SUBGROUP STRUCTURE OF FUNDAMENTAL GROUPS IN POSITIVE CHARACTERISTIC

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Abstract. Let Π be the étale fundamental group of a smooth affine curve over an algebraically closed field of characteristic $p > 0$. We establish a criterion for profinite freeness of closed subgroups of Π. Roughly speaking, if a closed subgroup of Π is "captured" between two normal subgroups, then it is free, provided it contains most of the open subgroups of index p. In the proof we establish a strong version of "almost ω-freeness" of Π and then apply the Haran-Shapiro induction.

1. Introduction

The étale fundamental group of a variety over a field of characteristic $p > 0$ is a mysterious group. Even in the case of smooth affine curves it is not yet completely understood. The prime-to-p part of the group is understood because of Grothendieck's Riemann Existence Theorem [19, XIII, Corollary 2.12]. The main difficulty arises from the wildly ramified covers.

Let C be a smooth affine curve over an algebraically closed field k of characteristic $p > 0$. Let X be the smooth completion of C, g the genus of X and $r = \mathrm{card}(X \smallsetminus C)$. Let $\pi_1(C)$ be the étale fundamental group of C. This is a profinite group and henceforth the terminology should be understood in the category of profinite groups, e.g. subgroups are closed, homomorphisms are continuous, etc. For a finite group G, the quasi-p subgroup $p(G)$ is the subgroup generated by all p-sylow subgroups of G. If $G = p(G)$ then G is said to be a quasi-p group.

Abhyankar's conjecture which was proved by Raynaud [17] and Harbater [8] classifies the finite quotients of $\pi_1(C)$, namely a finite group G is a quotient of $\pi_1(C)$ if and only if $G/p(G)$ is generated by $2g + r - 1$ elements. In particular, $\pi_1(C)$ is not finitely generated, hence it is not determined by the set of its finite quotients.

Recently people have tried to understand the structure of $\pi_1(C)$ by studying its subgroups. In [12] the second author shows that the commutator subgroup is free of countable rank, provided k is countable. In [14] Pacheco, Stevenson and Zalesskii make an attempt to understand the normal subgroups N of $\pi_1(C)$, again when k is countable. In [10] Harbater and Stevenson show that $\pi_1(C)$ is almost ω-free, in the sense that every finite embedding problem has a proper solution after restricting to an open subgroup (see also [11]). If k is uncountable, the result of [12] is extended in [13]. We prove a diamond theorem for $\pi_1(C)$ when k is countable, Theorem 1.1 below, by using Theorem 1.3 below which strengthens [10, Theorem 6].

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Let us explain the main result of this paper in detail. We say that a subgroup H of a profinite group Π lies in a Π-diamond if there exist $M_1, M_2 \lhd \Pi$ such that $M_1 \cap M_2 \le H$, but $M_1 \not\le H$ and $M_2 \not\le H$. Haran's diamond theorem states that a subgroup H of a free profinite group of infinite rank that lies in a diamond is free [6]. In [1] the first author extends the diamond theorem to free profinite groups of finite rank ≥ 2. Many subgroups lie in a diamond, notably, the commutator and any proper open subgroup of a normal subgroup.

The diamond theorem is general in the sense that most of the other criteria follow from it, and it applies also to non-normal subgroups, in contrast to Melnikov's theorem [5, Theorem 25.9.2]. Moreover, there are analogue of diamond theorem for Hilbertian fields [7] and other classes of profinite groups. In [2] Haran, Harbater, and the first author prove a diamond theorem for semi-free profinite groups, and in [3] Stevenson, Zalesskii and the first author establish a diamond theorem for $\pi_1(C)$, where C is a projective curve of genus at least 2 over an algebraically closed field of characteristic 0.

For a smooth affine curve Z, let $\bar{Z}$ denote the smooth completion of Z. For each $g \ge 0$, we define a subgroup of $\pi_1(C)$,

$$P_g(C) = \bigcap \{\pi_1(Z) \mid Z \to C \text{ is étale } \mathbb{Z}/p\mathbb{Z}\text{-cover and the genus of } \bar{Z} \ge g\}.$$

We note that $P_0(C)$ is the intersection of all open normal subgroups of index p and we have $P_{g+1}(C) \ge P_g(C)$.

Our main result is the following diamond theorem for subgroups of $\pi_1(C)$ that are contained in $P_g(C)$ for some $g \ge 0$.

Theorem 1.1. *Let k be a countable algebraically closed field of characteristic $p > 0$, let C be a smooth affine k-curve, let $\Pi = \pi_1(C)$, let g be a non-negative integer, and let M be a subgroup of Π. Assume $M \le P_g(C)$ and there exist normal subgroups M_1, M_2 of Π such that M contains $M_1 \cap M_2$ but contain neither M_1 or M_2. Then:*

(i) *For every finite simple group S the direct power S^∞ is a quotient of M.*
(ii) *Assume further that $[MM_i : M] \ne p$ for $i = 1, 2$. Then M is free of countable rank.*

This generalizes [12, Theorem 4.8]. Let Π' be the commutator subgroup of Π. Then $\Pi' \le P_0(C)$, so Π/Π' is a non finitely generated abelian profinite group, hence $\Pi/\Pi' = A \times B$ with A, B infinite. So Π' is the intersection of the preimages of A and B in Π, hence lies in a diamond as in the above theorem. Since A and B are infinite, the hypothesis of Theorem 1.1(ii) also holds for Π'.

We note that Theorem 1.1 implies that for every $g \ge 0$ the subgroup $P_g(C)$ is free. However we do not know whether Theorem 1.1 follows from this latter assertion, because it is not clear that M lies in a $P_g(C)$-diamond even if it contained in a $\pi_1(C)$-diamond.

Although a normal subgroup N of $\pi_1(C)$ of infinite index is not necessarily free, every proper open subgroup of N is free, provided it is contained in $P_g(C)$ for some g (cf. Lubotzky-v.d. Dries' theorem [5, Proposition 24.10.3]).

Corollary 1.2. *Let k be a countable algebraically closed field of characteristic $p > 0$, let C be a smooth affine k-curve, let $\Pi = \pi_1(C)$. Let N be a normal subgroup of Π of infinite index and let M be a proper open subgroup of N that is contained in $P_g(C)$ for some $g \ge 0$. Then M is free.*

A stronger form of this corollary appears in [14]. There instead of assuming $M \leq P_g(C)$, it is assumed that Π/N has a "big" p-sylow subgroup. However there is a gap in [14]. The gap can be fixed under the assumption that $N \leq P_g(C)$. Note that under the assumption $N \leq P_g(C)$, the assertion of Corollary 1.2 follows from Lubotzky-v.d. Dries' theorem for free groups because $P_g(C)$ is free. But in general, it seems that one cannot apply the group theoretical theorem directly.

If $[N : M] \neq p$ the corollary follows directly from Theorem 1.1 Part (ii). For the general case we need the diamond theorem for free groups, Melnikov's theorem on normal subgroups of free groups, and Theorem 1.1 Part(i). See §3.4 for the proof.

The proof of Theorem 1.1 contains two key ingredients. We first prove a geometric result on solvability of embedding problems for $\pi_1(C)$. This geometric result and the Haran-Shapiro induction, see [2] is used in the second step which is group theoretic in nature.

For the geometric result, we need a piece of notation. An open subgroup Π^o of Π is the étale fundamental group of an étale cover D of C. We say Π^o *corresponds to a curve of genus* g if the genus of the smooth completion of D is g.

Theorem 1.3. *Let k be an algebraically closed field of characteristic $p > 0$, let C be a smooth affine k-curve, $\Pi = \pi_1(C)$, g a positive integer, and $\mathcal{E} = (\mu\colon \Pi \to G, \alpha\colon \Gamma \to G)$ a finite embedding problem for Π. Then there exists an open normal index p subgroup Π^o of Π such that:*

(1) $\mu(\Pi^o) = G$.
(2) Π^o *corresponds to a curve of genus at least g.*
(3) *The restricted embedding problem $(\mu|_{\Pi^o} : \Pi^o \to G, \alpha : \Gamma \to G)$ is properly solvable.*

Harbater and Stevenson prove that $\Pi = \pi_1(C)$ is almost ω-free, i.e., there exists an open normal subgroup Π^o satisfying (1) and (3) [10, Theorem 6]. The proof of [10, Theorem 6] is based on "adding branch points". In [11] Jarden proves that a profinite group Γ is almost-ω free under the mild condition that A_n^m is a quotient of Γ for all sufficiently large n and m (here A_n is the alternating group). Since A_n^m is a quasi-p group, if $n \geq p$, A_n^m is a quotient of $\pi_1(C)$ by the result of Raynaud [17] and Harbater [8]. Hence Jarden's result implies [10, Theorem 6]. We note that both Harbater-Stevenson's and Jarden's methods do not give any bound on the index of Π^o in Π, which is essential for applications, e.g. to the proof of Theorem 1.1. Our proof relies on the constructions in [12]. We use an auxiliary curve of high genus to construct an étale p-cyclic cover of C such that the given embedding problem has a proper solution when restricted to the fundamental group of this cover. This fundamental group is the subgroup Π^o of Theorem 1.3. The hypothesis that $M \leq P_g(C)$ in Theorem 1.1 ensures that M is contained in Π^o. The existence of such Π^o for any given embedding problem is used in the second step.

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2. Solutions of embedding problems for the fundamental group

An *embedding problem* $\mathcal{E} = (\mu : \Pi \to G, \alpha : \Gamma \to G)$ for a profinite group Π consists of profinite group epimorphisms μ and α. We call $H = \ker \alpha$ the *kernel* of the embedding problem.

$$\begin{tikzcd}
 & & & \Pi \arrow[d, "\mu"] \arrow[dl, dashed, "\varphi"'] & \\
1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r, "\alpha"] & G \arrow[r] & 1
\end{tikzcd}$$

The embedding problem is *finite* (resp. *split*) if Γ is finite (resp. α splits). A *weak solution* of $\mathcal{E}$ is a homomorphism $\varphi\colon \Pi \to \Gamma$ such that $\alpha \circ \varphi = \mu$. A *proper solution* is a surjective weak solution φ.

Let k be an algebraically closed field of characteristic $p > 0$. Let C be a smooth affine k-curve, $\Pi = \pi_1(C)$, and $(\mu : \Pi \to G, \alpha : \Gamma \to G)$ a finite embedding problem for Π. Let H be the kernel of α.

2.1. Prime-to-p kernel. In the rest of the section we shall prove the existence of proper solutions of finite embedding problems with prime-to-p kernels when restricting to a normal subgroup of index p:

Proposition 2.1. *Let g be a positive integer and let $\mathcal{E} = (\mu : \Pi \to G, \alpha : \Gamma \to G)$ be a finite split embedding problem for Π with $H = \ker \alpha$ prime-to-p. Then there exists an open normal subgroup Π^o of Π of index p such that*

(1) *The image $\mu(\Pi^o) = G$;*
(2) *The subgroup Π^o corresponds to a curve of genus at least g;*
(3) *The restricted embedding problem, $(\mu|_{\Pi^o} : \Pi^o \to G, \alpha : \Gamma \to G)$, is properly solvable.*

Let K^{un} denote the compositum (in some fixed algebraic closure of $k(C)$) of the function fields of all finite étale covers of C. Let X be the smooth completion of C. By a strong version of Noether normalization theorem ([4, Corollary 16.18]), there exists a finite surjective k-morphism $\Phi_0 : C \to \mathbb{A}^1_x$ which is generically separable. Here x denotes the local coordinate of the affine line. The branch locus of Φ_0 is of codimension 1, hence Φ_0 is étale away from finitely many points. By translation we may assume none of these points map to $x = 0$. Note that Φ_0 extends to a finite surjective morphism $\Phi_X : X \to \mathbb{P}^1_x$. Let $\{r_1, \dots, r_N\} = \Phi_X^{-1}(\{x = 0\})$, then Φ_X is étale at $r_1, \dots, r_N$.

We use the construction in [12, Section 6], which we recall for the reader's convenience. Let $\Phi_Y : Y \to \mathbb{P}^1_y$ be a p-cyclic cover between smooth curves, locally given by $Z^p - Z - y^{-r} = 0$, for some r that is prime to p. The genus of Y is

$$g_Y = (p-1)(r-1)/2 \tag{1}$$

and Φ_Y is totally ramified at $y = 0$ and étale elsewhere (see [16]). Let F be the zero locus of $t - xy$ in $\mathbb{P}^1_x \times_{\mathrm{Spec}(k)} \mathbb{P}^1_y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[[t]])$. Let X_F and Y_F be the normalization of an irreducible dominating component of the product $X \times_{\mathbb{P}^1_x} F$ and $Y \times_{\mathbb{P}^1_y} F$, respectively. Let T be the normalization of an irreducible dominating component of $X_F \times_F Y_F$. We summarize the situation in Figure 2.1.

Lemma 2.2. *The fiber of the morphism $T \to X_F$ over $\mathrm{Spec}(k((t)))$ induces a $\mathbb{Z}/p\mathbb{Z}$-cover $T \times_{\mathrm{Spec}(k[[t]])} \mathrm{Spec}(k((t))) \to X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k((t)))$ which is étale over C.*

$$\begin{tikzcd}
 & & T \arrow[dl] \arrow[dr] & & \\
 & X_F \arrow[dl] \arrow[dr] & & Y_F \arrow[dl] \arrow[dr] & \\
X \arrow[dr] & & F \arrow[dl] \arrow[dr] & & Y \arrow[dl] \\
 & \mathbb{P}^1_x & & \mathbb{P}^1_y &
\end{tikzcd}$$

FIGURE 2.1

Proof. Note that $k(X_F) = k((t))k(X)$ because generically $t \neq 0$ and over $t \neq 0$ the morphism $F \to \mathbb{P}^1_x$ is the base change $\mathbb{P}^1_x \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k((t))) \to \mathbb{P}^1_x$. Also note that over $t \neq 0$ the local coordinates x and y satisfy the relation $y = t/x$ on T. Therefore, since Y_F is the base change of $Y \to \mathbb{P}^1_y$, it follows that $k(Y_F) = k((t))(x)[Z]/(Z^p - Z - (x/t)^r)$. Since T is a dominating component of $X_F \times_F Y_F$, we have

$$k(T) = k(X_F)k(Y_F) = k((t))k(X)(k(x)[Z]/(Z^p - Z - (x/t)^r)).$$

But $Z^p - Z - (x/t)^r$ is irreducible over $k(X)k((t))$, hence $k(T)/k(X_F)$ is a Galois extension with Galois group $\mathbb{Z}/p\mathbb{Z}$. Finally since the cover $T \times_{\mathrm{Spec}(k[[t]])} \mathrm{Spec}(k((t))) \to X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k((t)))$ is given by the equation $Z^p - Z - (x/t)^r = 0$, it is étale away from $x = \infty$. In particular, it is étale over C. $\square$

Since $\mathrm{Gal}(K^{un}/k(C)) = \pi_1(C)$, the surjection $\mu\colon \pi_1(C) \to G$ induces an irreducible normal G-cover $\Psi_X : W_X \to X$ which is étale over C. Since $g_Y = (p-1)(r-1)/2$, we can choose a prime-to-p integer r to be sufficiently large so that $g_Y \geq \max\{|H|, g\}$. Then there exist an irreducible smooth étale H-cover $\Psi_Y : W_Y \to Y$, since H is a prime-to-p group ([19, XIII, Corollary 2.12]).

Proposition 2.3. *There exist an irreducible normal Γ-cover $W \to T$ of $k[[t]]$-curves, a regular finite type $k[t, t^{-1}]$-algebra B contained in $k((t))$ and an open subset S of* $\mathrm{Spec}(B)$ *such that the Γ-cover $W \to T$ descends to a Γ-cover of connected normal projective* $\mathrm{Spec}(B)$*-schemes $W_B \to T_B$ and for any closed point s in S we have the following:*

(1) *The fiber T_s of T_B at s is a smooth irreducible $\mathbb{Z}/p\mathbb{Z}$-cover of X étale over C.*
(2) *The fiber $W_s \to T_s$ of $W_B \to T_B$ at s is a smooth irreducible Γ-cover such that the composition $W_s \to T_s \to X$ is étale over C.*
(3) *The quotient W_s/H is isomorphic to an irreducible dominating component of $W_X \times_X T_s$ as G-covers of T_s. In particular, $k(W_X)$ and $k(T_s)$ are linearly disjoint over $k(X)$.*

Proof. By [12, Proposition 6.4] there exists a normal irreducible Γ-cover $W \to T$ of $k[[t]]$-curves with prescribed properties. We claim that this cover is étale away from the points lying above $x = \infty$ in T.

Indeed, by Conclusion (1) of [12, Proposition 6.4], we have

$$W \times_T \widetilde{T_X} = \mathrm{Ind}_G^\Gamma(\widetilde{W_{XT} \times_T T_X}), \tag{2}$$

where tilde denotes the (t)-adic completion, $T_X = T \setminus \{x = 0\}$, W_{XT} is the normalization of an irreducible dominating component of $W_X \times_X T$, and for any variety V, $\mathrm{Ind}_G^\Gamma V = V^\Gamma / \sim$, where $(\gamma, v) \sim (\gamma', v')$ if and only if $\gamma' = \gamma g^{-1}$ and $v' = gv$, for some $g \in G$. Note that the left hand side of (2) is the (t)-adic completion of $W \times_T T_X$ hence the branch locus of $W \times_T T_X \to T_X$ and $W \times_T \tilde{T}_X \to \tilde{T}_X$ are same. On the other hand, the right hand side of (2) is the disjoint union of copies of $\widetilde{W_{XT} \times_T T_X}$, so the branch locus of $W \times_T \tilde{T}_X \to \tilde{T}_X$ maps to the branch locus of $W_X \to X$ under the morphism $T_X \to X$. Thus, since $W_X \to X$ is ètale away from $\{x = \infty\}$, so is $W \times_T T_X \to T_X$. Similarly, using (1') of [12, Proposition 6.4] and the fact that $W_Y \to Y$ is étale everywhere, we get that over $T_Y = T \setminus \{y = 0\}$ the cover $W \to T$ is étale. It follows from Conclusion (2) of [12, Proposition 6.4] that $W \to T$ is étale over $\{x = y = 0\}$, as needed.

By Conclusion (5) of [12, Proposition 6.4] $W/H \cong W_{XT}$ as G-covers of T, hence we identify them. By Lemma 2.2, away from $t = 0$, $T \to X \times_k \mathrm{Spec}(k[[t]])$ is a $\mathbb{Z}/p\mathbb{Z}$-cover.

Now we apply "Lefschetz's type principle", [12, Proposition 6.9], to the proper surjective morphisms of projective $k[[t]]$-curves

$$W \to W_{XT} \to T \to X \times_k \mathrm{Spec}(k[[t]])$$

and the points $r_1, \ldots, r_N$ in X lying over $x = \infty$. The proof of [12, Proposition 6.9] shows that there exists an open subset S of the spectrum of a $k[t, t^{-1}]$-algebra B such that the above morphisms descend to the morphisms of $\mathrm{Spec}(B)$-schemes

$$W_B \to W_{XT,B} \to T_B \to X \times_k \mathrm{Spec}(B)$$

and the fiber over every point s in S leads to covers of smooth irreducible curves $W_s \to T_s$ and $T_s \to X$ with the desired ramification properties and Galois groups. This proves Conclusions (1) and (2). For (3), we note that $W_s \to T_s$ factors through $W_{XT,s}$ and since W/H is isomorphic to W_{XT}, W_s/H is isomorphic to $W_{XT,s}$. So $W_{XT,s}$ is an irreducible G-cover of T_s. Also $W_{XT,s}$ is an irreducible dominating component of $W_X \times_X T_s$. Finally, the compositum $k(W_X)k(T_s)$ is the function field of $W_{XT,s}$ and it is a Galois extension of $k(T_s)$ with Galois group G. Also $\mathrm{Gal}(k(W_X)/k(X)) = G$, so $k(T_s)$ and $k(W_X)$ are linearly disjoint over $k(X)$.

Note that the statement of [12, Proposition 6.9] only asserts the existence of the fibers W_s, T_s, etc. and covering morphisms between them with the desired Galois group and ramification properties. Since the morphisms of $k[[t]]$-curves $W \to T$ and $T \to X_F$ are finite, they descend to morphisms of a regular finite type $k[t, t^{-1}]$-algebra B contained in $k((t))$. In the proof of [12, Proposition 6.9] it is further shown that there exists an open subset of $\mathrm{Spec}(B)$ such that fibers over any point in this open subset have the desired properties. ∎

Proof of Proposition 2.1. Using the notation of the above proposition, let $\Sigma \subset T_s$ be the preimage of $X \setminus C$ under the covering $T_s \to X$, let $T_s^o = T_s \setminus \Sigma$, and let $\Pi^o = \pi_1(T_s^o)$. By (1) of Proposition 2.3, $k(T_s^o)/k(C)$ is a Galois extension with Galois group $\mathbb{Z}/p\mathbb{Z}$. If $Z \to T_s^o$ is an étale cover of T_s^o then the composition with $T_s^o \to C$ gives an étale cover $Z \to C$. Conversely, if $Z \to C$ is an étale cover of C which factors through $T_s^o \to C$, then $Z \to T_s^o$ is also étale. So $\Pi^o = \mathrm{Gal}(K^{un}/k(T_s)) = \pi_1(T_s^o)$ is an index p normal subgroup of $\mathrm{Gal}(K^{un}/k(C)) = \pi_1(C)$.

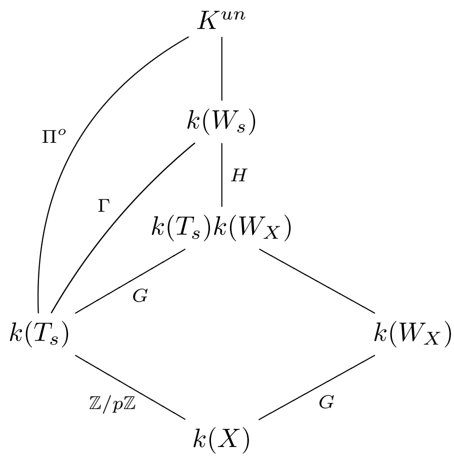
For (2), note that by construction of T (see Figure 2.1), T dominates Y_F, so over the generic point, the genus of the $k((t))$-curve $T \times_{\mathrm{Spec}(k[[t]]]} \mathrm{Spec}(k((t))$ is at least the genus of Y. But $g_Y \geq g$, so this genus is at least g. Since T_s is a smooth fiber of T_B which in turn descends from T, the genus of T_s is also at least g.

Since the G-cover $W_X \to X$ is induced by the epimorphism $\mu\colon \Pi \to G$, we have $\ker\mu = \mathrm{Gal}(K^{un}/k(W_X))$. So $(\ker\mu) \cap \Pi^o = \mathrm{Gal}(K^{un}/k(W_X)k(T_s))$. Hence

$$\mu(\Pi^o) = \Pi^o/((\ker\mu) \cap \Pi^o) = \mathrm{Gal}(k(W_X)k(T_s)/k(T_s)).$$

But $\mathrm{Gal}(k(W_X)k(T_s)/k(T_s)) = G$ by (3) of Proposition 2.3, so we get $\mu(\Pi^o) = G$, as needed.

Also from Proposition 2.3, we get a Γ-cover W_s of T_s such that $k(W_X) \subset k(W_s)$ and we obtain the following tower of field extensions:



Hence there is a surjection from $\Pi^o \twoheadrightarrow \Gamma = \mathrm{Gal}(k(W_s)/k(T_s))$ which dominates $\mu|_{\Pi^o}\colon \Pi^o \twoheadrightarrow G$. Hence W_s provides a proper solution to the embedding problem restricted to Π^o. $\square$

2.2. **Proof of Theorem 1.3.**

Let k be an algebraically closed field of characteristic $p > 0$, let C be a smooth affine k-curve, $\Pi = \pi_1(C)$, g a positive integer, and $\mathcal{E} = (\mu\colon \Pi \to G, \alpha\colon \Gamma \to G)$ a finite embedding problem for Π. Put $H = \ker\alpha$.

Since Π is projective ([18, Proposition 1]), $\mathcal{E}$ has a weak solution. When this happens, it is well known that there exists a finite split embedding problem $\mathcal{E}'$ with the same kernel H, such that a proper solution of $\mathcal{E}'$ induces a proper solution of $\mathcal{E}$ (even when restricted to a subgroup Π^o with $\mu(\Pi^o) = G$), see for example [9, Discussion after Lemma 2.3]. Hence, without loss of generality, we can assume that $\mathcal{E}$ splits.

Let $p(H)$ be the quasi-p subgroup of H. Since $p(H)$ is characteristic in H, it is normal in Γ. Then there is $\alpha'\colon \Gamma/p(H) \to G$ such that $\alpha = \alpha' \circ \alpha''$, where $\alpha''\colon \Gamma \to \Gamma/p(H)$ is the natural quotient map.

Since $\ker\alpha'$ is prime-to-p, by Proposition 2.1, there exists an index p normal subgroup Π^o of Π such that $\mu(\Pi^o) = G$, Π^o corresponds to a curve of genus at least

g, and the embedding problem

$$(\mu|_{\Pi^o} : \Pi^o \to G, \alpha' \colon \Gamma/p(H) \to G)$$

has a proper solution, say $\mu' \colon \Pi^o \to \Gamma/p(H)$.

The kernel of α'' is $p(H)$ which by definition is quasi-p. So using Pop's result ([15, Theorem B]) the embedding problem

$$(\mu' : \Pi^o \to \Gamma/p(H), \alpha'' \colon \Gamma \to \Gamma/p(H))$$

has a proper solution, say φ. Then

$$\alpha \circ \varphi = \alpha' \circ \alpha'' \circ \varphi = \alpha' \circ \mu' = \mu|_{\Pi^o}.$$

Hence φ is also a proper solution of $(\mu|_{\Pi^o} : \Pi^o \to G, \alpha \colon \Gamma \to G)$, as needed. □

3. Proof of Theorem 1.1

3.1. **The Haran-Shapiro induction.** We recall the notion of twisted wreath products and its application in solutions of split embedding problems via the Haran-Shapiro induction. We refer the reader to [2] for more details and proofs.

Let A, G_0, and G be finite groups with $G_0 \leq G$. Assume G_0 acts on A and for $a \in A$, $\rho \in G_0$ let a^ρ denote the action. Let $I = \mathrm{Ind}_{G_0}^G(A) = \{f \colon G \to A \mid f(\sigma\rho) = f(\sigma)^\rho,\ \forall \sigma \in G, \rho \in G_0\}$ be the induced G-group. Then $I \cong A^{(G:G_0)}$ as groups and G acts on I by the formula

$$f^\sigma(\tau) = f(\sigma\tau), \qquad \sigma, \tau \in G.$$

We denote the corresponding semidirect product by $A \wr_{G_0} G := I \rtimes G$, and we refer to it as the twisted wreath product. It comes with a canonical projection map $\alpha \colon A \wr_{G_0} G \to G$. If A_0 is a normal subgroup of A that is invariant under the action of G_0 we will say that $\bar{A} = A/A_0$ is a quotient of A through which the action of G_0 on A descends.

For the reader's convenience we formulate the following special case of [2, Proposition 4.6] that is built on [6, Theorem 2.2] which will be used.

Theorem 3.1. *Let $M \leq \Pi$ be profinite groups, let A, G_1 be finite groups together with an action of G_1 on A, and let*

$$\mathcal{E}_1(A) = (\mu \colon M \to G_1, \alpha_1 \colon A \rtimes G_1 \to G_1)$$

be a finite split embedding problem for M. Let D, Π_0, L be open subgroups of Π such that

1. *D is a normal subgroup of Π with $M \cap D \leq \ker \mu$,*
2. *$M \leq \Pi_0 \leq MD$,*
3. *L is a normal subgroup of Π with $L \leq \Pi_0 \cap D$.*

Put $G = \Pi/L$, $G_0 = \Pi_0/L \leq G$, and let $\varphi \colon \Pi \to G$ be the quotient map. Then there is an epimorphism $\varphi_1 \colon G_0 \to G_1$, through which the action of G_0 on A is defined. Moreover, assume that there exists a closed normal subgroup N of Π with $N \leq M \cap L$ such that there is no nontrivial quotient $\bar{A}$ of A through which the action of G_0 on A descends and for which the finite split embedding problem

$$\mathcal{E}(N, \bar{A}) = (\bar{\varphi} \colon \Pi/N \to G, \alpha \colon \bar{A} \wr_{G_0} G \to G),$$

where $\bar{\varphi}$ is the quotient map, is properly solvable. Then any proper solution of

$$\mathcal{E}(A) = (\varphi \colon \Pi \to G, \alpha \colon A \wr_{G_0} G \to G),$$

where φ is the quotient map, induces a proper solution of $\mathcal{E}_1(A)$.

3.2. **Special cases.** Let k be a countable algebraically closed field of characteristic $p > 0$, let C be an affine k-curve, let $\Pi = \pi_1(C)$, let g be a non-negative integer, and let $M \leq P_g(C)$ be a subgroup of Π. In particular $[\Pi : M] = \infty$. Assume that there exist normal subgroups M_1, M_2 of Π such that $M_1 \cap M_2 \leq M$ and $M_1, M_2 \not\leq M$.

Proposition 3.2. *Assume $[MM_1 : M] = \infty$ and $[MM_2 : M] \neq 1, p$. Let*

$$\mathcal{E}_1(A) = (\mu \colon M \to G_1, \alpha_1 \colon A \rtimes G_1 \to G_1)$$

be a finite split embedding problem for M. Then $\mathcal{E}_1$ is properly solvable.

Proof. There exists an open normal subgroup L of Π such that $M \cap L \leq \ker \mu$ and the following conditions hold:

(A) $[M_1ML : ML] = [M_1 : M_1 \cap (ML)] \geq 3p$.
(B) $[M_2ML : ML] = [M_2 : M_2 \cap (ML)] \neq 1, p$.
(C) $[\Pi : ML] \geq 3p$.

Note that any sufficiently small open normal subgroup L of Π satisfies (A) because $[M_1M : M] = \infty$, and (B) because $[M_2M : M] \neq 1, p$, and (C) because $[\Pi : M] = \infty$.

Let $G = \Pi/L$, $\varphi \colon \Pi \to G$ the natural epimorphism, and $G_0 = \varphi(M) = ML/L$. Let $D = L$ and $\Pi_0 = ML$, then by Theorem 3.1 there exists an epimorphism $\varphi_1 \colon G_0 \to G_1$ through which the action of G_0 on A is defined.

Thus

$$\mathcal{E}(A) = (\varphi \colon \Pi \to G, \alpha \colon A \wr_{G_0} G \to G)$$

is a finite split embedding problem for Π. Theorem 1.3 gives an open normal subgroup U of Π of index p that corresponds to a curve of genus at least g such that $\varphi(U) = G$ and such that the restricted embedding problem

$$\tilde{\mathcal{E}}(A) = \mathcal{E}|_U(A) = (\varphi|_U \colon U \to G, \alpha \colon A \wr_{G_0} G \to G) \tag{3}$$

has a proper solution $\psi \colon U \to A \wr_{G_0} G$. Since U corresponds to a curve of genus at least g we have $M \leq P_g(C) \leq U$. Also $\varphi(U) = G$ implies $UL = \Pi$.

$$\begin{tikzcd}
M_i \cap U \arrow[r, dash, "\leq p"] \arrow[d, dash] \arrow[dd, dashed, dash, bend right, "a_i"'] & M_i \arrow[d, dash] \\
M_i \cap (ML) \cap U \arrow[r, dash] \arrow[d, dash] & M_i \cap (ML) \arrow[dd, dash] \\
(M_i \cap U) \cap (M(L \cap U)) \arrow[d, dash] & \\
M_i \cap U \cap M \arrow[r, equal] & M_i \cap M
\end{tikzcd}$$

Let $a_i = [M_i \cap U : (M_i \cap U) \cap (M(L \cap U))]$, for $i = 1, 2$. Then $a_1 \geq \frac{[M_1 : M_1 \cap (ML)]}{p} \geq 3$. If $M_2 \leq U$, then $M_2 \cap U = M_2$, and we have

$$a_2 = [M_2 : M_2 \cap (M(L \cap U))] \geq [M_2 : M_2 \cap ML] > 1.$$

If $M_2 \not\leq U$, then $[M_2 : M_2 \cap U] = p$. Then we have

$$\begin{aligned} p \cdot [M_2 \cap U : M_2 \cap (ML) \cap U] &= [M_2 : M_2 \cap U][M_2 \cap U : M_2 \cap (ML) \cap U] \\ &= [M_2 : M_2 \cap (ML) \cap U] \\ &= [M_2 : M_2 \cap (ML)][M_2 \cap (ML) : M_2 \cap (ML) \cap U]. \end{aligned}$$

By (B) we get that the right hand side does not equal p, hence $[M_2 \cap U : M_2 \cap (ML) \cap U] > 1$, hence $a_2 > 1$. Since $UL = \Pi$, we have

$$[U : M(L \cap U)] \geq [U : U \cap ML] = [UML : ML] = [\Pi : ML] \geq 3p \geq 3.$$

The last paragraph implies that if we set $\tilde{\Pi} = U$, $\tilde{M}_i = M_i \cap U$, $\tilde{M} = M \cap U = M$, and $\tilde{L} = L \cap U$ then we have

($\tilde{\mathrm{A}}$) $[\tilde{M}_1 \tilde{M}\tilde{L} : \tilde{M}\tilde{L}] = [\tilde{M}_1 : \tilde{M}_1 \cap (\tilde{M}\tilde{L})] \geq 3$.
($\tilde{\mathrm{B}}$) $[\tilde{M}_2 \tilde{M}\tilde{L} : \tilde{M}\tilde{L}] = [\tilde{M}_2 : \tilde{M}_2 \cap (\tilde{M}\tilde{L})] \geq 2$.
($\tilde{\mathrm{C}}$) $[\tilde{\Pi} : \tilde{M}\tilde{L}] \geq 3$.

Moreover ψ is a proper solution of $\tilde{\mathcal{E}}(A)$.

Let $K_i = \varphi(\tilde{M}_i) \cong \tilde{M}_i\tilde{L}/\tilde{L}$, $i = 1, 2$. By the third isomorphism theorem ($\tilde{\mathrm{A}}$)-($\tilde{\mathrm{C}}$) can be reformulated as

(a) $[K_1 G_0 : G_0] \geq 3$;
(b) $[K_2 G_0 : G_0] \geq 2$;
(c) $[G : G_0] \geq 3$.

We show that $M \leq \tilde{\Pi}$, $\tilde{D} = \tilde{L}$, $\tilde{\Pi}_0 = \tilde{M}\tilde{L}$, and $\tilde{N} = \tilde{L} \cap \tilde{M}_1 \cap \tilde{M}_2$ satisfy the conditions of Theorem 3.1 to conclude that ψ induces a proper solution of $\mathcal{E}_1(A)$.

Indeed, let $\bar{A}$ be a nontrivial quotient of A through which the action of G_0 on A descends. We need to prove that

$$\mathcal{E}(\tilde{N}, \bar{A}) = (\bar{\varphi}\colon \tilde{\Pi}/\tilde{N} \to G, \bar{\alpha}\colon \bar{A} \wr_{G_0} G \to G)$$

is not properly solvable. Here $\bar{\varphi}$ is the quotient map (recall that $G = \Pi/L \cong \tilde{\Pi}/\tilde{L}$ and $\tilde{N} \leq \tilde{L}$).

Put $H = \bar{A} \wr_{G_0} G$. Assume by negation that there is an epimorphism $\rho\colon \tilde{\Pi} \to H$ with $\bar{\alpha} \circ \rho = \varphi$ and $\rho(\tilde{N}) = 1$. For $i = 1, 2$, let $H_i = \rho(\tilde{M}_i)$. Then H_i is normal in H and $\bar{\alpha}(H_i) = \varphi(\tilde{M}_i) = K_i$. By [5, Lemma 13.7.4(a)] there exists $h_1 \in H_1$ and $h_2 \in H_2$ such that $\bar{\alpha}(h_1) = 1$ and $[h_1, h_2] = h_1 h_2 h_1^{-1} h_2^{-1} \neq 1$.

Let $\gamma_i \in \tilde{M}_i$ be such that $\rho(\gamma_i) = h_i$. Then $\varphi(\gamma_1) = \bar{\alpha}(h_1) = 1$, so $\gamma_1 \in \tilde{L}$. Then

$$[\gamma_1, \gamma_2] \in [\tilde{L}, \tilde{M}_2] \cap [\tilde{M}_1, \tilde{M}_2] \leq \tilde{L} \cap (\tilde{M}_1 \cap \tilde{M}_2) = \tilde{N}.$$

(As $\tilde{N}_1, \tilde{N}_2$ are normal subgroups, we have $[\tilde{N}_1, \tilde{N}_2] \leq \tilde{N}_1 \cap \tilde{N}_2$, because $a^{-1}b^{-1}ab = a^{-1}a^b = b^{-a}b$ for $a \in \tilde{N}_1$ and $b \in \tilde{N}_2$.) We thus get that $[h_1, h_2] = [\rho(\gamma_1), \rho(\gamma_2)] \in \rho(\tilde{N}) = 1$. This contradiction implies that $\mathcal{E}(N, \bar{A})$ is not properly solvable, as needed. □

The following result is very well known.

Lemma 3.3. *Let Γ be a profinite group and G a finite group. Assume that G^n is a quotient of Γ for every $n \geq 1$. Then G^∞ is a quotient of Γ.*

Proof. We construct by induction a sequence of open normal subgroups $G = N_0 > N_1 > N_2 > \cdots$ of Γ such that $\Gamma/N_i = G^i$. Then $\Gamma/\cap_{i=1}^{\infty} N_i = G^\infty$.

Assume we constructed N_i. There exists an integer m that is bigger than the number of subgroups of $\Gamma/N_i = G^i$. Let V be an open normal subgroup of Γ with

$\Gamma/V = G^m$. Then $V = \cap_{k=1}^m V_k$, where V_k is the kernel of the projection onto the k-th factor of Γ/V. In particular, $\Gamma/V_k \cong G$ and $V_k V_l = \Gamma$ for $k \neq l$.

If there exists $1 \leq k \leq m$ such that $V_k N_i = \Gamma$, then $\Gamma/(V_k \cap N_i) \cong \Gamma/V_k \times \Gamma/N_i = G^{i+1}$. So we set $N_{i+1} = V_k \cap N_i$ and we are done. Otherwise, $V_k N_i \neq \Gamma$ for $k = 1, \ldots, m$. Since $(V_k N_i)(V_l N_i) = (V_k V_l) N_i = \Gamma N_i = \Gamma$, for $k \neq l$, the subgroups $V_1 N_i, \ldots, V_m N_i$ are distinct. But this yields that Γ/N_i has at least m subgroup, in contradiction to the choice of m. □

Proposition 3.4. *Let S be a finite simple group. Assume p divides the order of S and $[MM_1 : M] = \infty$ and $[MM_2 : M] \geq 2$. Then S^∞ is a quotient of M.*

Proof. By Lemma 3.3 it suffices to prove that S^n is a quotient of M for every n. Set $A = S^n$ and $G_1 = 1$. Then the embedding problem

$$\mathcal{E}_1(A) = (\mu\colon M \to G_1, \alpha_1\colon A \rtimes G_1 \to G_1) = (M \to 1, S^n \to 1)$$

is properly solvable if and only if S^n is a quotient of Π. As in the proof of Proposition 3.2 there exists an open normal subgroup L of Π such that

(A) $[M_1 ML : ML] = [M_1 : M_1 \cap (ML)] \geq 3$.
(B) $[M_2 ML : ML] = [M_2 : M_2 \cap (ML)] \geq 2$.
(C) $[\Pi : ML] \geq 3$.

Let $G = \Pi/L$, $\varphi\colon \Pi \to G$ the quotient map, $\Pi_0 = ML$, $G_0 = \varphi(M) = ML/L$, and $N = L \cap M_1 \cap M_2$. If we let $K_i = \varphi(M_i) = M_i L/L$, $i = 1, 2$, then, as in the proof of Proposition 3.2, we have

(a) $[K_1 G_0 : G_0] \geq 3$;
(b) $[K_2 G_0 : G_0] \geq 2$;
(c) $[G : G_0] \geq 3$.

Note that since p divides the order of S, S^n is quasi-p. Thus $\mathcal{E}(A) = (\varphi\colon \Pi \to G, \alpha\colon A \wr_{G_0} G \to G)$ is properly solvable.

Exactly the same argument as in the last two paragraphs of the proof of Proposition 3.2 gives that there is no quotient $\bar{A}$ of A through which the action of G_0 on A descends and for which $\mathcal{E}(N, \bar{A})$ is properly solvable. So Theorem 3.1 (with $D = L$) gives a proper solution of $\mathcal{E}_1$. □

Proposition 3.5. *Let S be a finite simple group. Assume p does not divide the order of S and $[MM_1 : M] = \infty$ and $[MM_2 : M] \geq 2$. Then S^∞ is a quotient of M.*

Proof. Again by Lemma 3.3 it suffices to show that the embedding problem

$$\mathcal{E}_1(A) = (\mu\colon M \to G_1, \alpha_1\colon A \rtimes G_1 \to G_1) = (M \to 1, S^n \to 1)$$

is properly solvable for all n. If $[MM_2 : M] \neq p$, then by Proposition 3.2 the embedding problem is properly solvable and we are done. Thus we assume $[MM_2 : M] = p$.

As in the proof of Proposition 3.2 we can choose an open normal subgroup L of Π satisfying (A) and (C) of the proof of Proposition 3.2 and the following (B') instead of (B):

(B') $[M_2 ML : ML] = [M_2 : M_2 \cap (ML)] = p$.

Then we again let $G = \Pi/L$, $\varphi\colon \Pi \to G$ the natural epimorphism, and $G_0 = \varphi(M) = ML/L$. Let $D = L$ and $\Pi_0 = ML$. By Theorem 3.1 there exists an epimorphism $\varphi_1\colon G_0 \to G_1$ through which the action of G_0 on A is defined. The same application as in the proof of Proposition 3.2 gives an open normal subgroup U of Π that corresponds to a curve of an arbitrary large genus g_U such that $M \leq U$, $\varphi(U) = G$, $UL = \Pi$ and the embedding problem

$$\tilde{\mathcal{E}}(A) = \mathcal{E}|_U(A) = (\varphi|_U : U \to G, \alpha \colon A \wr_{G_0} G \to G) \tag{4}$$

has a proper solution $\psi \colon U \to A \wr_{G_0} G$.

If $M_2 \leq U$, then

$$\begin{aligned} a_2 &:= [M_2 \cap U : (M_2 \cap U) \cap (M(L \cap U))] \\ &= [M_2 : M_2 \cap M(L \cap U)] \geq M_2 : M_2 \cap ML] = p > 1. \end{aligned}$$

Hence we can proceed as in the proof of Proposition 3.2 to conclude that $\mathcal{E}_1(A)$ is properly solvable. Note that condition (B) in the proof of Proposition 3.2 was only used to show that $a_2 > 1$.

Next assume that $M_2 \not\leq U$. Thus, since $[\Pi : U] = p$ we have $[M_2 : M_2 \cap U] = p$. Since $M \leq P_g(C) \leq U$ (by the choice of U) and since $[M_2 : M_2 \cap M] = p$ we have $M_2 \cap M = M_2 \cap U$. In particular $U/(M_2 \cap U) \cong \Pi/M_2$ hence U/M_2 is a quotient of Π.

Let X be the smooth completion of C, let g_Π be the genus of X and $r + 1 = |X \smallsetminus C|$. Let ν be the maximal positive integer such that S^ν is a generated by $2g+r$ elements. Since the maximal prime-to-p quotient of Π is free (pro-prime-to-p group) on $2g + r$ elements, ν is the maximal integer such that S^ν is a quotient of Π.

If g_U is sufficiently large, then $S^{\nu+n}$ is generated by $2g_U$ elements. Since the prime-to-p quotient of U is free of rank at least $2g_U$, there exists an epimorphism $\psi \colon U \to S^{\nu+n}$. Let $\Gamma = \psi(M_2 \cap U)$.

Since $M_2 \cap U \triangleleft U$ it follows that $\Gamma \triangleleft S^{\nu+n}$. As Γ is a normal subgroup of a direct power of a finite simple group, $\Gamma \cong S^{\nu_1}$ and $S^{\nu+n} = \Gamma \times \Lambda$, where $\Lambda \cong S^{\nu_2}$ and $\nu_1 + \nu_2 = \nu + n$. We have $\Lambda \cong \psi(U/(M_2 \cap U))$, hence Λ is a quotient of $U/(M_2 \cap U) \cong \Pi/M_2$, and hence of Π. Thus $\nu_2 \leq \nu$, so $\nu_1 \geq n$.

Let $\varphi \colon U \to \Gamma$ be the composition of ψ with the projection onto Γ. Then $\varphi(M_2 \cap U) = \psi(\Gamma \times 1) = \Gamma$. Since $M_2 \cap U \leq M \leq U$ we get that $\varphi(M) = \Gamma$. This finishes the proof because $\Gamma \cong S^{\nu_1}$ with $\nu_1 \geq n$. □

3.3. **Proof of Theorem 1.1.** First note that we can assume $[M_1 : M_1 \cap M] = [MM_1 : M] = \infty$. Indeed, if $[M_1 : M_1 \cap M] = [MM_1 : M] < \infty$, then there exists an open normal subgroup U of Π such that $U \cap M_1 \leq M \cap M_1 \leq M$. Since $[\Pi : M] = \infty$, we have $[U : U \cap M] = [UM : M] = \infty$. Thus we can replace M_1 by U and M_2 by M_1 to get the assumption.

Now Propositions 3.4 and 3.5 imply that for every finite simple group S, S^∞ is a quotient of M. This finishes the proof of the first part of the theorem.

The group Π is projective, hence M, being a subgroup, is also projective [5, Proposition 22.4.7]. Since k is countable, Π is of rank $\aleph_0$, hence $\mathrm{rank}(M) \leq \aleph_0$. Thus to prove that M is free of countable rank it suffices to properly solve any finite split embedding problem for M [9, Theorem 2.1]. By the assumption in the second part of the theorem $[MM_2 : M] \neq p$, hence Proposition 3.2 asserts that every finite split embedding problem for M is solvable, as needed. □

3.4. **Proof of Corollary 1.2.** Let k be a countable algebraically closed field of characteristic $p > 0$, let C be an affine k-curve, let $\Pi = \pi_1(C)$. Let N be a normal subgroup of infinite index and let M be a proper open subgroup of N that is contained in $P_g(C)$ for some $g \geq 0$. We have to prove that M is free.

Let U be an open normal subgroup of Π such that $N \cap U \leq M$. Then M is in the Π-diamond determined by N and U. Note that $[\Pi : M] = \infty$, so $[UM : M] = \infty$. Thus if $[N : M] \neq p$, then M is free by Part (ii) of Theorem 1.1 and we are done.

We thus assume $[N : M] = p$ and divide the proof into two cases.

Case A, assume $N \leq P_g(C)$. As $\Pi/P_g(C) = \mathbb{F}_p^\infty$, we have $\Pi/P_g(C) = V \times W$, where $V \cong W \cong \mathbb{F}_p^\infty$. Therefore by Part (ii) of Theorem 1.1 $P_g(C)$ is free. Then M is a proper open subgroup of a normal subgroup N of a free profinite group $P_g(C)$, thus free [5, Proposition 24.10.3].

Case B, assume $N \not\leq P_g(C)$. Then $M = P_g(C) \cap N$ (because $M \leq P_g(C) \cap N \lneqq N$). Thus M is a normal subgroup of the free profinite group $P_g(C)$. By Part (i) of Theorem 1.1, S^∞ is a quotient of M, for every finite simple group S. Thus by Melnikov's theorem [5, Proposition 25.9.9] M is free. □

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