(A) Independence and Lévy processes in quantum probability

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Abstract: Quantum probability describes the probabilistic foundations of quantum physics. Many concepts from classical probability have their counterpart in quantum probability. In these lectures we will study the notions of independence and independent increment processes. One surprising feature of quantum probability is the existence of several different notions of (stochastic) independence, as, e.g., tensor independence (which generalizes the notion of stochastic independence in classical probability), freeness, Boolean independence, or monotone independence. These lectures will give an introduction to the most common notions of independence and their basic theory, including the description of their convolutions and infinitely divisible measures. Then we will study the axiomatic approach due to Speicher, Ben Ghorbal, Schürmann, and Muraki, and the classification of the five universal notions of independence. Finally, we will study the relations between these notions and the theory of independent increment processes.

Lectures

- Lecture 1: Introduction to quantum probability, tensor independence, freeness, boolean and monotone independence
- Lecture 2: Fourier and Cauchy transform, infinite divisibility and continous convolution semigroups
- Lecture 3: Products of quantum probability spaces and of GNS representations, Fock spaces and constructions of independent increment processes
- Lecture 4: Independence and universal products, the classification of universal products of quantum probability spaces.
- Lecture 5 and 6: Relations between the universal products; dual groups, bialgebras, and quantum groups; Lévy processes and independent increment processes on dual groups, bialgebras, and quantum groups.

(B) Quantum Processes

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Abstract: This course will be on applications of the fundamental concepts of the classical theory of dynamical systems to the study of endomorphisms of operator algebras. Classical theory of continuous or measurable dynamics can be in a natural way extended to so-called noncommutative dynamical systems, i.e. endomorphisms of respectively C*-algebras or von Neumann algebras. In these series of lectures we present how the fundamental concepts such as the topological entropy or ergodic-theoretical notions extend to the noncommutative framework. A more specific plan is as follows:

Lectures

- Lectures 1-2: Basic setup of classical continuous and measurable dynamics; Gelfand theorem and general 'noncommutative mathematics'; automorphisms or endomorphisms of unital C*-algebras (respectively, von Neumann algebras) as quantum counterparts of continuous (measurable) dynamical systems; some examples of noncommutative dynamical systems.
- Lectures 3-4: Bowen's approach to classical topological entropy via finite approximations; Voiculescu's noncommutative topological entropy and its basic properties; the computation of the Voiculescu entropy of the shift on the Cuntz algebra.
- Lectures 5-6: Almost uniform convergence as a counterpart of almost everywhere convergence; Lance's almost uniform ergodic theorem for automorphisms of von Neumann algebras; noncommutative L^p-spaces and recent generalisations of multiparameter ergodic theorems of Tao.

(C) The Corona Problem

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Motivations

The space $H^\infty(\mathbb{D})$ is the collection of bounded analytic functions on the unit disk $\mathbb{D}$ and has been well-studied from the viewpoint of complex and harmonic analysis and the interaction with operator theory. Under the norm $\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)|$, $H^\infty(\mathbb{D})$ is a complex Banach algebra. With this Banach algebra, it is possible to ask questions that are a blend of analysis and algebra.

One such important question is the *Corona Problem*. This problem can be phrased purely as a function theoretic question. If one is given a finite collection of N functions $g_j \in H^\infty(\mathbb{D})$, such that

$$1 \geq \sum_{j=1}^{N} |g_j(z)|^2 \geq \delta^2 > 0 \qquad \forall z \in \mathbb{D},$$

then is it possible to find functions $f_j \in H^\infty(\mathbb{D})$ such that

$$\sum_{j=1}^{N} f_j(z) g_j(z) = 1 \quad \forall z \in \mathbb{D} \text{ and } \sup_{z\in\mathbb{D}} \sum_{j=1}^{N} |f_j(z)|^2 \leq C(\delta, N)?$$

The collection of functions g_j are typically called *Corona Data* and the collection f_j is called the *Corona Solution*. The condition that $1 \geq \sum_{j=1}^{N} |g_j(z)|^2 \geq \delta^2 > 0\ \forall z \in \mathbb{D}$ is called the *Corona Condition*. Using functional analysis, it is possible to change this problem into an algebraic question about the maximal ideals of $H^\infty(\mathbb{D})$ and a topological question about the density of the unit disk $\mathbb{D}$ in the maximal ideal space of $H^\infty(\mathbb{D})$.

The famous Carleson Corona Theorem, see [7], answered this question affirmatively. It was then also shown to be true by T. Wolff, see the proof in [10], who used deep connections with Carleson measures, solutions of the $\overline{\partial}$-equation, and the duality of H^1 and BMO. For modern treatments, see [10] and [12]. The tools and techniques Carleson implemented have become an integral part of analysis and have served as an impetus for much research in function theory, complex analysis, harmonic analysis, and operator theory for the past 50 years.

Overview of the Course

The course presented by Wick will focus on certain aspects of the Corona problem for multiplier algebras of Besov–Sobolev spaces. The Besov–Sobolev spaces of functions are a class of analytic functions that measure the smoothness of the function. The Besov–Sobolev spaces $B^2_\sigma(\mathbb{B}_n)$ of analytic functions on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ are the collection of functions that are analytic on the unit ball and such that for any integer $m \geq 0$ and any $0 \leq \sigma < \infty$

such that $m+\sigma > \frac{n}{2}$ we have the following norm being finite:

$$\|f\|^2_{B_2^\sigma(\mathbb{B}_n)} := \sum_{j=0}^{m-1} |f^{(j)}(0)|^2 + \int_{\mathbb{B}_n} |(1-|z|^2)^{m+\sigma} f^{(m)}(z)|^2 \frac{dV(z)}{(1-|z|^2)^{n+1}}.$$

One can show that these spaces are independent of m and are reproducing kernel Hilbert spaces, with obvious inner products. Moreover, there are natural generalizations of this norm to the scale $1 < p < \infty$, [14].

We then will discuss the multiplier algebras of the Besov–Sobolev spaces. For the Besov–Sobolev space $B^2_\sigma(\mathbb{B}_n)$, one defines the *multiplier algebra* $\mathsf{M}^2_\sigma(\mathbb{B}_n)$ as the collection of analytic functions φ that are pointwise multipliers of $B^2_\sigma(\mathbb{B}_n)$. Namely, $\varphi f \in B^2_\sigma(\mathbb{B}_n)$ for all $f \in B^2_\sigma(\mathbb{B}_n)$, and then norms $\mathsf{M}^2_\sigma(\mathbb{B}_n)$ by

$$\|\varphi\|_{\mathsf{M}^2_\sigma(\mathbb{B}_n)} := \sup_{f\in B^2_\sigma(\mathbb{B}_n)} \frac{\|\varphi f\|_{B^2_\sigma(\mathbb{B}_n)}}{\|f\|_{B^2_\sigma(\mathbb{B}_n)}}.$$

These spaces of functions contain all the well-known and studied examples of analytic functions, including the Dirichlet space, the Hardy space, and the Bergman space. For a certain range of values of σ, these spaces of functions have deep connections with operator theory. When $0 \leq \sigma \leq \frac{1}{2}$ the space of function the space $B^2_\sigma(\mathbb{B}_n)$ possess additional properties, and has numerous connections with interpolation theory and other problems in complex function theory, [1,2].

The ultimate goal of the course will be to discuss the background and ideas in the proof of the Corona theorem for the multiplier algebras $\mathsf{M}^2_\sigma(\mathbb{B}_n)$.

Theorem 0.1 (Costea, Sawyer, Wick [9]) *Let $0 \leq \sigma \leq \frac{1}{2}$. Suppose that $g_1, \ldots, g_N \in \mathsf{M}^2_\sigma(\mathbb{B}_n)$ satisfy*

$$0 < \delta \leq \sum_{j=1}^N |g_j(z)|^2 \leq 1 \quad \forall z \in \mathbb{B}_n.$$

There are $f_1, \ldots, f_N \in \mathsf{M}^2_\sigma(\mathbb{B}_n)$ such that

(i) $\sum_{j=1}^N f_j(z) g_j(z) = 1 \quad \forall z \in \mathbb{B}_n$;

(ii) $\sum_{j=1}^N \|f_j\|_{\mathsf{X}^2_\sigma(\mathbb{D}^n)} \leq C_{n,N,\sigma,\delta}$.

This theorem encompasses many extensions of L. Carleson's famous proof for $H^\infty(\mathbb{D})$, [7,8].

During the lectures, the following topics will be covered.

1. In the preliminary lectures, we will focus on the basic aspects of the Besov–Sobolev spaces of analytic functions and their multiplier algebras, topics will include the Carleson measures and their geometric characterization. In particular, the well known examples of the Hardy space $H^2(\mathbb{D})$ and $H^\infty(\mathbb{D})$ will be highlighted and contrasted with the other spaces of functions. Topics from [3–5, 10, 12–14] will be covered.

2. Illustrate the connections with operator theory and function theory when $0 \leq \sigma \leq \frac{1}{2}$. In particular some aspects of [1, 6, 11] will be discussed.

3. Discuss the tools of the proof of the Corona problem; topics to include are the $\overline{\partial}$-problem on $\mathbb{B}_n$, Charpentier solution operators for these equations, and estimates for these operators. Topics will include results from [9] and the references therein.

There are numerous exercises that will be provided during the course so that the interested student can learn the basics of the topics covered. These exercises will be designed so that an interested student can go from a limited background in the area to a relatively deep understanding of the material. Additionally, open problems and future directions of research will be pointed out and highlighted during the course.

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