde{T_Y}} \mathcal{K}_Y^i = \mathrm{Ind}_H^\Gamma W_Y \times_Y \mathcal{K}_Y^i$ which is isomorphic to disjoint union of card(Γ) copies of $\mathcal{K}_Y^i$, since $W_Y \to Y$ is étale over s. Hence $\widetilde{W^o}$ restricted to $\cup_{i=1}^N(\mathcal{K}_{X,r_i} \cup \mathcal{K}_Y^i)$ is a Γ-cover induced from the trivial cover. Let $\hat{W}_i$ be a Γ-cover of $\hat{T}_i$ induced from the trivial cover. Then their union $\hat{W}$ is an object in $\Gamma\mathcal{M}(\cup_{i=1}^N \hat{T}_i)$ which when restricted to $\cup_{i=1}^N(\mathcal{K}_{X,r_i} \cup \mathcal{K}_Y^i)$ obviously is a Γ-cover induced from the trivial cover. So after fixing an isomorphism between the two trivial Γ-covers of $\cup_{i=1}^N(\mathcal{K}_{X,r_i} \cup \mathcal{K}_Y^i)$, we can apply Lemma 6.3 and obtain an object W in $\Gamma\mathcal{M}(T)$ which induces the covers $\widetilde{W^o}$ and $\hat{W}$ on $\widetilde{T^o}$ and $\cup_{i=1}^N \hat{T}_i$ respectively. Hence we get conclusions (1) to (4) of the proposition. So it remains to prove that W is irreducible and normal and that conclusion (5) holds. To prove the irreducibility of W, first note that G and H generate Γ. Consider Γ^o, the stabilizer of the identity component of W. So W has card(Γ/Γ^o) irreducible components. Since G is the stabilizer of the identity component of $\widetilde{W_X}$ and H is the stabilizer of the identity component of $\widetilde{W_Y}$, G and H are contained in Γ^o. Hence $\Gamma^o = \Gamma$. Hence W is irreducible. To show that W is normal it is enough to show that for each closed point σ of T $W_\sigma = W \times_T \mathrm{Spec}(\hat{\mathcal{O}}_{T,\sigma})$ is normal. If $\sigma = \tau_i$ for some i then W_σ is isomorphic to the disjoint union of some copies of $\hat{T}_i$ and hence is normal. Otherwise σ belongs to T_X (or T_Y). So W_σ is isomorphic to $\mathrm{Ind}_G^\Gamma \widetilde{W_{XT} \times_T T_X} \times_{T_X} \mathrm{Spec}(\hat{\mathcal{O}}_{T_X,\sigma})$, which is a union of copies of $\mathrm{Spec}(\hat{\mathcal{O}}_{W_{XT}\times_T T_X,\sigma'})$, where σ' are points of $W_{XT} \times_T T_X$ lying above σ. But $W_{XT} \times_T T_X$ is normal. A similar argument holds in the case when $\sigma \in T_Y$. Next we shall show that W/H and W_{XT} restricts to same G-cover on the patches $\widetilde{T_X}$, $\widetilde{T_Y}$ and $\hat{T}_i$ for all i. Conclusion (5) will then follow from the assertion in Lemma 6.3 about the equivalence of categories (with $\mathcal{M}$ replaced by $G\mathcal{M}$). Clearly, both W/H and W_{XT} restrict to trivial G-cover of $\hat{T}_i$. $W_{XT} \times_T \widetilde{T_X} = W_{XT} \times_T \widetilde{T} \times_{\widetilde{T}} \widetilde{T_X} = \widetilde{W_{XT}} \times_{\widetilde{T}} \widetilde{T_X}$ and this is same as $\widetilde{W_{XT} \times_T T_X}$ since T_X is an open subscheme of T. On the other hand $W/H \times_T \widetilde{T_X} = (W \times_T \widetilde{T_X})/H$. But by (1) this is same as $\widetilde{W_{XT} \times_T T_X}$. Finally, note that the image of T_Y under the morphism $T \to X$ is the generic point of X. So the G-cover $W_{XT} \to T$ is trivial over the subscheme T_Y. Hence $W_{XT} \times_T \widetilde{T_Y} = \mathrm{Ind}_{\{e\}}^G \widetilde{T_Y}$. On the other hand by (1′), $W/H \times_T \widetilde{T_Y} = (\mathrm{Ind}_H^\Gamma \widetilde{W_{TY}})/H$. But $(\mathrm{Ind}_H^\Gamma \widetilde{W_{TY}})/H$ is the same as $\mathrm{Ind}_{H/H}^{\Gamma/H} \widetilde{T_Y}$ since $W_Y/H = Y$. But Γ/H is G. □

6.2. **Proof of the main theorem.** In this subsection the main theorem (Theorem 6.12) of this section will be proved using the patching results stated above.

Lemma 6.5. *Let T, X and Y be as in Lemma 6.4. Let D be an irreducible smooth projective k-curve. Assume that $\Phi_X\colon X \to \mathbb{P}^1_x$ factors through D, i.e., there exist $\Phi'_X\colon X \to D$ and $\Theta\colon D \to \mathbb{P}^1_x$ such that $\Phi_X = \Theta\circ\Phi'_X$. Also assume that Φ_Y and Φ'_X are abelian covers. For any $k[[t]]$-scheme V, let V^g denote its generic fiber. Then the morphism $(\Phi'_X \times \Phi_Y \times Id_{\mathrm{Spec}(k[[t]])})|_T$ from T to its image in $D \times_k \mathbb{P}^1_y \times_k \mathrm{Spec}(k[[t]])$ induces an abelian cover of projective $k((t))$-curves $T^g \to D \times_k \mathrm{Spec}(k((t)))$.*

Proof. We need to show that the function field $k(T)$ of T is an abelian extension of the field $k(D) \otimes_k k((t))$. Note that $k(T)$ is the compositum of $L_1 = k(X) \otimes_k k((t))$ and L_2. Here L_2 is the function field of a dominating irreducible component of

$$(Y \times_k \mathrm{Spec}(k((t)))) \times_{\mathbb{P}^1_y \times_k \mathrm{Spec}(k((t)))} (D \times_k \mathrm{Spec}(k((t))))$$

where the morphism $D \times_k \mathrm{Spec}(k((t))) \to \mathbb{P}^1_y \times_k \mathrm{Spec}(k((t)))$ is the composition of $D \times_k \mathrm{Spec}(k((t))) \to \mathbb{P}^1_x \times_k \mathrm{Spec}(k((t)))$ with the morphism $\mathbb{P}^1_x \times_k \mathrm{Spec}(k((t))) \to \mathbb{P}^1_y \times_k \mathrm{Spec}(k((t)))$ defined in local coordinates by sending y to t/x. Since L_1 is a base change of the finite extension $k(X)/k(x)$ by $k(D) \otimes_k k((t))$ and L_2 is a base change of the finite extension $k(Y)/k(y)$ by $k(D) \otimes_k k((t))$ after identifying y with t/x, we have $L_1 \cap L_2 = k(D) \otimes_k k((t))$. Hence L_1 and L_2 are linearly disjoint over $k(D) \otimes_k k((t))$. Moreover $\mathrm{Gal}(L_1/k(D) \otimes_k k((t)))$ is isomorphic to $\mathrm{Gal}(k(X)/k(D))$ and $\mathrm{Gal}(L_2/k(D) \otimes_k k((t)))$ is isomorphic to $\mathrm{Gal}(k(Y)/k(y))$. Hence these groups are abelian, since the latter groups are so by assumption. Using the fact that the Galois group of the compositum of linearly disjoint Galois field extensions is the direct sum of the Galois groups, we get that $\mathrm{Gal}(k(T)/k(D) \otimes_k k((t)))$ is a direct sum of abelian groups, and hence is abelian. □

We shall see a variation of the following result, which is a special case of [Ha2, Proposition 2.6, Corollary 2.7]. These results help in descending covers of $k[[t]]$-schemes to analogous covers of k-schemes.

Proposition 6.6. (Harbater) *Let k be an algebraically closed field. Let X_0^s be a smooth projective connected smooth k-curve. Let $\zeta^1, \cdots, \zeta^r \in X_0^s$. Let X_0 and X_1 be irreducible normal projective $k[[t]]$-curves. Suppose X_1 has generically smooth closed fiber. Let $\psi\colon X_1 \to X_0$ be a G-cover which induces $\psi^g\colon X_1^g \to X_0^g$ over the generic point of* $\mathrm{Spec}(k[[t]])$. *Assume $X_0 = X_0^s \times_k \mathrm{Spec}(k[[t]])$. Also assume ψ^g is a smooth G-cover étale away from $\{\zeta_1, \cdots, \zeta_r\}$ where $\zeta_j = \zeta^j \times_k k((t)) \in X_0^g$ for $1 \le j \le r$. Then there exists a smooth connected G-cover $\psi^s\colon X_1^s \to X_0^s$ étale away from $\{\zeta^1, \cdots, \zeta^r\}$.*

The above result uses Lemma 2.4(b) of [Ha2] and Proposition 5 of [Ha1]. These results are stated below (without proof) for the reader's convenience.

Lemma 6.7. *Let S be the spectrum of a complete regular local ring with algebraically closed residue field, V an irreducible normal scheme, $\phi\colon V \to S$ a surjective proper morphism such that the fiber over the closed point is generically smooth. Then the closed fiber of ϕ is connected.*

Proposition 6.8. *Let ϵ be a closed point of a noetherian normal scheme S and $\phi\colon V \to S$ a proper morphism that is generically smooth over ϵ. If the fiber V_ϵ is geometrically connected and V is geometrically unibranched along V_ϵ then for all closed points e in a nonempty open subset of S, the fiber V_e is geometrically irreducible.*

The proof of the following result is also similar to the one given in [Ha2].

Proposition 6.9. *Let k be an algebraically closed field. Let X_0, X_1, X_2, X_3 be irreducible normal projective $k[[t]]$-curves such that for $i > 0$, X_i have generically smooth closed fibers. For $i = 1, 2$ and 3, let $\psi_i\colon X_i \to X_{i-1}$ be proper surjective $k[[t]]$-morphisms and $\psi_i^g\colon X_i^g \to X_{i-1}^g$ be the induced morphisms on the generic fibers. Assume $X_0^g = X_0^s \times_k k((t))$ for some smooth projective k-curve X_0^s. Let*

$\zeta^1, \cdots, \zeta^r \in X_0^s$ and $\zeta_j = \zeta^j \times_k k((t)) \in X_0^g$ for $1 \le j \le r$, so that $\psi_1^g \circ \psi_2^g \circ \psi_3^g$ is étale away from $\{\zeta_1, \cdots, \zeta_r\}$. Let ψ_1 be an A-cover, ψ_2 be a G-cover, ψ_3 be an H-cover and $\psi_2 \circ \psi_3$ be a Γ-cover. Then there exist X_1^s, X_2^s and X_3^s connected smooth projective k-curves and morphisms $\psi_i^s \colon X_i^s \to X_{i-1}^s$ so that $\psi_1^s \circ \psi_2^s \circ \psi_3^s$ is étale away from $\{\zeta^1, \cdots, \zeta^r\}$ and ψ^1 is an A-cover, ψ^2 is a G-cover, ψ^3 is an H-cover and $\psi^2 \circ \psi^3$ is a Γ-cover.

Proof. Since all the three groups are finite, the covers ψ_i for $i = 1, \cdots, 3$ descend to B-morphisms, where $B \subset k[[t]]$ is a regular finite type $k[t]$-algebra. That is, there exist connected normal projective B-schemes X_i^B and morphisms $\psi_i^B \colon X_i^B \to X_{i-1}^B$ where ψ_1^B is an A-cover, ψ_2^B is a G-cover, ψ_3^B is an H-cover and $\psi_2^B \circ \psi_3^B$ is a Γ-cover and ψ_i^B induces ψ_i. Moreover for $E = \mathrm{Spec}(B[t^{-1}])$, $X_i^E = X_i^B \times_B E$ are normal projective E-schemes and X_0^E is isomorphic to $X_0^s \times_k E$. The induced morphisms ψ_i^E are such that ψ_1^E is an A-cover, ψ_2^E is a G-cover, ψ_3^E is an H-cover, $\psi_2^E \circ \psi_3^E$ is a Γ-cover and $\psi_1^E \circ \psi_2^E \circ \psi_3^E$ is ramified only over $\{\zeta_E^1, \cdots, \zeta_E^r\}$. To complete the proof, we shall show that there exists a nonempty open subset E' of E so that the fiber of ψ_i^E over each closed point of E' is irreducible and nonempty. First we note that by Lemma 6.7 the closed fibers of $X_i \to \mathrm{Spec}(k[[t]])$ are connected, since by assumption the closed fibers are generically smooth. Hence the fibers of ψ_i^B over $(t = 0)$ are connected because ψ_i^B induces ψ_i. Since X_i^B's are normal, they are unibranched along the respective fibers over $(t = 0)$. Hence by Proposition 6.8, we have a nonempty open subset of $\mathrm{Spec}(B)$, and hence an open subset E' of $E = \mathrm{Spec}(B) \setminus (t = 0)$, such that for all closed points $e \in E'$ the fibers X_i^e of $X_i^E \to E$ over e are irreducible. Next, we shall show that there exists a nonempty open subset S of E' such that the restriction morphism $X_i^S \to S$ is smooth of relative dimension 1. Since k is algebraically closed $k(X_0^s)$ is separably generated over k. Hence $k(X_0^{E'})$ is separably generated over $k(E')$. Moreover, since ψ_i^E are finite separable morphisms (in fact their composition is étale away from $\{\zeta_E^1, \cdots, \zeta_E^r\}$), $k(X_i^{E'})$ is separably generated over $k(E')$. Since $X_i^{E'} \to E'$ is a morphism of integral schemes of relative dimension 1 and is generically separable, the relative sheaf of differentials is free of rank 1 at the generic point ([Eis, Corollary 16.17a]). Hence there exists an open subset S of E' such that the morphism $X_i^S \to S$ is smooth of relative dimension 1. Hence, the fiber X_i^s at each point $s \in S \subset E'$ is a smooth, irreducible curve. □

Lemma 6.10. *Given any positive integer l, there exists a Galois cover $Y \to \mathbb{P}_y^1$ ramified only at $y = 0$, where it is totally ramified, with genus of $Y > l$ and Galois group $(\mathbb{Z}/p\mathbb{Z})^n$ for some n. In particular it is an abelian cover.*

Proof. Let Y' be the normal cover of $\mathbb{P}_y^1$ defined by the equation $u^{p^n} - u - y^{p^n+1} = 0$. To see that this is an irreducible polynomial in $k(y)[u]$, by Gauss lemma, it is enough to show it to be irreducible in $k[y, u]$. But in fact, it is irreducible in $k(u)[y]$ since the $(p^n+1)^{th}$ roots of $u^{p^n} - u$ do not belong to $k(u)$. Also Y' is étale everywhere except at $y = \infty$ and since there is only one point in Y' lying above ∞ it is totally ramified there. So by translation we can get Y, a cover of $\mathbb{P}_y^1$, which is totally ramified at $y = 0$ and étale elsewhere. Also the genus of Y, by the Hurwitz formula, is given by the equation $2g(Y) - 2 = (p^n + 1)(g(\mathbb{P}_u^1) - 2) + \deg(R)$ where R is the ramification divisor of the morphism $Y \to \mathbb{P}_u^1$. It is well known that $\deg(R) = \sum_{P \in Y} e_P - 1$ where e_P is the ramification index at the point $P \in Y$ if the morphism is tamely ramified at P. Now branch locus of Y as a cover of $\mathbb{P}_u^1$ is given by $u^{p^n} - u = 0$

and $u = \infty$. For each point $P \in Y$ lying above a branch point other than ∞, $e_P = p^n + 1$, hence $\deg(R) \geq p^n p^n = p^{2n}$. So we get the following inequality for genus of Y.

$$\begin{aligned} &2g(Y) - 2 \geq -2(p^n + 1) + p^{2n} \\ \Rightarrow &g(Y) \geq p^n(p^n - 2)/2 \end{aligned}$$

Clearly $g(Y)$ could be made arbitrary large. Also note that $\mathrm{Gal}(k(Y)/k(y)) \cong (\mathbb{Z}/p\mathbb{Z})^n$. Hence Y is an abelian cover of $\mathbb{P}^1_y$. □

6.2.1. *The case of the affine line.*

Theorem 6.11. *The following split embedding problem has a proper solution*

$$\begin{tikzcd} & & & \pi_1^c(\mathbb{A}^1) \arrow[dl, dashed] \arrow[d] & \\ 1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r] & G \arrow[r] \arrow[d] & 1 \\ & & & 1 & \end{tikzcd}$$

Here H is a prime to p-group and a minimal normal subgroup of Γ and $\pi_1^c(\mathbb{A}^1)$ is the commutator of the algebraic fundamental group of the affine line over an algebraically closed field k of characteristic p.

Proof. Let x denote the local coordinate for the given affine line. As in the previous sections we shall denote this affine line by $\mathbb{A}^1_x$. Let K^{un} denote the compositum, in some fixed algebraic closure of $k(x)$, of the function fields of all Galois étale covers of $\mathbb{A}^1_x$. And let K^{ab} be the subfield of K^{un} obtained by considering only abelian étale covers. In these terms $\pi_1^c(\mathbb{A}^1_x)$ is $\mathrm{Gal}(K^{un}/K^{ab})$. Note that $K^{ab}/k(x)$ is a pro-p field extension (Theorem 3.4 or Corollary 3.5). Giving a surjection from $\pi_1^c(\mathbb{A}^1_x)$ to G is equivalent to giving a Galois extension $M \subset K^{un}$ of K^{ab} with Galois group G. Since K^{ab} is an algebraic extension of $k(x)$ and M is a finite extension of K^{ab}, we can find a finite abelian extension $L \subset K^{ab}$ of $k(x)$ and $L' \subset K^{un}$ a G-Galois extension of L so that $M = K^{ab}L'$. Let $D = \mathbb{P}^1_x$ be the projective x-line, X the normalization of D in L and $\Phi_X : X \to D$ the normalization morphism. Then X is an abelian cover of D branched only at $x = \infty$, and the function field $k(X)$ of X is L. Let $U = X \setminus \Phi^{-1}(\{x = \infty)\}$. Let W_X be the normalization of X in L' and Ψ_X be the corresponding normalization morphism. Then Ψ_X is étale over U and $k(W_X) = L'$. Let $\{r_1, \cdots, r_N\} = \Phi_X^{-1}(\{x = 0\})$, then Φ_X is étale at $r_1, \cdots, r_N$. Let $\Phi_Y : Y \to \mathbb{P}^1_y$ be an abelian cover étale everywhere except at $y = 0$, where it is totally ramified such that the genus of Y is at least 2 and more than the number of generators for H. Let s be the point of Y lying above $y = 0$. The existence of such a Y is guaranteed by Lemma 6.10. Since H is a prime-to-p group, and Y has high genus by [SGAI, XIII, Corollary 2.12, page 392], there exists an irreducible étale H-cover W'_Y of Y. Let $\Psi_Y : W_Y \to Y$ denote the covering morphism. Now applying Proposition 6.4 we obtain an irreducible normal Γ-cover $W \to T$ satisfying conclusions (1) to (5) of Proposition 6.4. Also, by Lemma 6.5, we know that the morphism $T \to F$, where F is the locus of $xy - t = 0$ in $D \times_k \mathbb{P}^1_y \times_k \mathrm{Spec}(k[[t]])$, induces an abelian cover $T \times_{\mathrm{Spec}(k[[t]])} \mathrm{Spec}(k((t))) \to D \times_k \mathrm{Spec}(k((t)))$. For any $k[[t]]$-scheme V, let V^g denote its generic fiber. Since the branch locus of $W^g \to T^g$ is determined by the branch locus of $W \to T$ on the patches, from conclusions (1), (1′) and (2) of Proposition 6.4, we conclude that $W^g \to T^g$ is ramified only at points of T^g lying above $x = \infty$. This is because $W_{YT} \to T_Y$ is étale everywhere and $W_{XT} \to T_X$ is étale away from the points which map to $x = \infty$ under the composite $W_{XT} \to T_X \to X \to D$. Also $T^g \to D \times_k \mathrm{Spec}(k((t)))$ is ramified only at points above $x = \infty$, since $T \to F$ is ramified only at points above $x = \infty$ and $y = 0$ (which is the same as $t = 0$ and possibly $x = \infty$). So, on the generic fiber (i.e., $t \neq 0$) these two points get identified. Hence we obtain the following tower of covers.

$$\begin{tikzcd} W^g \arrow[d] \\ W^g/H \cong W^g_{XT} \arrow[d, "G\text{-cover ramified only at points lying above } x=\infty"] \\ T^g \arrow[d, "\text{abelian cover ramified only at } x=\infty"] \\ D \times_k \mathrm{Spec}(k((t))) \end{tikzcd}$$

Figure 2

Now applying Proposition 6.4, the above tower of $k((t))$-covers descends to a tower of k-covers with the same ramification properties and group actions.

$$\begin{tikzcd} W^s \arrow[d] \\ W^s/H \cong W^s_{XT} \arrow[d, "G\text{-cover ramified only at points lying above } x=\infty"] \\ T^s \arrow[d, "\text{abelian cover ramified only at } x=\infty"] \\ D \end{tikzcd}$$

Figure 3

Here $-^s$, as in Proposition 6.9, denotes specialization to the base field k. So to complete the proof, it is enough to show that the Galois group of $k(W^s)K^{ab}$ over K^{ab} is Γ, where $k(W^s)$ is the function field of W^s. Note that $k(W_X) \subset k(W^s_{XT}) = k(W_X)k(T^s)$. So $k(W^s_{XT})K^{ab} = k(W_X)K^{ab}$, since $k(T^s) \subset K^{ab}$. By assumption the Galois group of $k(W_X)K^{ab}$ over K^{ab} is G. So it is enough to show that the Galois group of $k(W^s)K^{ab}$ over $k(W^s_{XT})K^{ab}$ is H. Note that $k(W^s_{XT})K^{ab}/k(W^s_{XT})$ is a pro-p extension since it is a base change of the pro-p extension $K^{ab}/k(x)$ and $k(W^s)$ is a prime-to-p-extension of $k(W^s_{XT})$ (in fact the Galois group is H). Hence $k(W^s)$ and $K^{ab}k(W^s_{XT})$ are linearly disjoint over $k(W^s_{XT})$ and $\mathrm{Gal}(k(W^s)K^{ab}/k(W^s_{XT})K^{ab}) = H$. Thus the field $k(W^s)K^{ab}$ provides a proper solution to the given embedding problem as required. $\square$

6.2.2. *The general case.* Now we shall deal with the general case which require some modifications at the end and a slight modification in the beginning to facilitate the application of the patching results.

Theorem 6.12. *The following split embedding problem has a proper solution*

$$\begin{tikzcd} & & & \pi_1^c(C) \arrow[dl, dashed] \arrow[d] & \\ 1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r] & G \arrow[r] \arrow[d] & 1 \\ & & & 1 & \end{tikzcd}$$

Here H is a prime-to-p group and a minimal normal subgroup of Γ and $\pi_1^c(C)$ is the commutator of the algebraic fundamental group of a smooth affine curve C over an algebraically closed field k of characteristic p.

Proof. As in the proof of the affine line case (Theorem 6.11), let K^{un} denote the compositum of the function fields of all Galois étale covers of C. And let K^{ab} be the subfield of K^{un} obtained by considering only abelian covers. Let D be the smooth compactification of C. As before from the embedding problem, we obtain an abelian cover $\Phi'_X\colon X \to D$ which is étale over C and a G-cover $\Psi_X\colon W_X \to X$ étale over $U = {\Phi'_X}^{-1}(C)$ which remains a G-cover after base change to the maximal abelian étale "pro-cover" of C. Since k is algebraically closed, $k(C)/k$ has a separating transcendence basis. By a stronger version of Noether normalization (for instance, see [Eis, Corollary 16.18]), there exists a finite proper k-morphism from C to $\mathbb{A}^1_x$ which is generically separable. The branch locus of such a morphism is of codimension at most 1, hence this morphism is étale away from finitely many points. After translation of $\mathbb{A}^1_x$, if necessary, we may assume that none of these points map to $x = 0$. This morphism extends to a finite proper morphism $\Theta\colon D \to \mathbb{P}^1_x$. Let $\Phi_X\colon X \to \mathbb{P}^1_x$ be the composition $\Theta \circ \Phi'_X$. Let $\{r_1, \cdots, r_N\} = \Phi_X^{-1}(\{x = 0\})$, then Φ_X is étale at $r_1, \cdots, r_N$. Also note that $\Theta^{-1}(\{x = \infty\}) = D \setminus C$. From here on we can again obtain a Γ-cover of X which dominates W_X as in the affine line case. But now the field extension $K^{ab}/k(C)$ is not a pro-p group, as was the case when C was the affine line. And hence the linear disjointness argument to lift this cover to a Γ-cover of the maximal abelian pro-cover does not go through. But nevertheless, the prime-to-p part is a finitely generated group (Theorem 3.4). And we shall use this fact to obtain a cover which does lift to a Γ-cover. Let $l > 0$ be any integer. Let $\Phi_Y\colon Y \to \mathbb{P}^1_y$ be an abelian cover étale everywhere except over $y = 0$, where it is totally ramified, such that the genus of Y is at least 2 and more than the number of generators for H^l, i.e., the product of H with itself l times. Let s be the point lying above $y = 0$. Since H^l is prime-to-p, and Y has high genus by [SGAI, XIII, Corollary 2.12, page 392], there exists an irreducible étale

H^l-cover W_Y of Y. Let $\Gamma^{(l)}$ be the semidirect product $H^l \rtimes G$, where the action of G on H^l is the component-wise action of G on H. Let $\Psi_Y \colon W_Y \to Y$ denote the covering morphism. Again applying Proposition 6.4 with H replaced by H^l we obtain an irreducible normal $\Gamma^{(l)}$-cover $W \to T$ satisfying conclusions (1) to (5) of Proposition 6.3 with H replaced by H^l and Γ replaced by $\Gamma^{(l)}$ everywhere. Also, by Lemma 6.5, we know that the morphism $T \to F$, where F is the locus of $xy - t = 0$ in $D \times_k \mathbb{P}^1_y \times_k \operatorname{Spec}(k[[t]])$, induces an abelian cover $T \times_{\operatorname{Spec}(k[[t]])} \operatorname{Spec}(k((t))) \to D \times_k \operatorname{Spec}(k((t)))$. Again, for any $k[[t]]$-scheme V, let V^g denote its generic fiber. As in affine line case, we get the following tower of covers of $k[[t]]$-schemes:

$$\begin{tikzcd}
W^g \arrow[d] \\
W^g/H^l \cong W^g_{XT} \arrow[d, "G\text{-cover étale at points lying above } C\times_k \operatorname{Spec}(k((t)))"] \\
T^g \arrow[d, "\text{abelian cover étale over } C\times_k \operatorname{Spec}(k((t)))"] \\
D \times_k \operatorname{Spec}(k((t)))
\end{tikzcd}$$

FIGURE 4

However, now W^g is a $\Gamma^{(l)}$-cover of T^g and an H^l-cover of W^g_{XT}. So now we can get many disjoint H-covers of W^g_{XT} which leads to Γ-covers of T^g. And this helps in obtaining a Γ-cover of maximal étale abelian "procover" of C. But first applying Proposition 6.9, the above tower descends to a tower of k-covers with the same ramification properties and group actions as above.

$$\begin{tikzcd}
W^s \arrow[d] \\
W^s/H^l \cong W^s_{XT} \arrow[d, "G\text{-cover étale at points lying above } C"] \\
T^s \arrow[d, "\text{abelian cover étale over } C"] \\
D
\end{tikzcd}$$

FIGURE 5

For each i, with $1 \leq i \leq l$, let ${}^iW^s$ denote the quotient of W^s by the subgroup $H_i = H \times \cdots \times \hat{H} \times H \times \cdots \times H$ of H^l with i^{th} factor of H missing. Note that H_i is a subgroup of $\Gamma^{(l)}$ and hence acts on W^s. Also note that H_i is a normal subgroup of $\Gamma^{(l)}$ and $\Gamma^{(l)}/H_i \cong \Gamma$ for each i. So each of these ${}^iW^s$ is a Γ-cover of T^s. Moreover, by construction, these Γ-covers are linearly disjoint over $W^s_{XT} \cong W^s/H^l$. So to complete the proof, it is enough to show that for at least one i, the Galois group of $k({}^iW^s)K^{ab}$ over K^{ab} is Γ, where $k({}^iW^s)$ is the function field of ${}^iW^s$. As observed previously $k(W^s_{XT})K^{ab} = k(W_X)K^{ab}$ and by assumption the Galois group of $k(W_X)K^{ab}$ over K^{ab} is G. So it is enough to show that the Galois group of $k({}^iW^s)K^{ab}$ over $k(W^s_{XT})K^{ab}$ is H for some i. Since H is a minimal normal subgroup of Γ, $H \cong \mathbb{S} \times \mathbb{S} \times \cdots \times \mathbb{S}$ for some simple group $\mathbb{S}$. If $\mathbb{S}$ is nonabelian then $\mathbb{S}$ and hence H is perfect. $\mathrm{Gal}(k({}^iW^s)/k(W^s_{XT}))$ is perfect and $k(W^s_{XT})K^{ab}/k(W^s_{XT})$ is a pro-abelian field extension, so they are linearly disjoint. Hence $\mathrm{Gal}(k({}^iW^s)K^{ab}/k(W^s_{XT}K^{ab}) \cong H$. If $\mathbb{S}$ is abelian then $\mathbb{S} \cong \mathbb{Z}/q\mathbb{Z}$ for some prime q different from p. By Grothendieck's result on the prime-to-p part of the fundamental group (see Theorem 3.4), there are only finitely many nontrivial surjections from $\pi_1(C)$ to H. These epimorphisms correspond to the H-covers Z_j of D which are étale over C. Since we could have chosen l to be any integer, let l be an integer greater than the number of such H-covers of D. $Z_j \times_D W^s_{XT}$ may still be irreducible H-cover of W^s_{XT} for some j. We choose an i such that ${}^iW^s$ is different from $Z_j \times_D W^s_{XT}$ for all j. For such an i, $k({}^iW^s)$ is linearly disjoint with $k(W^s_{XT})K^{ab}$ over $k(W_{XT})$, since subfields of $k(W^s_{XT})K^{ab}$ which are finite extensions of $k(W^s_{XT})$ are in bijective correspondence with the covers of W^s_{XT} obtained from base change of étale covers of C. So we have found a proper solution to the given embedding problem. □

This concludes the proof of the main theorem (Theorem 4.8) of this thesis.

7. A few more consequences and a conjecture

In this section a few more consequences of the proof of the main theorem will be stated. Let G_p be the maximal p-torsion quotient group of $\pi_1(C)$, i.e., $G_p = \varprojlim Gal(k(Y)/k(C))$, where $Y \to C$ is a Galois étale cover with the Galois group $(\mathbb{Z}/p\mathbb{Z})^l$ for some l. Let K_p be the kernel of the quotient map.

$$1 \to K_p \to \pi_1(C) \to G_p \to 1$$

The same argument as was used to prove $\pi_1^c(C)$ to be a projective group in Theorem 4.6, also proves the projectivity of K_p. That every split quasi-p perfect embedding problem and every split abelian p-group embedding problem for K_p have solutions also follows in the same way as for $\pi_1^c(C)$. Even the prime-to-p embedding problem for K_p has a solution. To see this note that the Galois group of the cover $Y \to \mathbb{P}^1_y$ in the Lemma 6.10 is a p-torsion group. Hence the proof of the general case can be modified to get a Γ-cover dominating the given G-cover after passing to an appropriate $(\mathbb{Z}/p\mathbb{Z})^n$-cover of C. Then the standard linear disjointness argument as in the affine line case in Theorem 6.11 provides solution to every prime-to-p embedding problem. In this whole argument G_p can be replaced by G_{p^n} (and consequently K_p can be replaced by K_{p^n}) for every positive integer n. Hence we obtain the following result.

Theorem 7.1. *For every positive integer n, K_{p^n}, as defined above, is a profinite free group of countably infinite rank.*

The main theorem (Theorem 4.8) requires the base field to be countable so a natural question is what happens when the base field is uncountable. For uncountable base field it is easy to see that the rank of the commutator subgroup is the cardinality of the field. Moreover, for these fields as well one solution to any of the embedding problems mentioned in Proposition 4.7 is guaranteed by Theorem 5.3, Theorem 5.5 and Theorem 6.12. But to assert the freeness of the commutator subgroup when the base field k is uncountable one has to exhibit $\operatorname{card} k$ solutions to these embedding problems. The author believes that this should be the case.

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