

# DILATIONS, WANDERING SUBSPACES, AND INNER FUNCTIONS

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ABSTRACT. The objective of this paper is to study wandering subspaces for commuting tuples of bounded operators on Hilbert spaces. It is shown that, for a large class of analytic functional Hilbert spaces  $\mathcal{H}_K$  on the unit ball in  $\mathbb{C}^n$ , wandering subspaces for restrictions of the multiplication tuple  $M_z = (M_{z_1}, \dots, M_{z_n})$  can be described in terms of suitable  $\mathcal{H}_K$ -inner functions. We prove that  $\mathcal{H}_K$ -inner functions are contractive multipliers and deduce a result on the multiplier norm of quasi-homogenous polynomials as an application. Along the way we prove a refinement of a result of Arveson on the uniqueness of minimal dilations of pure row contractions.

## 1. INTRODUCTION

Let  $T = (T_1, \dots, T_n)$  be an  $n$ -tuple of commuting bounded linear operators on a complex Hilbert space  $\mathcal{H}$ . A closed subspace  $\mathcal{W} \subset \mathcal{H}$  is called a *wandering subspace* for  $T$  if

$$\mathcal{W} \perp T^{\mathbf{k}}\mathcal{W} \quad (\mathbf{k} \in \mathbb{N}^n \setminus \{\mathbf{0}\}).$$

We say that  $\mathcal{W}$  is a *generating wandering subspace* for  $T$  if in addition

$$\mathcal{H} = \overline{\text{span}}\{T^{\mathbf{k}}\mathcal{W} : \mathbf{k} \in \mathbb{N}^n\}.$$

Wandering subspaces were defined by Halmos in [10]. One of the main observations from [10] is the following. Let  $\mathcal{E}$  be a Hilbert space and let  $M_z : H_{\mathcal{E}}^2(\mathbb{D}) \rightarrow H_{\mathcal{E}}^2(\mathbb{D})$  be the operator of multiplication with the argument on the  $\mathcal{E}$ -valued Hardy space  $H_{\mathcal{E}}^2(\mathbb{D})$  on the unit disc  $\mathbb{D}$ . Suppose that  $\mathcal{S}$  is a non-trivial closed  $M_z$ -invariant subspace of  $H_{\mathcal{E}}^2(\mathbb{D})$ . Then

$$\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$$

is a wandering subspace for  $M_z|_{\mathcal{S}}$  such that

$$M_z^p\mathcal{W} \perp M_z^q\mathcal{W}$$

for all  $p \neq q$  in  $\mathbb{N}$  and

$$\mathcal{S} = \overline{\text{span}}\{z^m\mathcal{W} : m \in \mathbb{N}\}.$$

Hence

$$\mathcal{S} = \bigoplus_{m=0}^{\infty} z^m\mathcal{W}$$

and up to unitary equivalence

$$M_z|_{\mathcal{S}} \text{ on } \mathcal{S} \cong M_z \text{ on } H_{\mathcal{W}}^2(\mathbb{D}).$$

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In particular, we have  $\mathcal{S} = V(H_{\mathcal{W}}^2(\mathbb{D}))$ , where  $V : H_{\mathcal{W}}^2(\mathbb{D}) \rightarrow H_{\mathcal{E}}^2(\mathbb{D})$  is an isometry and  $VM_z = M_zV$ . One can show (see Lemma V.3.2 in [14] for details and more precise references) that any such intertwining isometry  $V$  acts as the multiplication operator  $V = M_{\Theta} : H_{\mathcal{W}}^2(\mathbb{D}) \rightarrow H_{\mathcal{E}}^2(\mathbb{D})$ ,  $f \mapsto \Theta f$ , with a bounded analytic function  $\Theta \in H_{B(\mathcal{W}, \mathcal{E})}^{\infty}(\mathbb{D})$  such that  $\Theta$  possesses isometric boundary values almost everywhere. In this case

$$\mathcal{S} = \Theta H_{\mathcal{W}}^2(\mathbb{D})$$

and (cf. Theorem 5.2 below)

$$(1.1) \quad \mathcal{S} \ominus z\mathcal{S} = \Theta\mathcal{W}.$$

Thus the wandering subspaces of  $M_z$  on  $H_{\mathcal{E}}^2(\mathbb{D})$  can be described using the Beurling-Lax-Halmos representation of  $M_z$ -invariant subspaces of  $H_{\mathcal{E}}^2(\mathbb{D})$ .

Much later, in Aleman, Richter and Sundberg [1], it was shown that every  $M_z$ -invariant closed subspace of the Bergman space on the unit disc  $\mathbb{D}$  is generated by a wandering subspace. More precisely, let  $\mathcal{S}$  be a closed  $M_z$ -invariant subspace of the Bergman space  $L_a^2(\mathbb{D})$ . Then

$$\mathcal{S} = [\mathcal{S} \ominus z\mathcal{S}],$$

where the notation  $[M]$  is used for the smallest closed  $M_z$ -invariant subspace containing a given set  $M \subset L_a^2(\mathbb{D})$ . The above result of Aleman, Richter and Sundberg has been extended by Shimorin (see [22] and [23]) who replaced the multiplication operator  $M_z$  on the Bergman space by left invertible Hilbert space operators satisfying suitable operator inequalities.

In this paper we study wandering subspaces for commuting tuples of operators on Hilbert spaces. More precisely, let  $T = (T_1, \dots, T_n)$  be a commuting tuple of bounded operators on a Hilbert space  $\mathcal{H}$ . Suppose that  $\mathcal{W} = \mathcal{H} \ominus \sum_{i=1}^n T_i\mathcal{H}$  is a generating wandering subspace for  $T$ . We are interested in the following general question: given a closed  $T$ -invariant subspace  $\mathcal{S} \subset \mathcal{H}$ , are there natural conditions which ensure that  $T|_{\mathcal{S}} = (T_1|_{\mathcal{S}}, \dots, T_n|_{\mathcal{S}})$  has a generating wandering subspace again?

In view of the known one-variable results it seems natural to study this problem first in the particular case where  $T$  is the tuple  $M_z = (M_{z_1}, \dots, M_{z_n})$  consisting of the multiplication operators with the coordinate functions on some classical reproducing kernel Hilbert spaces such as the Hardy space, the Bergman space or the Drury-Arveson space on the unit ball  $\mathbb{B}^n$  of  $\mathbb{C}^n$ .

The main purpose of this paper, however, is to parameterize the wandering subspaces and, in particular, to extend the representation (1.1) to a large class of commuting operator tuples. The above question concerning the existence of generating wandering subspaces, even for classical reproducing kernel Hilbert spaces over the unit ball in  $\mathbb{C}^n$ , seems to be more elusive.

Our primary motivation for studying wandering subspaces comes from recent results on Beurling-Lax-Halmos type representations of invariant subspaces of commuting tuples of operators (see Theorem 4.1 below or [4] and [20]). Our study is also motivated by Hedenmalm's theory [11] of Bergman inner functions for shift-invariant subspaces of the Bergman space on the unit disc  $\mathbb{D}$ . This concept has been further generalized by Olofsson [16, 17] to obtain parameterizations of wandering subspaces of shift-invariant subspaces for the weighted Bergman

spaces on  $\mathbb{D}$  corresponding to the kernels

$$K_m(z, w) = (1 - z\bar{w})^{-m} \quad (m \in \mathbb{N}).$$

Our observations heavily depend on the existence of dilations for commuting row contractions (see Section 2). For instance, let  $T = (T_1, \dots, T_n)$  be a pure commuting contractive tuple on a Hilbert space  $\mathcal{H}$ . Let  $\Pi : \mathcal{H} \rightarrow H_n^2(\mathcal{E})$  be the Arveson dilation of  $T$ , and let  $\tilde{\Pi} : \mathcal{H} \rightarrow H_n^2(\tilde{\mathcal{E}})$  be an arbitrary dilation of  $T$  (see Section 2). Then our main uniqueness result, which may be of independent interest, yields an isometry  $V : \mathcal{E} \rightarrow \tilde{\mathcal{E}}$  such that the following diagram commutes (Corollary 3.3):

$$\begin{array}{ccc} & & H_n^2(\mathcal{E}) \\ & \nearrow \Pi & \downarrow I_{H_n^2} \otimes V \\ \mathcal{H} & \xrightarrow{\tilde{\Pi}} & H_n^2(\tilde{\mathcal{E}}) \end{array}$$

In the one-dimensional case  $n = 1$ , some of our observations concerning wandering subspaces are closely related to results of Shimorin [22, 23], Ball and Bolotnikov [4, 5, 6] and Olofsson [15, 16, 17].

In Section 2 we define the notion of a minimal dilation for pure commuting row contractions  $T$  and show that the Arveson dilation is a minimal dilation of  $T$ . In Section 3 we show that minimal dilations are uniquely determined and that each dilation of a pure commuting row contraction factorizes through its minimal dilation. If  $\mathcal{S}$  is a closed invariant subspace for  $T$ , then by dualizing the minimal dilation of the restriction  $T|_{\mathcal{S}}$  one obtains a representation of  $\mathcal{S}$  as the image of a partially isometric module map  $\Pi : H_n^2(\mathcal{E}) \rightarrow H$  defined on a vector-valued Drury-Arveson space. In Section 4 the uniqueness and factorization results for minimal dilations are used to prove corresponding results for the representation  $\Pi$ . In Section 5 we show that any representation  $\Pi : H_n^2(\mathcal{E}) \rightarrow H$  of a  $T$ -invariant subspace  $\mathcal{S}$  induces a unitary representation of the associated wandering subspace  $\mathcal{W} = \mathcal{S} \ominus \sum_{i=1}^n T_i \mathcal{S}$ . Finally, in Section 6 we show that, in the particular case that  $T \in \mathcal{B}(\mathcal{H})^n$  is the multiplication tuple  $M_z = (M_{z_1}, \dots, M_{z_n})$  on a contractive analytic functional Hilbert space  $H(K)$  on  $\mathbb{B}^n$ , the above representation of the wandering subspace  $\mathcal{W} = \mathcal{S} \ominus \sum_{i=1}^n M_{z_i} \mathcal{S}$  is given by a suitably defined  $H(K)$ -inner function. We show that  $H(K)$ -inner functions are contractive multipliers and apply these results to deduce that the norm and the multiplier norm for quasi-homogeneous polynomials on the Drury-Arveson space coincide. We conclude with an example showing that in contrast to the one-dimensional case in dimension  $n > 1$ , even for the nicest analytic functional Hilbert spaces on  $\mathbb{B}^n$  such as the Hardy space, the Bergman space or the Drury-Arveson space, there are  $M_z$ -invariant subspaces which do not possess a generating wandering subspace.

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## 2. MINIMAL DILATIONS

We begin with a brief introduction to the theory of dilations for commuting row contractions.

Let  $T = (T_1, \dots, T_n)$  be an  $n$ -tuple of bounded linear operators on a complex Hilbert space  $\mathcal{H}$ . We denote by  $P_T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$  the completely positive map defined by

$$P_T(X) = \sum_{i=1}^n T_i X T_i^*.$$

An  $n$ -tuple  $T$  is called a *row contraction* or simply a *contractive tuple* if

$$P_T(I_{\mathcal{H}}) \leq I_{\mathcal{H}}.$$

If  $T$  is a row contraction, then

$$I_{\mathcal{H}} \geq P_T(I_{\mathcal{H}}) \geq P_T^2(I_{\mathcal{H}}) \geq \dots \geq P_T^m(I_{\mathcal{H}}) \geq \dots \geq 0.$$

Hence the limit

$$P_{\infty}(T) = \text{SOT-} \lim_{m \rightarrow \infty} P_T^m(I_{\mathcal{H}})$$

exists and satisfies the inequalities  $0 \leq P_{\infty}(T) \leq I_{\mathcal{H}}$ . A row contraction  $T$  is called *pure* (cf. [3] or [18]) if  $P_{\infty}(T) = 0$ .

Let  $\lambda \geq 1$  and let  $K_{\lambda} : \mathbb{B}^n \times \mathbb{B}^n \rightarrow \mathbb{C}$  be the positive definite function defined by

$$K_{\lambda}(\mathbf{z}, \mathbf{w}) = \left(1 - \sum_{i=1}^n z_i \bar{w}_i\right)^{-\lambda}.$$

Then the Drury-Arveson space  $H_n^2$ , the Hardy space  $H^2(\mathbb{B}^n)$ , the Bergman space  $L_a^2(\mathbb{B}^n)$ , and the weighted Bergman spaces  $L_{a,\alpha}^2(\mathbb{B}^n)$  with  $\alpha > -1$ , are the reproducing kernel Hilbert spaces with kernel  $K_{\lambda}$  where  $\lambda = 1, n, n+1$  and  $n+1+\alpha$ , respectively. The tuples  $M_{\mathbf{z}} = (M_{z_1}, \dots, M_{z_n})$  of multiplication operators with the coordinate functions on these reproducing kernel Hilbert spaces are examples of pure commuting contractive tuples of operators.

Let  $K$  be a positive definite function on  $\mathbb{B}^n$  holomorphic in the first and anti-holomorphic in the second variable. Then the functional Hilbert space  $\mathcal{H}_K$  with reproducing kernel  $K$  consists of analytic functions on  $\mathbb{B}^n$ . For any Hilbert space  $\mathcal{E}$ , the  $\mathcal{E}$ -valued functional Hilbert space with reproducing kernel

$$\mathbb{B}^n \times \mathbb{B}^n \rightarrow \mathcal{B}(\mathcal{E}), \quad (\mathbf{z}, \mathbf{w}) \mapsto K(\mathbf{z}, \mathbf{w})I_{\mathcal{E}}$$

can canonically be identified with the tensor product Hilbert space  $\mathcal{H}_K \otimes \mathcal{E}$ . To simplify the notation, we often identify  $H_n^2 \otimes \mathcal{E}$  with the  $\mathcal{E}$ -valued Drury-Arveson space  $H_n^2(\mathcal{E})$ .

Let  $T$  be a commuting row contraction on  $\mathcal{H}$  and let  $\mathcal{E}$  be an arbitrary Hilbert space. An isometry  $\Gamma : \mathcal{H} \rightarrow H_n^2(\mathcal{E})$  is called a *dilation* of  $T$  if

$$M_{z_i}^* \Gamma = \Gamma T_i^* \quad (i = 1, \dots, n).$$

Since  $M_{\mathbf{z}} \in \mathcal{B}(H_n^2(\mathcal{E}))^n$  is a pure row contraction and since a compression of a pure row contraction to a co-invariant subspace remains pure, any commuting row contraction possessing a dilation of the above type is necessarily pure.

Let  $\Gamma : \mathcal{H} \rightarrow H_n^2(\mathcal{E})$  be a dilation of  $T$ . Since the  $C^*$ -subalgebra of  $\mathcal{B}(H_n^2)$  generated by  $(M_{z_1}, \dots, M_{z_n})$  has the form

$$C^*(M_z) = \overline{\text{span}}\{M_z^{\mathbf{k}} M_z^{*\mathbf{l}} : \mathbf{k}, \mathbf{l} \in \mathbb{N}^n\}$$

(see Theorem 5.7 in [3]), the space

$$M = \overline{\text{span}}\{z^{\mathbf{k}} \Gamma h : \mathbf{k} \in \mathbb{N}^n, h \in \mathcal{H}\}$$

is the smallest reducing subspace for  $M_z$  on  $H_n^2(\mathcal{E})$  containing the image of  $\Gamma$ . As a reducing subspace for  $M_z \in \mathcal{B}(H_n^2(\mathcal{E}))^n$  the space  $M$  has the form

$$M = \bigvee_{\mathbf{k} \in \mathbb{N}^n} z^{\mathbf{k}} \mathcal{L} = H_n^2(\mathcal{L}) \quad \text{with} \quad \mathcal{L} = M \cap \mathcal{E}.$$

We call  $\Gamma$  a *minimal dilation* of  $T$  if

$$H_n^2(\mathcal{E}) = \overline{\text{span}}\{z^{\mathbf{k}} \Gamma h : \mathbf{k} \in \mathbb{N}^n, h \in \mathcal{H}\}.$$

We briefly recall a canonical way to construct minimal dilations for pure commuting contractive tuples (cf. [3]). Let  $T$  be a pure commuting contractive tuple on  $\mathcal{H}$ . Define

$$\mathcal{E}_c = \overline{\text{ran}}(I_{\mathcal{H}} - P_T(I_{\mathcal{H}})), \quad D = (I_{\mathcal{H}} - P_T(I_{\mathcal{H}}))^{\frac{1}{2}}.$$

Then the operator  $\Pi_c : \mathcal{H} \rightarrow H_n^2(\mathcal{E}_c)$  defined by

$$(\Pi_c h)(z) = D(I_{\mathcal{H}} - \sum_{i=1}^n z_i T_i^*)^{-1} h \quad (z \in \mathbb{B}^n, h \in \mathcal{H}).$$

is a dilation of  $T$ . Let

$$M = \bigvee_{\mathbf{k} \in \mathbb{N}^n} z^{\mathbf{k}} \mathcal{L} = H_n^2(\mathcal{L}) \quad \text{with} \quad \mathcal{L} = M \cap \mathcal{E}_c$$

be the smallest reducing subspace for  $M_z \in \mathcal{B}(H_n^2(\mathcal{E}_c))^n$  which contains the image of  $\Pi_c$  and let  $P_{\mathcal{E}_c}$  be the orthogonal projection of  $H_n^2(\mathcal{E}_c)$  onto the subspace consisting of all constant functions. Since

$$P_{\mathcal{E}_c} = I_{H_n^2(\mathcal{E}_c)} - \sum_{i=1}^n M_{z_i} M_{z_i}^*,$$

we obtain that

$$Dh = P_{\mathcal{E}_c}(\Pi_c h) \in H_n^2(\mathcal{L}) \cap \mathcal{E}_c = \mathcal{L}$$

for each  $h \in \mathcal{H}$ . Hence  $\Pi_c$  is a minimal dilation of  $T$ .

## 3. UNIQUENESS OF MINIMAL DILATIONS

Using a refinement of an idea of Arveson [3], we obtain the following sharpened uniqueness result for minimal dilations of pure commuting contractive tuples.

**THEOREM 3.1.** *Let  $T \in \mathcal{B}(\mathcal{H})^n$  be a pure commuting contractive tuple on a Hilbert space  $\mathcal{H}$  and let  $\Pi_i : \mathcal{H} \rightarrow H_n^2(\mathcal{E}_i)$ ,  $i = 1, 2$ , be a pair of minimal dilations of  $T$ . Then there exists a unitary operator  $U \in \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$  such that*

$$\Pi_2 = (I_{H_n^2} \otimes U)\Pi_1.$$

**Proof.** As an application of Theorem 8.5 in [3] one can show that there is a unitary operator  $W : H_n^2(\mathcal{E}_1) \rightarrow H_n^2(\mathcal{E}_2)$  which intertwines the tuples  $M_z$  on  $H_n^2(\mathcal{E}_1)$  and  $H_n^2(\mathcal{E}_2)$ . We prefer to give a more direct proof containing some simplifications which are possible in the pure case.

Let  $\mathcal{B} = C^*(M_z) \subset \mathcal{B}(H_n^2)$  be the  $C^*$ -algebra generated by the multiplication tuple  $(M_{z_1}, \dots, M_{z_n})$  on the scalar-valued Drury-Arveson space  $H_n^2$ . Denote by  $\mathcal{A}$  the unital subalgebra of  $\mathcal{B}$  consisting of all polynomials in  $(M_{z_1}, \dots, M_{z_n})$ .

Let  $\Pi : \mathcal{H} \rightarrow H_n^2(\mathcal{F})$  be a minimal dilation of  $T$ . The map  $\varphi : \mathcal{B} \rightarrow \mathcal{B}(\mathcal{H})$  defined by

$$\varphi(X) = \Pi^*(X \otimes I_{\mathcal{F}})\Pi$$

is completely positive and unital. For each polynomial  $p \in \mathbb{C}[z_1, \dots, z_n]$  and each operator  $X \in \mathcal{B}$ , we have  $\varphi(p(M_z)) = p(T)$  and, since  $\text{ran}(\Pi)^\perp$  is invariant for  $M_z$ , it follows that

$$\varphi(p(M_z)X) = \Pi^*(p(M_z)X \otimes 1_{\mathcal{F}})\Pi = \varphi(p(M_z))\varphi(X).$$

Hence  $\varphi : \mathcal{B} \rightarrow \mathcal{B}(\mathcal{H})$  is an  $\mathcal{A}$ -morphism in the sense of Arveson (Definition 6.1 in [3]). The minimality of  $\Pi$  as a dilation of  $T$  implies that the map  $\pi : \mathcal{B} \rightarrow \mathcal{B}(H_n^2(\mathcal{F}))$  defined by

$$\pi(X) = X \otimes I_{\mathcal{F}}$$

is a minimal Stinespring dilation of  $\varphi$ . Now let  $\Pi_i : \mathcal{H} \rightarrow H_n^2(\mathcal{E}_i)$ ,  $i = 1, 2$ , be a pair of minimal dilations of  $T$  and let  $\varphi_i : \mathcal{B} \rightarrow \mathcal{B}(\mathcal{H})$ ,  $\pi_i : \mathcal{B} \rightarrow \mathcal{B}(H_n^2(\mathcal{E}_i))$  be the maps induced by  $\Pi_i$  as explained above. Then  $\varphi_1 = \varphi_2$  on  $\mathcal{A}$ , and hence a direct application of Lemma 8.6 in [3] shows that there is a unitary operator  $W : H_n^2(\mathcal{E}_1) \rightarrow H_n^2(\mathcal{E}_2)$  such that  $W(X \otimes I_{\mathcal{E}_1}) = (X \otimes I_{\mathcal{E}_2})W$  for all  $X \in \mathcal{B}$  and such that

$$W\Pi_1 = \Pi_2.$$

Since  $W$  and  $W^*$  both intertwine the tuples  $M_z \otimes I_{\mathcal{E}_1}$  and  $M_z \otimes I_{\mathcal{E}_2}$ , it follows as a very special case of the functional commutant lifting theorem for the Drury-Arveson space (see Theorem 5.1 in [7] or Theorem 3.7 in [2]) that  $W$  and  $W^*$  are induced by multipliers, say  $W = M_a$  and  $W^* = M_b$ , where  $a : \mathbb{B}^n \rightarrow \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$  and  $b : \mathbb{B}^n \rightarrow \mathcal{B}(\mathcal{E}_2, \mathcal{E}_1)$  are operator-valued multipliers. Since

$$M_{ab} = M_a M_b = I_{H_n^2(\mathcal{E}_2)} \quad \text{and} \quad M_{ba} = M_b M_a = I_{H_n^2(\mathcal{E}_1)},$$

it follows that  $a(\mathbf{z}) = b(\mathbf{z})^{-1}$  are invertible operators for all  $\mathbf{z} \in \mathbb{B}^n$ . Moreover, since

$$K(\mathbf{z}, \mathbf{w})\eta = (M_a M_a^* K(\cdot, \mathbf{w})\eta)(\mathbf{z}) = a(\mathbf{z})K(\mathbf{z}, \mathbf{w})a(\mathbf{w})^*\eta,$$

holds for all  $\mathbf{z}, \mathbf{w} \in \mathbb{B}^n$  and  $\eta \in \mathcal{E}_2$ , it follows that  $a(\mathbf{z})a(\mathbf{w})^* = I_{\mathcal{E}_2}$  for all  $\mathbf{z}, \mathbf{w} \in \mathbb{B}^n$ . Hence the operators  $a(\mathbf{z})$  are unitary with  $a(\mathbf{z}) = a(\mathbf{w})$  for all  $\mathbf{z}, \mathbf{w} \in \mathbb{B}^n$ . Let  $U = a(\mathbf{z})$

be the constant value of the multiplier  $a$ . Then  $U \in \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$  is a unitary operator with  $W = I_{H_n^2} \otimes U$ . Hence the proof is complete.  $\blacksquare$

REMARK 3.2. Let  $\Pi_c : \mathcal{H} \rightarrow H_n^2(\mathcal{E}_c)$  be the Arveson dilation of a pure commuting row contraction  $T$  as in Section 2 and let  $\Pi : \mathcal{H} \rightarrow H_n^2(\mathcal{E})$  be an arbitrary minimal dilation of  $T$ . Then there is a unitary operator  $U \in \mathcal{B}(\mathcal{E}_c, \mathcal{E})$  such that

$$\Pi = (I_{H_n^2} \otimes U)\Pi_c.$$

In this sense the Arveson dilation  $\Pi_c$  is the unique minimal dilation of  $T$ . In the sequel we call  $\Pi_c$  the canonical dilation of  $T$ .

As a consequence of Theorem 3.1, we obtain the following factorization result (see Theorem 4.1 in [19] for the single-variable case).

COROLLARY 3.3. (*Canonical factorizations of dilations*) Let  $T \in \mathcal{B}(\mathcal{H})^n$  be a pure commuting contractive tuple and let  $\Pi : \mathcal{H} \rightarrow H_n^2(\mathcal{E})$  be a dilation of  $T$ . Then there exists an isometry  $V \in \mathcal{B}(\mathcal{E}_c, \mathcal{E})$  such that

$$\Pi = (I_{H_n^2} \otimes V)\Pi_c.$$

*Proof.* As shown in Section 2 there is a closed subspace  $\mathcal{F} \subseteq \mathcal{E}$  such that

$$H_n^2(\mathcal{F}) = \overline{\text{span}}\{z^{\mathbf{k}}\Pi\mathcal{H} : \mathbf{k} \in \mathbb{N}^n\}.$$

Then by definition  $\Pi : \mathcal{H} \rightarrow H_n^2(\mathcal{F})$  is a minimal dilation of  $T$ . By Theorem 3.1, there exists a unitary operator  $U : \mathcal{E}_c \rightarrow \mathcal{F}$  such that

$$\Pi = (I_{H_n^2} \otimes U)\Pi_c.$$

Clearly,  $U$  regarded as an operator with values in  $\mathcal{E}$ , defines an isometry  $V$  with the required properties.  $\blacksquare$

#### 4. JOINT INVARIANT SUBSPACES

In this section we study the structure of joint invariant subspaces of pure commuting row contractions.

We begin with the following characterization of invariant subspaces from [21] (Theorem 3.2). The proof follows as an elementary application of the above dilation results.

THEOREM 4.1. Let  $T = (T_1, \dots, T_n)$  be a pure commuting contractive tuple on  $\mathcal{H}$  and let  $\mathcal{S}$  be a closed subspace of  $\mathcal{H}$ . Then  $\mathcal{S}$  is a joint  $T$ -invariant subspace of  $\mathcal{H}$  if and only if there exists a Hilbert space  $\mathcal{E}$  and a partial isometry  $\Pi \in \mathcal{B}(H_n^2(\mathcal{E}), \mathcal{H})$  with  $\Pi M_{z_i} = T_i \Pi$  for  $i = 1, \dots, n$  and

$$\mathcal{S} = \Pi(H_n^2(\mathcal{E})).$$

*Proof.* We indicate the main ideas. Let  $\mathcal{S} \subset \mathcal{H}$  be a closed invariant subspace for  $T$ . Since

$$\langle P_{(T|\mathcal{S})}^m(I_{\mathcal{S}})x, x \rangle = \sum_{|\mathbf{k}|=m} \frac{|\mathbf{k}|!}{\mathbf{k}!} \|P_{\mathcal{S}} T^{*\mathbf{k}} x\|^2 \xrightarrow{m} 0$$

for each  $x \in \mathcal{S}$ , the restriction  $T|_{\mathcal{S}}$  is a pure commuting row contraction again. Let  $\Pi_{\mathcal{S}} : \mathcal{S} \rightarrow H_n^2(\mathcal{E})$  be a dilation of  $T|_{\mathcal{S}}$  and let  $i_{\mathcal{S}} : \mathcal{S} \rightarrow \mathcal{H}$  be the inclusion map. Then

$$\Pi = i_{\mathcal{S}} \circ \Pi_{\mathcal{S}}^* : H_n^2(\mathcal{E}) \rightarrow \mathcal{H}$$

is a partial isometry with the required properties. The reverse implication obviously holds. ■

REMARK 4.2. Let  $\mathcal{S} \subset \mathcal{H}$  be a closed invariant subspace of a pure commuting row contraction  $T \in \mathcal{B}(\mathcal{H})^n$  and let  $\Pi_{\mathcal{S}} : \mathcal{S} \rightarrow H_n^2(\mathcal{E})$  be a minimal dilation of the restriction  $T|_{\mathcal{S}}$ . Then the map

$$\Pi : H_n^2(\mathcal{E}) \xrightarrow{\Pi_{\mathcal{S}}^*} \mathcal{S} \hookrightarrow \mathcal{H}$$

is a partial isometry with  $\text{ran} \Pi = \mathcal{S}$  such that  $\Pi$  intertwines  $M_z$  on  $H_n^2(\mathcal{E})$  and  $T$  on  $\mathcal{H}$  componentwise. Any map  $\Pi$  arising in this way will be called a canonical representation of  $\mathcal{S}$ . Note that in this situation  $\Pi^*|_{\mathcal{S}} = \Pi_{\mathcal{S}}$  is a minimal dilation of  $T|_{\mathcal{S}}$ .

If  $\Pi : H_n^2(\mathcal{E}) \rightarrow \mathcal{H}$  and  $\tilde{\Pi} : H_n^2(\tilde{\mathcal{E}}) \rightarrow \mathcal{H}$  are two canonical representations of  $\mathcal{S}$ , then by Theorem 3.1 there is a unitary operator  $U \in \mathcal{B}(\tilde{\mathcal{E}}, \mathcal{E})$  such that  $\tilde{\Pi} = \Pi(I_{H_n^2} \otimes U)$ .

By dualizing Corollary 3.3 one obtains the following uniqueness result.

THEOREM 4.3. Let  $T \in \mathcal{B}(\mathcal{H})^n$  be a pure commuting contractive tuple and let  $\Pi : H_n^2(\mathcal{E}) \rightarrow \mathcal{H}$  be a canonical representation of a closed  $T$ -invariant subspace  $\mathcal{S} \subset \mathcal{H}$ . If  $\tilde{\Pi} : H_n^2(\tilde{\mathcal{E}}) \rightarrow \mathcal{H}$  is a partial isometry with  $\mathcal{S} = \tilde{\Pi}H_n^2(\tilde{\mathcal{E}})$  and  $\tilde{\Pi}M_{z_i} = T_i\tilde{\Pi}$  for  $i = 1, \dots, n$ , then there exists an isometry  $V : \mathcal{E} \rightarrow \tilde{\mathcal{E}}$  such that

$$\tilde{\Pi} = \Pi(I_{H_n^2} \otimes V^*).$$

Proof. Since  $\tilde{\Pi}^*$  is a partial isometry with  $\ker \tilde{\Pi}^* = (\text{ran} \tilde{\Pi})^\perp = \mathcal{S}^\perp$ , the map  $\tilde{\Pi}^* : \mathcal{S} \rightarrow H_n^2(\tilde{\mathcal{E}})$  is an isometry. The adjoint of this isometry intertwines the tuples  $M_z \in \mathcal{B}(H_n^2(\tilde{\mathcal{E}}))^n$  and  $T|_{\mathcal{S}}$ . Hence  $\tilde{\Pi}^* : \mathcal{S} \rightarrow H_n^2(\tilde{\mathcal{E}})$  is a dilation of  $T|_{\mathcal{S}}$ . Since  $\Pi^* : \mathcal{S} \rightarrow H_n^2(\mathcal{E})$  is a minimal dilation of  $T|_{\mathcal{S}}$ , Corollary 3.3 implies that there is an isometry  $V : \mathcal{E} \rightarrow \tilde{\mathcal{E}}$  such that

$$\tilde{\Pi}^*|_{\mathcal{S}} = (I_{H_n^2} \otimes V)\Pi^*|_{\mathcal{S}}.$$

By taking adjoints and using the fact that  $\text{ran} \Pi = \text{ran} \tilde{\Pi} = \mathcal{S}$ , we obtain that

$$\tilde{\Pi} = \Pi(I_{H_n^2} \otimes V^*).$$

Thus the proof is complete. ■

COROLLARY 4.4. Let  $T \in \mathcal{B}(\mathcal{H})^n$  be a pure commuting contractive tuple and let  $\mathcal{S} \subset \mathcal{H}$  be a closed  $T$ -invariant subspace. Suppose that

$$\Pi_j : H_n^2(\mathcal{E}_j) \rightarrow \mathcal{H} \quad (j = 1, 2)$$

are partial isometries with range  $\mathcal{S}$  such that  $\Pi_j$  intertwines  $M_z$  on  $H_n^2(\mathcal{E}_j)$  and  $T$  on  $\mathcal{H}$  for  $j = 1, 2$ . Then there exists a partial isometry  $V : \mathcal{E}_1 \rightarrow \mathcal{E}_2$  such that

$$\Pi_1 = \Pi_2(I_{H_n^2} \otimes V).$$



**Proof.** Let  $\mathcal{S} = \Pi H_n^2(\mathcal{E})$  be a canonical representation of  $\mathcal{S}$ . Theorem 4.3 implies that

$$\Pi_j = \Pi(I_{H_n^2} \otimes V_j^*)$$

for some isometry  $V_j : \mathcal{E} \rightarrow \mathcal{E}_j$ ,  $j = 1, 2$ . Therefore,

$$\begin{aligned} \Pi_1 &= \Pi(I_{H_n^2} \otimes V_1^*) \\ &= (\Pi_2(I_{H_n^2} \otimes V_2))(I_{H_n^2} \otimes V_1^*) \\ &= \Pi_2(I_{H_n^2} \otimes V_2 V_1^*) \\ &= \Pi_2(I_{H_n^2} \otimes V), \end{aligned}$$

where  $V = V_2 V_1^* : \mathcal{E}_1 \rightarrow \mathcal{E}_2$  is a partial isometry. This completes the proof.  $\blacksquare$

## 5. PARAMETERIZATIONS OF WANDERING SUBSPACES

In this section we consider parameterizations of wandering subspaces for pure commuting row contractions  $T \in \mathcal{B}(\mathcal{H})^n$  which in the case of the one-variable shift  $T = M_z \in \mathcal{B}(H_{\mathcal{E}}^2(\mathbb{D}))$  on the Hardy space of the unit disc reduce to the representation (1.1). As suggested by Theorem 4.1 the isometric intertwiner  $V : H_{\mathcal{W}}^2(\mathbb{D}) \rightarrow H_{\mathcal{E}}^2(\mathbb{D})$  from the introduction is replaced by a partial isometry  $\Pi : H_n^2(\mathcal{E}) \rightarrow H$  intertwining  $M_z \in \mathcal{B}(H_n^2(\mathcal{E}))^n$  and  $T \in \mathcal{B}(\mathcal{H})^n$ .

We begin with an elementary but crucial observation concerning the uniqueness of wandering subspaces for commuting tuples of operators.

Let  $\mathcal{W}$  be a wandering subspace for a commuting tuple  $T \in \mathcal{B}(\mathcal{H})^n$ . Set

$$\mathcal{G}_{T, \mathcal{W}} = \bigvee_{\mathbf{k} \in \mathbb{N}^n} T^{\mathbf{k}} \mathcal{W}.$$

An elementary argument shows that

$$\mathcal{G}_{T, \mathcal{W}} \ominus \sum_{i=1}^n T_i \mathcal{G}_{T, \mathcal{W}} = \bigvee_{\mathbf{k} \in \mathbb{N}^n} T^{\mathbf{k}} \mathcal{W} \ominus \bigvee_{\mathbf{k} \in \mathbb{N}^n \setminus \{\mathbf{0}\}} T^{\mathbf{k}} \mathcal{W} = \mathcal{W}.$$

It follows that

$$\mathcal{W} = \bigcap_{i=1}^n (\mathcal{G}_{T, \mathcal{W}} \ominus T_i \mathcal{G}_{T, \mathcal{W}}).$$

Consequently, we have the following result.

**PROPOSITION 5.1.** *Let  $T \in \mathcal{B}(\mathcal{H})^n$  be a commuting tuple of bounded operators on a Hilbert space  $\mathcal{H}$  and let  $\mathcal{W}$  be a wandering subspace for  $T$ . Then*

$$\mathcal{W} = \bigcap_{i=1}^n (\mathcal{G}_{T, \mathcal{W}} \ominus T_i \mathcal{G}_{T, \mathcal{W}}) = \mathcal{G}_{T, \mathcal{W}} \ominus \sum_{i=1}^n T_i \mathcal{G}_{T, \mathcal{W}}.$$

*In particular, if  $\mathcal{W}$  is a generating wandering subspace for  $T$ , then*

$$\mathcal{W} = \bigcap_{i=1}^n (\mathcal{H} \ominus T_i \mathcal{H}) = \mathcal{H} \ominus \sum_{i=1}^n T_i \mathcal{H}.$$

Starting point for our description of wandering subspaces is the following general observation.

**THEOREM 5.2.** *Let  $T \in \mathcal{B}(\mathcal{H})^n$  be a commuting tuple of bounded operators on a Hilbert space  $\mathcal{H}$  and let  $\Pi : H_n^2(\mathcal{E}) \rightarrow \mathcal{H}$  be a partial isometry with  $\Pi M_{z_i} = T_i \Pi$  for  $i = 1, \dots, n$ . Then  $\mathcal{S} = \Pi(H_n^2(\mathcal{E}))$  is a closed  $T$ -invariant subspace,  $\mathcal{W} = \mathcal{S} \ominus \sum_{i=1}^n T_i \mathcal{S}$  is a wandering subspace for  $T|_{\mathcal{S}}$  and*

$$\mathcal{W} = \Pi((\ker \Pi)^\perp \cap \mathcal{E}).$$

*Proof.* Define  $\mathcal{F} = (\ker \Pi)^\perp \cap \mathcal{E}$ . Obviously the range  $\mathcal{S}$  of  $\Pi$  is a closed  $T$ -invariant subspace and  $\mathcal{W} = \mathcal{S} \ominus \sum_{i=1}^n T_i \mathcal{S}$  is a wandering subspace for  $T|_{\mathcal{S}}$ . To prove the claimed representation of  $\mathcal{W}$  note first that

$$\mathcal{W} = \Pi(H_n^2(\mathcal{E})) \ominus \sum_{i=1}^n T_i \Pi(H_n^2(\mathcal{E})) = \Pi(H_n^2(\mathcal{E})) \ominus \sum_{i=1}^n \Pi M_{z_i}(H_n^2(\mathcal{E})).$$

For  $f \in \mathcal{F}$ ,  $h \in H_n^2(\mathcal{E})$  and  $i = 1, \dots, n$ , we have

$$\langle \Pi f, \Pi z_i h \rangle = \langle \Pi^* \Pi f, z_i h \rangle = \langle f, z_i h \rangle = 0.$$

Conversely, each element in  $\mathcal{W}$  can be written as  $\Pi f$  with  $f \in (\ker \Pi)^\perp$ . But then, for  $h \in H_n^2(\mathcal{E})$  and  $i = 1, \dots, n$ , we obtain

$$\langle f, z_i h \rangle = \langle \Pi f, \Pi z_i h \rangle = 0.$$

To conclude the proof it suffices to recall that  $H_n^2(\mathcal{E}) \ominus \sum_{i=1}^n M_{z_i}(H_n^2(\mathcal{E})) = \mathcal{E}$ . This identity is well known, but also follows directly from Proposition 5.1, since  $\mathcal{E}$  is a generating wandering subspace for  $M_z \in \mathcal{B}(H_n^2(\mathcal{E}))^n$ .  $\blacksquare$

If  $T \in \mathcal{B}(\mathcal{H})^n$  is a pure commuting row contraction, then by Theorem 4.1 each closed  $T$ -invariant subspace  $\mathcal{S} \subset \mathcal{H}$  admits a representation  $\mathcal{S} = \Pi(H_n^2(\mathcal{E}))$  as in the hypothesis of the preceding theorem.

**COROLLARY 5.3.** *In the setting of Theorem 5.2 the identity*

$$\bigvee_{\mathbf{k} \in \mathbb{N}^n} T^{\mathbf{k}} \mathcal{W} = \overline{\Pi H_n^2(\mathcal{F})}$$

*holds with  $\mathcal{F} = (\ker \Pi)^\perp \cap \mathcal{E}$ .*

**Proof.** Since  $\mathcal{W} = \Pi \mathcal{F}$ , we have

$$\bigvee_{\mathbf{k} \in \mathbb{N}^n} T^{\mathbf{k}} \mathcal{W} = \bigvee_{\mathbf{k} \in \mathbb{N}^n} T^{\mathbf{k}} \Pi \mathcal{F} = \bigvee_{\mathbf{k} \in \mathbb{N}^n} \Pi M_z^{\mathbf{k}} \mathcal{F} = \overline{\Pi \left( \bigvee_{\mathbf{k} \in \mathbb{N}^n} M_z^{\mathbf{k}} \mathcal{F} \right)} = \overline{\Pi H_n^2(\mathcal{F})}.$$

This completes the proof.  $\blacksquare$

## 6. CONTRACTIVE ANALYTIC HILBERT SPACES AND INNER FUNCTIONS

Let  $K : \mathbb{B}^n \times \mathbb{B}^n \rightarrow \mathbb{C}$  be a positive definite function and let  $\mathcal{H}_K$  be the functional Hilbert space with reproducing kernel  $K$ . We say that  $\mathcal{H}_K$  is a *contractive analytic Hilbert space* (cf. [20], [21]) over  $\mathbb{B}^n$  if  $\mathcal{H}_K$  consists of analytic functions on  $\mathbb{B}^n$  and if the multiplication tuple  $M_z = (M_{z_1}, \dots, M_{z_n})$  is a pure row contraction on  $\mathcal{H}_K$ .

Typical and important examples of contractive analytic Hilbert spaces include the Drury-Arveson space, the Hardy space and the weighted Bergman spaces over  $\mathbb{B}^n$  (cf. Proposition 4.1 in [21]).

Let  $\mathcal{H}_K$  be a contractive analytic Hilbert space and let  $\mathcal{E}$  and  $\mathcal{E}_*$  be arbitrary Hilbert spaces. An operator-valued map  $\Theta : \mathbb{B}^n \rightarrow \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$  is said to be a *K-multiplier* if

$$\Theta f \in \mathcal{H}_K \otimes \mathcal{E}_* \quad \text{for every } f \in H_n^2 \otimes \mathcal{E}.$$

The set of all  $K$ -multipliers is denoted by  $\mathcal{M}(H_n^2 \otimes \mathcal{E}, \mathcal{H}_K \otimes \mathcal{E}_*)$ . If  $\Theta \in \mathcal{M}(H_n^2 \otimes \mathcal{E}, \mathcal{H}_K \otimes \mathcal{E}_*)$ , then the multiplication operator  $M_\Theta : H_n^2 \otimes \mathcal{E} \rightarrow \mathcal{H}_K \otimes \mathcal{E}_*$  defined by

$$(M_\Theta f)(\mathbf{w}) = (\Theta f)(\mathbf{w}) = \Theta(\mathbf{w})f(\mathbf{w})$$

is bounded by the closed graph theorem. We shall call a multiplier  $\Theta$  partially isometric or isometric if the induced multiplication operator  $M_\Theta$  has the corresponding property.

The space of  $K$ -multipliers can be described in the following way (cf. Corollary 4.3 in [21]). Let  $X$  be in  $\mathcal{B}(H_n^2 \otimes \mathcal{E}, \mathcal{H}_K \otimes \mathcal{E}_*)$ . Then  $X = M_\Theta$  for some  $\Theta \in \mathcal{M}(H_n^2 \otimes \mathcal{E}, \mathcal{H}_K \otimes \mathcal{E}_*)$  if and only if

$$X(M_{z_i} \otimes I_{\mathcal{E}}) = (M_{z_i} \otimes I_{\mathcal{E}_*})X \quad \text{for } i = 1, \dots, n.$$

**DEFINITION 6.1.** Let  $\Theta : \mathbb{B}^n \rightarrow \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$  be an operator-valued function and let  $\mathcal{H}_K$  be a contractive analytic Hilbert space as above. Then  $\Theta$  is said to be a *K-inner function* if  $\Theta x \in \mathcal{H}_K \otimes \mathcal{E}_*$  with  $\|\Theta x\|_{\mathcal{H}_K \otimes \mathcal{E}_*} = \|x\|_{\mathcal{E}}$  for all  $x \in \mathcal{E}$  and if

$$\Theta \mathcal{E} \perp M_z^{\mathbf{k}}(\Theta \mathcal{E}) \quad \text{for all } \mathbf{k} \in \mathbb{N}^n \setminus \{0\}.$$

The notion of  $K$ -inner functions for the particular case of weighted Bergman spaces on  $\mathbb{D}$  is due to A. Olofsson [16]. His definition of Bergman inner functions was motivated by earlier observations [11] of H. Hedenmalm concerning invariant subspaces and wandering subspaces of the Bergman space on  $\mathbb{D}$ .

The following result should be compared with Theorem 6.1 in [16] or Theorem 3.3 in [12].

**THEOREM 6.2.** *Each K-inner function  $\Theta : \mathbb{B}^n \rightarrow \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$  is a contractive K-multiplier.*

**Proof.** Let  $\Theta : \mathbb{B}^n \rightarrow \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$  be a  $K$ -inner function. Then  $\mathcal{W} = \Theta \mathcal{E} \subset \mathcal{H}_K \otimes \mathcal{E}_*$  is a generating wandering subspace for the restriction of  $M_z \in \mathcal{B}(\mathcal{H}_K \otimes \mathcal{E}_*)^n$  to the closed invariant subspace

$$\mathcal{S} = \bigvee_{\mathbf{k} \in \mathbb{N}^n} (M_z^{\mathbf{k}} \mathcal{W}) \subset \mathcal{H}_K \otimes \mathcal{E}_*.$$

By Theorem 4.1 and the remarks preceding Definition 6.1, there is a partially isometric  $K$ -multiplier  $\hat{\Theta} \in \mathcal{M}(H_n^2 \otimes \hat{\mathcal{E}}, \mathcal{H}_K \otimes \mathcal{E}_*)$  such that  $\mathcal{S}$  is the range of the induced multiplication

operator  $M_{\hat{\Theta}}$ . Define  $\mathcal{F} = (\ker M_{\hat{\Theta}})^\perp \cap \hat{\mathcal{E}}$ . A straightforward application of Proposition 5.1 and Theorem 5.2 yields that

$$M_{\hat{\Theta}} : \mathcal{F} \rightarrow \mathcal{W}$$

is a unitary operator. Since also  $M_\Theta : \mathcal{E} \rightarrow \mathcal{W}$  is a unitary operator, it follows that there is a unitary operator  $U : \mathcal{E} \rightarrow \mathcal{F}$  such that  $\Theta(z) = \hat{\Theta}(z)U$  for all  $z \in \mathbb{B}^n$ . For each function  $f \in H_n^2 \otimes \mathcal{E} \subset \mathcal{O}(\mathbb{B}^n, \mathcal{E})$ , it follows that

$$\Theta(z)f(\mathbf{w}) = \hat{\Theta}(z)Uf(\mathbf{w})$$

for all  $z, \mathbf{w} \in \mathbb{B}^n$ . Evaluating this identity for  $z = \mathbf{w}$ , we obtain that  $\Theta f = \hat{\Theta}Uf$  for all  $f \in H_n^2(\mathcal{E})$ . Since  $H_n^2(\mathcal{E}) \rightarrow H_n^2(\hat{\mathcal{E}})$ ,  $f \mapsto Uf$ , is isometric and since  $\hat{\Theta} \in \mathcal{M}(H_n^2 \otimes \hat{\mathcal{E}}, \mathcal{H}_K \otimes \mathcal{E}_*)$  is a contractive  $K$ -multiplier, it follows that also  $\Theta$  is a contractive  $K$ -multiplier.  $\blacksquare$

In the scalar case  $\mathcal{E} = \mathcal{E}_* = \mathbb{C}$  the preceding theorem implies that each  $K$ -inner function  $\Theta : \mathbb{B}^n \rightarrow \mathcal{B}(\mathbb{C}) \cong \mathbb{C}$  satisfies the estimates

$$1 = \|\Theta\|_{\mathcal{H}_K} \leq \|\Theta\|_{\mathcal{M}(H_n^2, \mathcal{H}_K)} \|1\|_{H_n^2} \leq 1.$$

Hence the norm of  $\Theta$ , and also of each scalar multiple of  $\Theta$ , as an element in  $\mathcal{H}_K$  coincides with its norm as a multiplier from  $H_n^2$  to  $\mathcal{H}_K$ . We apply this observation to a natural class of examples.

A polynomial  $p = \sum_{\mathbf{k} \in \mathbb{N}^n} a_{\mathbf{k}} z^{\mathbf{k}}$  is called *quasi-homogeneous* if there are a tuple  $\mathbf{m} = (m_1, \dots, m_n)$  of positive integers  $m_i$  and an integer  $\ell \geq 0$  such that  $\sum_{i=1}^n m_i k_i = \ell$  for all  $\mathbf{k} \in \mathbb{N}^n$  with  $a_{\mathbf{k}} \neq 0$ . In this case  $p$  is said to be  $\mathbf{m}$ -quasi-homogeneous of degree  $\ell$ . Let us denote by  $R_{\mathbf{m}}(\ell)$  the set of all  $\mathbf{m}$ -quasi-homogeneous polynomials of degree  $\ell$ .

**COROLLARY 6.3.** *Suppose that  $\mathcal{H}_K$  is a contractive analytic Hilbert space on  $\mathbb{B}^n$  such that the monomials  $z^{\mathbf{k}}$  ( $\mathbf{k} \in \mathbb{N}^n$ ) form an orthogonal basis of  $\mathcal{H}_K$ . Let  $p \in \mathbb{C}[z_1, \dots, z_n]$  be a quasi-homogeneous polynomial. Then*

$$\|p\|_{\mathcal{H}_K} = \|p\|_{\mathcal{M}(H_n^2, \mathcal{H}_K)}.$$

*If  $\|p\|_{\mathcal{H}_K} = 1$ , then  $p$  is a  $K$ -inner function.*

**Proof.** Suppose that  $p \in R_{\mathbf{m}}(\ell)$  is  $\mathbf{m}$ -quasi-homogeneous of degree  $\ell$  with  $\|p\|_{\mathcal{H}_K} = 1$ . Then  $z^{\mathbf{k}}p$  is  $\mathbf{m}$ -quasi-homogeneous of degree  $\ell + \sum_{i=1}^n m_i k_i$  for  $\mathbf{k} \in \mathbb{N}^n$ . Since by hypothesis

$$\mathcal{H}_K = \bigoplus_{\ell} R_{\mathbf{m}}(\ell),$$

it follows that  $p$  is a  $K$ -inner function. The remaining assertions follow from the remarks following Theorem 6.2.  $\blacksquare$

If  $\Theta : \mathbb{B}^n \rightarrow \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$  is a  $K$ -inner function, then  $\mathcal{W} = \Theta\mathcal{E} \subset \mathcal{H}_K \otimes \mathcal{E}_*$  is a closed subspace which is the generating wandering subspace for  $M_z$  restricted to  $\mathcal{S} = \bigvee_{\mathbf{k} \in \mathbb{N}^n} M_z^{\mathbf{k}}\mathcal{W}$ . Hence in the setting of Corollary 6.3 each closed  $M_z$ -invariant subspace  $\mathcal{S} = \bigvee_{\mathbf{k} \in \mathbb{N}^n} M_z^{\mathbf{k}}p \subset \mathcal{H}_K$  generated by a quasi-homogeneous polynomial  $p$  is generated by the wandering subspace  $\mathcal{W} = \mathbb{C}p = \mathcal{S} \ominus \sum_{i=1}^n M_{z_i}\mathcal{S}$ .

Corollary 6.3 applies in particular to the functional Hilbert spaces  $H(K_\lambda)$  ( $\lambda \geq 1$ ) with reproducing kernel  $K_\lambda(\mathbf{z}, \mathbf{w}) = (1 - \sum_{i=1}^n z_i \bar{w}_i)^{-\lambda}$ , since these spaces satisfy all hypotheses for  $\mathcal{H}_K$  contained in Corollary 6.3.

REMARK 6.4. It is well known (cf. Corollary 3.3 in [24]) that the unit sphere  $\partial\mathbb{B}^n$  is contained in the approximate point spectrum  $\sigma_\pi(M_z, H_n^2)$  of the multiplication tuple  $M_z \in \mathcal{B}(H_n^2)^n$ . Since the approximate point spectrum satisfies an analytic spectral mapping theorem (see Section 2.6 in [8] for the relevant definitions and the spectral mapping theorem), in dimension  $n > 1$

$$0 \in p(\sigma_\pi(M_z, H_n^2)) = \sigma_\pi(M_p, H_n^2)$$

for each homogeneous polynomial  $p$  of positive degree. Hence, for each such polynomial, the subspace  $pH_n^2 \subset H_n^2$  is non-closed. An elementary argument, using the fact that the inclusion  $H_n^2 \subset H(K_\lambda)$  is continuous, shows that also  $pH_n^2 \subset H(K_\lambda)$  is not closed. It follows that in dimension  $n > 1$  there is no chance to show that  $K_\lambda$ -inner multipliers have the expansive multiplier property proved in [17] for operator-valued Bergman inner functions on the unit disc.

By combining Theorem 4.1 (or Theorem 3.2 in [21]), Theorem 4.3 and Corollary 4.4 we obtain the following characterization of invariant subspaces of vector-valued contractive analytic Hilbert spaces.

THEOREM 6.5. *Let  $\mathcal{H}_K$  be a contractive analytic Hilbert space over  $\mathbb{B}^n$  and let  $\mathcal{E}_*$  be an arbitrary Hilbert space. Then a closed subspace  $\mathcal{S} \subset \mathcal{H}_K \otimes \mathcal{E}_*$  is invariant for  $M_z \otimes I_{\mathcal{E}_*}$  if and only if there exists a Hilbert space  $\mathcal{E}$  and a partially isometric  $K$ -multiplier  $\Theta \in \mathcal{M}(H_n^2 \otimes \mathcal{E}, \mathcal{H}_K \otimes \mathcal{E}_*)$  with*

$$\mathcal{S} = \Theta H_n^2(\mathcal{E}).$$

*If  $\mathcal{S} = \tilde{\Theta} H_n^2(\tilde{\mathcal{E}})$  is another representation of the same type, then there exists a partial isometry  $V : \mathcal{E} \rightarrow \tilde{\mathcal{E}}$  such that*

$$\Theta(z) = \tilde{\Theta}(z)V \quad (z \in \mathbb{B}^n).$$

*Furthermore, if  $\mathcal{S} = \Theta_c H_n^2(\mathcal{E}_c)$  is a canonical representation of  $\mathcal{S}$  in the sense of Remark 4.2, then*

$$\Theta_c(z) = \Theta(z)V_c \quad (z \in \mathbb{B}^n)$$

*for some isometry  $V_c : \mathcal{E}_c \rightarrow \mathcal{E}$ .*

The first part of the preceding theorem for the particular case  $\mathcal{H}_K = H_n^2$  is the Drury-Arveson space is a result of McCullough and Trent [13], which generalizes the classical Beurling-Lax-Halmos theorem to the multivariable case. The last part seems to be new even in the case of the Drury-Arveson space.

By applying Theorem 5.2 we obtain a generalization of a result of Olofsson (Theorem 4.1 in [16]) to a quite general multivariable setting.

THEOREM 6.6. *Let  $\mathcal{H}_K$  be a contractive analytic Hilbert space on  $\mathbb{B}^n$  and let  $\mathcal{E}_*$  be an arbitrary Hilbert space. Let  $\mathcal{S} = \Theta H_n^2(\mathcal{E})$  be a closed  $M_z$ -invariant subspace of  $\mathcal{H}_K \otimes \mathcal{E}_*$  represented by a partially isometric multiplier  $\Theta \in \mathcal{M}(H_n^2 \otimes \mathcal{E}, \mathcal{H}_K \otimes \mathcal{E}_*)$  as in Theorem 6.5. Then  $\Theta_0 : \mathbb{B}^n \rightarrow \mathcal{B}(\mathcal{F}, \mathcal{E}_*)$ ,  $\Theta_0(z) = \Theta(z)|_{\mathcal{F}}$ , where*

$$\mathcal{F} = \{\eta \in \mathcal{E} : M_\Theta^* M_\Theta \eta = \eta\},$$

defines a  $K$ -inner function such that the wandering subspace  $\mathcal{W} = \mathcal{S} \ominus \sum_{i=1}^n M_{z_i} \mathcal{S}$  of  $M_z$  restricted to  $\mathcal{S}$  is given by

$$\mathcal{W} = \Theta_0 \mathcal{F}.$$

The wandering subspace  $\mathcal{W}$  is generating for  $M_z|_{\mathcal{S}}$  if and only if

$$\mathcal{S} = \overline{\Theta H_n^2(\mathcal{F})}.$$

**Proof.** Since  $M_{\Theta}|_{(\ker M_{\Theta})^{\perp}}$  is an isometry and since

$$\mathcal{F} = (\ker M_{\Theta})^{\perp} \cap \mathcal{E},$$

the result follows as an application of Theorem 5.2 and Corollary 5.3 with  $T = M_z \in \mathcal{B}(\mathcal{H}_K \otimes \mathcal{E}_*)^n$ .  $\blacksquare$

By using once again the fact that  $M_{\Theta}$  is a partial isometry one can reformulate the necessary and sufficient condition for  $\mathcal{W}$  to be a generating wandering subspace for  $M_z|_{\mathcal{S}}$  given in Theorem 6.6.

**THEOREM 6.7.** *In the setting of Theorem 6.6 the space  $\mathcal{W} = \Theta \mathcal{F}$  is a generating wandering subspace for  $M_z|_{\mathcal{S}}$  if and only if*

$$(\ker M_{\Theta})^{\perp} \cap (H_n^2(\mathcal{F}))^{\perp} = \{0\}.$$

**Proof.** For  $f \in (\ker M_{\Theta})^{\perp}$  and  $h \in H_n^2(\mathcal{F})$ , the identity

$$\langle f, h \rangle = \langle f, P_{(\ker M_{\Theta})^{\perp}} h \rangle = \langle \Theta f, \Theta P_{(\ker M_{\Theta})^{\perp}} h \rangle = \langle \Theta f, \Theta h \rangle$$

holds. Using this observation one easily obtains the identity

$$\Theta H_n^2(\mathcal{E}) \ominus \overline{\Theta H_n^2(\mathcal{F})} = \Theta[(\ker M_{\Theta})^{\perp} \cap (H_n^2(\mathcal{F}))^{\perp}].$$

Since by Theorem 6.6 the space  $\mathcal{W} = \Theta \mathcal{F}$  is a generating wandering subspace for  $M_z|_{\mathcal{S}}$  if and only if the space on the left-hand side is the zero space, the assertion follows.  $\blacksquare$

We conclude by giving an example which shows that in the multivariable setting, even for the nicest analytic functional Hilbert spaces on  $\mathbb{B}^n$ , there are  $M_z$ -invariant subspaces which do not possess a generating wandering subspace.

**EXAMPLE 6.8.** For  $a \in \mathbb{B}^n$ , define  $\mathcal{S}_a = \{f \in H_n^2 : f(a) = 0\}$ . For  $a \neq 0$ , the wandering subspace  $\mathcal{W}_a = \mathcal{S}_a \ominus \sum_{i=1}^n M_{z_i} \mathcal{S}_a$  for  $M_z$  restricted to  $\mathcal{S}_a$  is one-dimensional (see Theorem 4.3 in [9]). Hence, if  $n > 1$ , then the common zero sets

$$Z(\mathcal{S}_a) = \{a\} \neq Z(\mathcal{W}_a) = Z\left(\bigvee_{\mathbf{k} \in \mathbb{N}^n} z^{\mathbf{k}} \mathcal{W}_a\right)$$

of  $\mathcal{S}_a$  and the invariant subspace generated by  $\mathcal{W}_a$  are different. Thus, for  $n > 1$  and  $a \neq 0$ , the restriction of  $M_z$  to  $\mathcal{S}_a$  has no generating wandering subspace. Since  $z_1, \dots, z_n \in \mathcal{S}_0 \ominus \sum_{i=1}^n M_{z_i} \mathcal{S}_0$  and since  $\mathcal{S}_0 = \bigvee \{z^{\mathbf{k}} z_j : \mathbf{k} \in \mathbb{N}^n \text{ and } j = 1, \dots, n\}$ , we obtain that  $\mathcal{S}_0$  possesses the generating wandering subspace  $\mathcal{W}_0 = \text{span}\{z_1, \dots, z_n\}$ . By using Corollary 4.6 in [9] one sees that the above observations remain true if  $H_n^2$  is replaced by the Hardy space  $H^2(\mathbb{B}^n)$  or the Bergman space  $L_a^2(\mathbb{B}^n)$ .

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