INVARIANT SUBSPACES AND THE C_{00}-PROPERTY OF BROWNIAN SHIFTS

NILANJAN DAS, SOMA DAS, AND JAYDEB SARKAR

Abstract. We introduce Brownian shifts on vector-valued Hardy spaces and describe their invariant subspaces. We then consider the restriction of Brownian shifts to their invariant subspaces and classify when they are unitarily equivalent. Additionally, we prove an asymptotic property stating that normalized Brownian shifts belong to the classical C_{00}-class.

Contents

1. Introduction

Let $\sigma > 0$ and $\theta \in [0, 2\pi)$. The *Brownian shift* of covariance σ and angle θ is the bounded linear operator $B_{\sigma,e^{i\theta}} : H^2(\mathbb{T}) \oplus \mathbb{C} \to H^2(\mathbb{T}) \oplus \mathbb{C}$, defined by

$$B_{\sigma,e^{i\theta}} = \begin{bmatrix} S & \sigma(1 \otimes 1) \\ 0 & e^{i\theta} \end{bmatrix},$$

where $S : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ is the shift operator on $H^2(\mathbb{T})$ and $H^2(\mathbb{T})$ denotes the Hardy space of complex-valued square integrable functions on the unit circle $\mathbb{T}$ in $\mathbb{C}$. Recall that

$$Sf = zf,$$

for all $f \in H^2(\mathbb{T})$. Moreover, the operator $1 \otimes 1 : \mathbb{C} \to H^2(\mathbb{T})$ is defined by $((1 \otimes 1)\alpha)(z) = \alpha$ for all $\alpha \in \mathbb{C}$ and $z \in \mathbb{T}$. Brownian shifts were introduced by Agler and Stankus in the context of m-isometries [1, Definition 5.5]. These operators are related to the time-shift operators associated with Brownian motion processes.

2020 *Mathematics Subject Classification.* 46J15, 30H10, 30J05, 60J65, 47A08.

Key words and phrases. Invariant subspaces, Brownian shifts, model spaces, inner functions, Hardy spaces, perturbations.

Determining the lattice of closed subspaces that are invariant under a given bounded linear operator is always an interesting problem. When the underlying operator is simple—particularly for naturally occurring operators—the problem becomes even more intriguing. In the case of the Brownian shifts, Agler and Stankus resolved the invariant subspace problem [1]. In this paper, we introduce Brownian shifts acting on vector-valued Hardy spaces, and solve the invariant subspace problem for them. These are generalizations of the settings and results of Agler and Stankus. Furthermore, we investigate the restrictions of Brownian shifts to their invariant subspaces in vector-valued Hardy spaces and determine when they are unitarily similar.

Let us first introduce Brownian shifts on vector-valued Hardy spaces. Given a Hilbert space E (all Hilbert spaces in this paper are separable and over $\mathbb{C}$), denote by $H^2_E(\mathbb{T})$ the E-valued Hardy space over $\mathbb{T}$. By S_E, we refer to the shift operator on $H^2_E(\mathbb{T})$. When $E = \mathbb{C}$, we simply write $H^2_{\mathbb{C}}(\mathbb{T})$ as $H^2(\mathbb{T})$, and $S_{\mathbb{C}}$ as S. The *Brownian shift on* $H^2_E(\mathbb{T}) \oplus E$ of covariance σ and angle θ is the bounded linear operator

$$B^E_{\sigma,e^{i\theta}} = \begin{bmatrix} S_E & \sigma i_E \\ 0 & e^{i\theta} I_E \end{bmatrix} : H^2_E(\mathbb{T}) \oplus E \to H^2_E(\mathbb{T}) \oplus E,$$

where $i_E : E \to H^2_E(\mathbb{T})$ is the inclusion map defined by $(i_E x)(z) = x$ for all $x \in E$ and $z \in \mathbb{T}$. Clearly, in the scalar case $E = \mathbb{C}$, we have $B^{\mathbb{C}}_{\sigma,e^{i\theta}} = B_{\sigma,e^{i\theta}}$. We often refer to $B_{\sigma,e^{i\theta}}$ as a Brownian shift on $H^2(\mathbb{T})$. Throughout the paper, E will denote an arbitrary but fixed Hilbert space, with the possibility that $E = \mathbb{C}$. One motivation for studying Brownian shifts on vector-valued Hardy spaces is that they are unitarily equivalent to the Brownian shifts tensored with identity operators. That is,

$$B^E_{\sigma,e^{i\theta}} \text{ on } H^2_E(\mathbb{T}) \oplus E \cong B_{\sigma,e^{i\theta}} \otimes I_E \text{ on } (H^2(\mathbb{T}) \oplus \mathbb{C}) \otimes E,$$

where "$\cong$"denotes unitary equivalence between operators. Invariant subspaces of Brownian shifts can be partitioned into two distinct types:

Definition 1.1. Let $\mathcal{M}$ be a closed subspace of $H^2_E(\mathbb{T}) \oplus E$ that is invariant under $B^E_{\sigma,e^{i\theta}}$. We say that:

1. $\mathcal{M}$ is Type I if $\mathcal{M} \subseteq H^2_E(\mathbb{T}) \oplus \{0\}$.
2. $\mathcal{M}$ is Type II if $\mathcal{M} \nsubseteq H^2_E(\mathbb{T}) \oplus \{0\}$.

Our first goal is to classify the invariant subspaces of Brownian shifts, which will exhibit fundamentally different structures in the Type I and Type II cases. To proceed, we first recall the basic and commonly used terminology: Given a Hilbert space E_*, let $H^\infty_{\mathcal{B}(E_*,E)}(\mathbb{T})$ denote the Banach space of $\mathcal{B}(E_*, E)$-valued bounded analytic functions on the open unit disc $\mathbb{D}$, where $\mathcal{B}(E_*, E)$ is the space of bounded linear operators from E_* to E (we write $H^\infty_{\mathcal{B}(E_*,E)}(\mathbb{T})$ as $H^\infty(\mathbb{T})$ whenever $E_* = E = \mathbb{C}$). Each $\Phi \in H^\infty_{\mathcal{B}(E_*,E)}(\mathbb{T})$ induces a bounded linear operator $M_\Phi : H^2_{E_*}(\mathbb{T}) \to H^2_E(\mathbb{T})$, defined by

$$M_\Phi f = \Phi f,$$

for all $f \in H^2_{E_*}(\mathbb{T})$. It is important to note that

$$M_\Phi S_{E_*} = S_E M_\Phi.$$

A function $\Phi \in H^\infty_{\mathcal{B}(E_*,E)}(\mathbb{T})$ is called *inner* if M_Φ is an isometry. This is equivalent to the condition that $\Phi(z)$ is an isometry from E_* to E for almost every $z \in \mathbb{T}$. Recall that for an inner function $\Phi \in H^\infty_{\mathcal{B}(E_*,E)}(\mathbb{T})$, the *model space* $\mathcal{K}_\Phi$ (cf. [9]) is defined by

$$\mathcal{K}_\Phi := H^2_E(\mathbb{T}) \ominus \Phi H^2_{E_*}(\mathbb{T}).$$

Fix a Brownian shift $B^E_{\sigma,e^{i\theta}}$. Given an inner function $\Phi \in H^\infty_{\mathcal{B}(E_*,E)}(\mathbb{T})$, we set

$$\mathcal{G}_\Phi = \left\{ \begin{bmatrix} g \\ y \end{bmatrix} \in H^2_E(\mathbb{T}) \oplus E : y \in E, g = \frac{\Phi x - \sigma y}{z - e^{i\theta}} \in H^2_E(\mathbb{T}) \text{ for some } x \in E_* \right\}. \tag{1.1}$$

In Theorems 2.1 and 2.6, we establish the following invariant subspace theorem for Brownian shifts: Let $\mathcal{M}$ be a nonzero closed subspace of $H^2_E(\mathbb{T}) \oplus E$. Then, the following are true:

(1) $\mathcal{M}$ is a Type I invariant subspace of $B^E_{\sigma,e^{i\theta}}$ if and only if

$$\mathcal{M} = \Phi H^2_{E_1}(\mathbb{T}) \oplus \{0\},$$

for some inner function $\Phi \in H^\infty_{\mathcal{B}(E_1,E)}(\mathbb{T})$ and nonzero Hilbert space E_1.

(2) $\mathcal{M}$ is a Type II invariant subspace of $B^E_{\sigma,e^{i\theta}}$ if and only if there exists an inner function $\Phi \in H^\infty_{\mathcal{B}(E_2,E)}(\mathbb{T})$ for some nonzero Hilbert space E_2, and a nonzero subset $\mathcal{G} \subseteq \mathcal{G}_\Phi$ such that

$$\mathcal{M} = \langle \mathcal{G} \rangle \oplus \left(\Phi H^2_{E_2}(\mathbb{T}) \oplus \{0\} \right).$$

Given a set $\mathcal{N}$ in a Hilbert space $\mathcal{H}$, we denote by $\langle \mathcal{N} \rangle$ the closed linear span of the elements of $\mathcal{N}$.

Definition 1.2. The representations of $\mathcal{M}$ in (1) and (2) above are referred to as the *canonical representations* of Type I and Type II invariant subspaces of $B^E_{\sigma,e^{i\theta}}$, respectively.

The inner function Φ in the canonical representation given in part (2) also exhibits a certain boundary behavior, similar to the scalar case studied by Agler and Stankus. We refer the reader to Remark 2.4 for more details. See also Section 5 for a detailed analysis of how the full-length invariant subspace theorem of Agler and Stankus can be recovered using the above result and Remark 2.4.

We take the above invariant subspace result to the next natural step. Specifically, given a pair of closed subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ of $H^2_E(\mathbb{T}) \oplus E$ that are invariant under the Brownian shifts $B^E_{\sigma_1,e^{i\theta_1}}$ and $B^E_{\sigma_2,e^{i\theta_2}}$, respectively, we consider the restriction operators $B^E_{\sigma_1,e^{i\theta_1}}$ and $B^E_{\sigma_2,e^{i\theta_2}}$ on $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, and determine when they are unitarily equivalent. In Theorem 3.1, we prove: There exists a unitary $U : \mathcal{M}_1 \to \mathcal{M}_2$ such that

$$U B^E_{\sigma_1,e^{i\theta_1}}\big|_{\mathcal{M}_1} = B^E_{\sigma_2,e^{i\theta_2}}\big|_{\mathcal{M}_2} U,$$

if and only if any one of the following conditions is true:

(1) Both $\mathcal{M}_1$ and $\mathcal{M}_2$ are Type I, and if $\mathcal{M}_i = \Phi_i H^2_{E_i}(\mathbb{T}) \oplus \{0\}$ for $i = 1, 2$, then $\dim E_1 = \dim E_2$.

(2) Both $\mathcal{M}_1$ and $\mathcal{M}_2$ are Type II. Furthermore, $\theta_1 = \theta_2$, and if $\mathcal{M}_j = \langle \mathcal{G}_j \rangle \oplus (\Phi_j H^2_{E_j}(\mathbb{T}) \oplus \{0\})$ is the canonical representation of $\mathcal{M}_j$, $j = 1, 2$, then there exist a pair of unitaries $U_{\mathcal{G}} : \langle \mathcal{G}_1 \rangle \to \langle \mathcal{G}_2 \rangle$ and $U_E : E_1 \to E_2$, such that

$$U_E x_1' = x_2',$$

whenever

$$U_{\mathcal{G}} \begin{bmatrix} g_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} g_2 \\ y_2 \end{bmatrix},$$

where $\begin{bmatrix} g_j \\ y_j \end{bmatrix} \in \langle \mathcal{G}_j \rangle$ with

$$g_j = \frac{\Phi_j x'_j - \sigma_j y_j}{z - e^{i\theta_j}},$$

for some unique $x'_j \in E_j$, $j = 1, 2$.

We refer to Theorem 5.2 for the scalar version of the above result.

Given such a pair of invariant subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ as above, we say that $\mathcal{M}_1$ and $\mathcal{M}_2$ are *unitarily equivalent* if there exists a unitary operator U satisfying the above intertwining relation. Similarly, one can define unitarily equivalent invariant subspaces of the shift operator on the Hardy space, the Bergman space, the Dirichlet space, and many more. Invariant subspaces of the shift on $H^2(\mathbb{T})$ are always unitarily equivalent, whereas they are never unitarily equivalent for the Bergman or Dirichlet spaces [11]. From this perspective, the Brownian shift exhibits a mixture of invariant subspaces—a property that is highly distinctive compared to other classical operators (see the examples in Section 7).

Now we turn to an asymptotic property of Brownian shifts. Let T be a contraction on a Hilbert space $\mathcal{H}$. We say that T is pure, denoted by $T \in C_{\cdot 0}$, if

$$SOT - \lim_{m \to \infty} T^{*m} = 0.$$

Furthermore, we say that T satisfies the C_{00}-property, which we simply write as $T \in C_{00}$, if both T and T^* are pure.

Operators in class $C_{\cdot 0}$ or C_{00} are of interest. The asymptotic property is often useful in representing these operators [9]. Examples of operators in C_{00} include strict contractions. Moreover, any operator can be scaled by a scalar so that the resulting operator belongs to C_{00}. However, scaling an operator is not always a desirable method for revealing its structure. In our present context, we first prove that Brownian shifts are not power-bounded and, hence, in particular, are not even similar to contractions. Next, we highlight a peculiar property of Brownian shifts: namely, we prove in Theorem 4.2 that for any covariance $\sigma > 0$ and angle $\theta \in [0, 2\pi)$, the normalized operator

$$\frac{1}{\|B^E_{\sigma, e^{i\theta}}\|} B^E_{\sigma, e^{i\theta}} \in C_{00}.$$

There are many reasons to study Brownian shifts, as also pointed out by Agler and Stankus in their paper [1]. For instance, Brownian shifts play a crucial role in understanding the structure of 2-isometries, a notion introduced by Agler decades ago (cf. [7]). We refer the reader to [1] for representations of 2-isometries and to [8] for some recent developments. In addition, we highlight that a Brownian shift on $H^2(\mathbb{T})$ can be thought of as a rank-one perturbation of an isometry. Indeed:

$$B_{\sigma, e^{i\theta}} = B_s + R, \tag{1.2}$$

where

$$B_s = \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix},$$

is an isometry, and

$$R = \begin{bmatrix} 0 & \sigma(1 \otimes 1) \\ 0 & e^{i\theta} - 1 \end{bmatrix},$$

is a rank one operator on $H^2(\mathbb{T}) \oplus \mathbb{C}$. The theory of perturbed operators and their invariant subspaces is certainly of interest. From this perspective, the result of Agler and Stankus on the invariant subspaces of Brownian shifts is particularly notable. Subsequently, the present work aims to shed new light on the general theory of operators and functions for Brownian shifts.

The remainder of the paper is organized as follows. In Section 2, we describe the invariant subspaces of Brownian shifts and classify them into two types (following Agler-Stankus): Type I and Type II. In the next section, Section 3, we determine when such pairs of invariant subspaces are unitarily equivalent. In Section 4, we prove that normalized Brownian shifts are always in C_{00}. Section 5 presents the results obtained this far in the context of the scalar-valued Hardy space. In particular, it highlights the unitary equivalence result for Brownian shifts on $H^2(\mathbb{T})$. Section 6 presents the structure of reducing subspaces for $B^E_{\sigma,e^{i\theta}}$, and points out that Brownian shifts on $H^2(\mathbb{T})$ are irreducible. In the final section, Section 7, we illustrate our results with some concrete examples.

2. Invariant subspaces

This section presents a complete description of the invariant subspaces of Brownian shifts $B^E_{\sigma,e^{i\theta}}$ on $H^2_E(\mathbb{T}) \oplus E$. In particular, this description recovers the invariant subspaces obtained by Agler and Stankus in the scalar-valued Hardy space setting. Moreover, our proof technique is new, even in the case considered by Agler and Stankus.

We begin with Type I invariant subspaces. Before proceeding, we note that $\mathcal{M} \subseteq H^2_E(\mathbb{T}) \oplus \{0\}$ is a nonzero closed subspace if and only if $\mathcal{M} = \mathcal{M}_0 \oplus \{0\}$ for some nonzero closed subspace $\mathcal{M}_0$ of $H^2_E(\mathbb{T})$.

Theorem 2.1. *Let $\mathcal{M}$ be a nonzero closed subspace of $H^2_E(\mathbb{T}) \oplus E$. Assume that $\mathcal{M} \subseteq H^2_E(\mathbb{T}) \oplus \{0\}$. Then $B^E_{\sigma,e^{i\theta}}(\mathcal{M}) \subseteq \mathcal{M}$ if and only if*

$$\mathcal{M} = \Phi H^2_{E_1}(\mathbb{T}) \oplus \{0\},$$

for some inner function $\Phi \in H^\infty_{\mathcal{B}(E_1,E)}(\mathbb{T})$ and nonzero Hilbert space E_1.

Proof. We know that $\mathcal{M} = \mathcal{M}_0 \oplus \{0\}$ for some nonzero closed subspace $\mathcal{M}_0 \subseteq H^2_E(\mathbb{T})$. Since

$$B^E_{\sigma,e^{i\theta}}\big|_{H^2_E(\mathbb{T})\oplus\{0\}} = \begin{bmatrix} S_E & 0 \\ 0 & 0 \end{bmatrix},$$

the fact that $\mathcal{M}$ is invariant under $B^E_{\sigma,e^{i\theta}}\big|_{H^2_E(\mathbb{T})\oplus\{0\}}$ is equivalent to $\mathcal{M}_0$ being invariant under S_E on $H^2_E(\mathbb{T})$. The classical Beurling-Lax-Halmos theorem (cf. [5, Theorem 2.1, p. 239]) guarantees that this is same as saying

$$\mathcal{M}_0 = \Phi H^2_{E_1}(\mathbb{T}),$$

for some inner function $\Phi \in H^\infty_{\mathcal{B}(E_1,E)}(\mathbb{T})$ and nonzero Hilbert space E_1. The converse follows from the upper triangular representation of the Brownian shifts, and we thus conclude the proof. □

We now proceed to the other type of invariant subspaces. For a fixed Brownian shift $B^E_{\sigma,e^{i\theta}}$ and an inner function $\Phi \in H^\infty_{\mathcal{B}(E_2,E)}(\mathbb{T})$, recall the construction of $\mathcal{G}_\Phi$ from (1.1):

$$\mathcal{G}_\Phi = \left\{ \begin{bmatrix} g \\ y \end{bmatrix} \in H^2_E(\mathbb{T}) \oplus E : y \in E, g = \frac{\Phi x - \sigma y}{z - e^{i\theta}} \in H^2_E(\mathbb{T}) \text{ for some } x \in E_2 \right\}.$$

In the following, we prove that the set $\mathcal{G}_\Phi$ is special:

Lemma 2.2. *Let $\Phi \in H^\infty_{\mathcal{B}(E_2,E)}(\mathbb{T})$ be an inner function. If $\begin{bmatrix} g \\ y \end{bmatrix} \in \mathcal{G}_\Phi$ is a nonzero element, then $g \in \mathcal{K}_\Phi$ and $y \neq 0$. Moreover, $\mathcal{G}_\Phi$ is a closed subspace of $H^2_E(\mathbb{T}) \oplus E$.*

Proof. Suppose $g = \frac{\Phi x - \sigma y}{z - e^{i\theta}} \in H^2_E(\mathbb{T})$ for some $x \in E_2$ and $y \in E$. We have $zg + \sigma y = e^{i\theta} g + \Phi x$. Let $\{f_j : j \geq 1\}$ be an orthonormal basis for E_2. Then for any $n \geq 1$ and $j \geq 1$, we have (note that $\Phi(0)^* \in \mathcal{B}(E, E_2)$)

$$\langle y, \Phi(z^n f_j)\rangle = \langle \Phi(0)^* y, z^n f_j\rangle = 0,$$

and similarly, we also have (note that $x \in E_2$) $\langle \Phi x, \Phi(z^n f_j)\rangle = 0$. Then

$$\langle zg + \sigma y, \Phi(z^n f_j)\rangle = \langle e^{i\theta} g + \Phi x, \Phi(z^n f_j)\rangle,$$

implies

$$\langle zg, \Phi(z^n f_j)\rangle + \sigma\langle y, \Phi(z^n f_j)\rangle = e^{i\theta}\langle g, \Phi(z^n f_j)\rangle + \langle \Phi x, \Phi(z^n f_j)\rangle,$$

which gives

$$\langle M^*_\Phi g, z^{n-1} f_j\rangle = e^{i\theta}\langle M^*_\Phi g, z^n f_j\rangle. \tag{2.1}$$

Let

$$M^*_\Phi g = \sum_{j=0}^\infty x_j z^j,$$

where $x_j \in E_2$ for all $j \geq 0$. If any one of these x_j's is nonzero, then by (2.1) we get

$$\|M^*_\Phi g\| = \infty,$$

which is not possible. This shows that $M^*_\Phi g = 0$, thereby proving the claim that $g \in \mathcal{K}_\Phi$. Next, assume that $y = 0$. Then $zg = \Phi x + e^{i\theta} g$, which implies

$$\|zg\|^2 = \|\Phi x\|^2 + \|g\|^2,$$

as $g \in \mathcal{K}_\Phi$. As $\|zg\| = \|g\|$ and $\|\Phi x\| = \|x\|$, we conclude that $x = 0$, and consequently $g = 0$.

Now we turn to prove that $\mathcal{G}_\Phi$ is a closed subspace. It is easy to see that $\mathcal{G}_\Phi$ is indeed a subspace. To show that it is closed, we pick a sequence $\left\{\begin{bmatrix} g_n \\ y_n \end{bmatrix}\right\} \in \mathcal{G}_\Phi$ such that $\begin{bmatrix} g_n \\ y_n \end{bmatrix} \to \begin{bmatrix} g \\ y \end{bmatrix} \in H^2_E(\mathbb{T}) \oplus E$. Equivalently, $g_n \to g$ in $H^2_E(\mathbb{T})$ and $y_n \to y$ in E. Now for each g_n, there is $x_n \in E_2$ such that

$$zg_n + \sigma y_n = \Phi x_n + e^{i\theta} g_n,$$

which means $\{\Phi x_n\}$ is convergent, and this implies $\{x_n\}$ is a Cauchy sequence in E_2. Thus $x_n \to x \in E_2$, and hence $\Phi x_n \to \Phi x$. From the above identity, by passing over to the limit we see that

$$zg + \sigma y = \Phi x + e^{i\theta} g,$$

that is, $g = \frac{\Phi x - \sigma y}{z - e^{i\theta}}$. This completes the proof of the lemma. □

Given a closed subspace $\mathcal{M} \subseteq H^2_E(\mathbb{T}) \oplus E$, the *defect space* of $\mathcal{M}$ is defined by

$$\mathcal{D}_{\mathcal{M}} = \mathcal{M} \ominus (\mathcal{M} \cap (H^2_E(\mathbb{T}) \oplus \{0\})).$$

Theorem 2.3. *Let $\mathcal{M}$ be a nonzero closed subspace of $H^2_E(\mathbb{T}) \oplus E$. Assume that $\mathcal{M} \nsubseteq H^2_E(\mathbb{T}) \oplus \{0\}$. If $B^E_{\sigma,e^{i\theta}}(\mathcal{M}) \subseteq \mathcal{M}$, then there exists an inner function $\Phi \in H^\infty_{\mathcal{B}(E_2,E)}(\mathbb{T})$ for some nonzero Hilbert space E_2, and a set $\mathcal{G} \subseteq \mathcal{G}_\Phi$ such that*

$$\mathcal{M} = \mathcal{D}_{\mathcal{M}} \oplus \left(\Phi H^2_{E_2}(\mathbb{T}) \oplus \{0\}\right),$$

and

$$\mathcal{D}_{\mathcal{M}} = \langle \mathcal{G} \rangle.$$

Moreover, if $\begin{bmatrix} g \\ y \end{bmatrix} \in \mathcal{D}_{\mathcal{M}}$, then there exists unique $x \in E_2$ with $\|x\| = \sigma\|y\|$ such that $g = \frac{\Phi x - \sigma y}{z - e^{i\theta}}$.

Proof. Set

$$\mathcal{M}_0 := \mathcal{M} \cap (H^2_E(\mathbb{T}) \oplus \{0\}),$$

and decompose $\mathcal{M}$ as

$$\mathcal{M} = \mathcal{D}_{\mathcal{M}} \oplus \mathcal{M}_0.$$

Since $\mathcal{M} \nsubseteq H^2_E(\mathbb{T}) \oplus \{0\}$, $\mathcal{D}_{\mathcal{M}} \neq \{0\}$. We need to show that $\mathcal{M}_0$ is also a nonzero subspace of $\mathcal{M}$. If possible, let $\mathcal{M}_0 = \{0\}$. By the assumption, there exists

$$F = \begin{bmatrix} f \\ x' \end{bmatrix} \in \mathcal{M}$$

such that $x' \neq 0$. Let us observe that

$$B^E_{\sigma,e^{i\theta}} \begin{bmatrix} f \\ x' \end{bmatrix} - e^{i\theta} \begin{bmatrix} f \\ x' \end{bmatrix} = \begin{bmatrix} zf + \sigma x' \\ e^{i\theta} x' \end{bmatrix} - e^{i\theta} \begin{bmatrix} f \\ x' \end{bmatrix} = \begin{bmatrix} (z - e^{i\theta})f + \sigma x' \\ 0 \end{bmatrix} \in \mathcal{M}_0.$$

As $\mathcal{M}_0 = \{0\}$, we have $(z - e^{i\theta})f + \sigma x' = 0$, that is,

$$\frac{\sigma}{e^{i\theta} - z} x' = f \in H^2_E(\mathbb{T}),$$

which is a contradiction, as $(e^{i\theta} - z)^{-1}$ is not square integrable over $\mathbb{T}$. Thus, $\mathcal{M}_0 \neq \{0\}$. As $B^E_{\sigma,e^{i\theta}}(\mathcal{M}) \subseteq \mathcal{M}$ and $\mathcal{M}_0 \subseteq H^2_E(\mathbb{T}) \oplus \{0\}$, in view of the upper triangular block matrix representation of $B^E_{\sigma,e^{i\theta}}$, it follows that

$$B^E_{\sigma,e^{i\theta}}(\mathcal{M}_0) \subseteq \mathcal{M}_0.$$

Theorem 2.1 ensures the existence of a nonzero Hilbert space E_2 and an inner function $\Phi \in H^\infty_{\mathcal{B}(E_2,E)}(\mathbb{T})$ such that

$$\mathcal{M}_0 = \Phi H^2_{E_2}(\mathbb{T}) \oplus \{0\}.$$

We can therefore rewrite $\mathcal{M}$ as

$$\mathcal{M} = \mathcal{D}_{\mathcal{M}} \oplus \left(\Phi H^2_{E_2}(\mathbb{T}) \oplus \{0\}\right).$$

Now for any nonzero $\begin{bmatrix} g \\ y \end{bmatrix} \in \mathcal{D}_{\mathcal{M}}$, it is easy to check that $g \in \mathcal{K}_\Phi$ and $y \neq 0$ in E. Moreover, we have

$$B^E_{\sigma, e^{i\theta}} \begin{bmatrix} g \\ y \end{bmatrix} = \begin{bmatrix} zg + \sigma y \\ e^{i\theta} y \end{bmatrix} = e^{i\theta} \begin{bmatrix} g \\ y \end{bmatrix} \oplus \begin{bmatrix} (z - e^{i\theta})g + \sigma y \\ 0 \end{bmatrix} \in \mathcal{M},$$

which, in particular, implies

$$(z - e^{i\theta})g + \sigma y \in \Phi H^2_{E_2}(\mathbb{T}),$$

and hence, there exists a unique $h \in H^2_{E_2}(\mathbb{T})$ (note that Φ is an inner function) such that

$$zg + \sigma y = e^{i\theta} g + \Phi h. \tag{2.2}$$

Let $\{f_j : j \geq 1\}$ be an orthonormal basis for E_2. Then for any $n \geq 1$ and $j \geq 1$, we have (note that $\Phi(0)^* \in \mathcal{B}(E, E_2)$)

$$\langle y, \Phi(z^n f_j) \rangle = \langle \Phi(0)^* y, z^n f_j \rangle = 0,$$

and (note again that Φ is an inner function)

$$\langle \Phi h, \Phi(z^n f_j) \rangle = \langle h, z^n f_j \rangle.$$

By (2.2), we now have

$$\begin{aligned} \langle h, z^n f_j \rangle &= \langle \Phi h, \Phi(z^n f_j) \rangle \\ &= \langle (zg + \sigma y) - e^{i\theta} g, \Phi(z^n f_j) \rangle \\ &= \langle zg, \Phi(z^n f_j) \rangle - e^{i\theta} \langle g, \Phi(z^n f_j) \rangle \\ &= \langle g, \Phi(z^{n-1} f_j) \rangle - e^{i\theta} \langle g, \Phi(z^n f_j) \rangle \\ &= 0, \end{aligned}$$

as $g \in \mathcal{K}_\Phi$. In other words, $h \in E_2$. Let us rename h by x. Since $y \in E$, by (2.2), we have

$$\|g\|^2 + \sigma^2 \|y\|^2 = \|g\|^2 + \|x\|^2,$$

and consequently $\|x\| = \sigma \|y\|$. Finally, again by (2.2), we have $g = \frac{\Phi x - \sigma y}{z - e^{i\theta}}$, completing the proof of the theorem. □

We emphasize that for a Type II invariant subspace $\mathcal{M}$, as described in the above theorem, both $\mathcal{D}_{\mathcal{M}}$ and $\Phi H^2_{E_2}(\mathbb{T})$ are nonzero.

The description of invariant subspaces of Brownian shifts on $H^2(\mathbb{T})$ by Agler and Stankus also includes a certain boundary property of the associated inner functions. A similar result holds in the setting of vector-valued Hardy spaces. However, to establish this, we need to use the identification of $H^2_E(\mathbb{T})$ with the Hardy space $H^2_E(\mathbb{D})$ of E-valued square-summable analytic functions on $\mathbb{D}$ [9, Chapter V].

Remark 2.4. We remain with the setting of Theorem 2.3. We have

$$\Phi x - \sigma y = (z - e^{i\theta}) g,$$

where $g \in \mathcal{K}_\Phi$, $y \in E \setminus \{0\}$, and $x \in E_2$. We treat g as an element of $H^2_E(\mathbb{D})$ and write the power series expansion $g(z) = \sum_{n=0}^\infty x_n z^n$ on $\mathbb{D}$ (note that $x_n \in E$ for all n). Then

$$\|g(z)\| \leq \sum_{n=0}^\infty \|x_n\| |z|^n \leq \left(\sum_{n=0}^\infty \|x_n\|^2 \right)^{\frac{1}{2}} \left(\sum_{n=0}^\infty |z|^{2n} \right)^{\frac{1}{2}} = \|g\|_2 \frac{1}{\sqrt{1 - |z|^2}},$$

for all $z \in \mathbb{D}$. Therefore, we have

$$\|\Phi(z)x - \sigma y\| \leq |z - e^{i\theta}| \frac{\|g\|_2}{\sqrt{1-|z|^2}},$$

which after putting $z = re^{i\theta}$ becomes

$$\|\Phi(re^{i\theta})x - \sigma y\| \leq \|g\|_2 \sqrt{\frac{1-r}{1+r}}.$$

Finally, letting $r \to 1-$ we get that $\Phi(e^{i\theta})x := \lim_{r\to 1-} \Phi(re^{i\theta})x$ exists and

$$\Phi(e^{i\theta})x = \sigma y.$$

In particular, if $E = \mathbb{C}$, then $\Phi(e^{i\theta})$ refers to the existence of the radial limit value of Φ at $e^{i\theta}$.

Before we state and prove a converse to Theorem 2.3, we recall the following observation from the setting of Lemma 2.2:

$$\langle \mathcal{G} \rangle \perp \Phi H^2_{E_2}(\mathbb{T}) \oplus \{0\},$$

for all nonzero $\mathcal{G} \subseteq \mathcal{G}_\Phi$. Moreover, if $\begin{bmatrix} g \\ y \end{bmatrix} \in \mathcal{G}$ is nonzero, then $y \neq 0$. In view of this, we now present a converse to Theorem 2.3:

Theorem 2.5. *Let $\Phi \in H^\infty_{\mathcal{B}(E_2,E)}(\mathbb{T})$ be an inner function. Then $\langle \mathcal{G} \rangle \oplus \big(\Phi H^2_{E_2}(\mathbb{T}) \oplus \{0\}\big)$ is a Type II invariant subspace of $B^E_{\sigma,e^{i\theta}}$ for every nonzero subset $\mathcal{G}$ of $\mathcal{G}_\Phi$.*

Proof. For any $f \in H^2_{E_2}(\mathbb{T})$, we have

$$B^E_{\sigma,e^{i\theta}} \begin{bmatrix} \Phi f \\ 0 \end{bmatrix} = \begin{bmatrix} \Phi z f \\ 0 \end{bmatrix} \in \Phi H^2_{E_2}(\mathbb{T}) \oplus \{0\},$$

and for $\begin{bmatrix} g \\ y \end{bmatrix} \in \mathcal{G}$,

$$B^E_{\sigma,e^{i\theta}} \begin{bmatrix} g \\ y \end{bmatrix} = \begin{bmatrix} zg + \sigma y \\ e^{i\theta} y \end{bmatrix} = e^{i\theta} \begin{bmatrix} g \\ y \end{bmatrix} + \begin{bmatrix} (z - e^{i\theta})g + \sigma y \\ 0 \end{bmatrix} = e^{i\theta} \begin{bmatrix} g \\ y \end{bmatrix} + \begin{bmatrix} \Phi x \\ 0 \end{bmatrix},$$

for some $x \in E_2$. Therefore,

$$B^E_{\sigma,e^{i\theta}}(\text{span}(\mathcal{G})) \subseteq \mathcal{M}.$$

Since $B^E_{\sigma,e^{i\theta}}$ is a bounded linear operator, $\mathcal{M}$ is a closed subspace, we have $B^E_{\sigma,e^{i\theta}}(\langle \mathcal{G} \rangle) \subseteq \mathcal{M}$. This proves that $\langle \mathcal{G} \rangle \oplus \big(\Phi H^2_{E_2}(\mathbb{T}) \oplus \{0\}\big)$ is a Type II invariant subspace of $B^E_{\sigma,e^{i\theta}}$. ☐

Summarizing Theorems 2.3 and 2.5, we obtain the following characterization of Type II invariant subspaces of Brownian shifts:

Theorem 2.6. *Let $\mathcal{M} \nsubseteq H^2_E(\mathbb{T}) \oplus \{0\}$ be a nonzero closed subspace of $H^2_E(\mathbb{T}) \oplus E$. Then $\mathcal{M}$ is invariant under $B^E_{\sigma,e^{i\theta}}$ if and only if there exists an inner function $\Phi \in H^\infty_{\mathcal{B}(E_2,E)}(\mathbb{T})$ for some nonzero Hilbert space E_2, and a nonzero subset $\mathcal{G} \subseteq \mathcal{G}_\Phi$ such that*

$$\mathcal{M} = \langle \mathcal{G} \rangle \oplus \left(\Phi H^2_{E_2}(\mathbb{T}) \oplus \{0\}\right).$$

The results of this section, when specialized to the case $E = \mathbb{C}$ (that is, the scalar case), recover the results of Agler and Stankus. We will elaborate on this in Section 5.

3. Unitary equivalence

Recall that the nonzero invariant subspaces of S in $H^2(\mathbb{T})$ are given by $\varphi H^2(\mathbb{T})$, where φ runs over all inner functions from $H^\infty(\mathbb{T})$ [2]. Given a pair of S-invariant subspaces $\varphi_1 H^2(\mathbb{T})$ and $\varphi_2 H^2(\mathbb{T})$ for some inner functions $\varphi_1, \varphi_2 \in H^\infty(\mathbb{T})$, we consider the restrictions $S|_{\varphi_1 H^2(\mathbb{T})}$ and $S|_{\varphi_2 H^2(\mathbb{T})}$ of S on $\varphi_1 H^2(\mathbb{T})$ and $\varphi_2 H^2(\mathbb{T})$, respectively. It is now easy to see that there exists a unitary operator $U : \varphi_1 H^2(\mathbb{T}) \to \varphi_2 H^2(\mathbb{T})$ such that

$$US|_{\varphi_1 H^2(\mathbb{T})} = S|_{\varphi_2 H^2(\mathbb{T})} U.$$

Therefore, as far as operators are concerned, restrictions of S on its invariant subspaces do not yield anything new. This prompts the question of distinguishing the restrictions of Brownian shifts on their invariant subspaces. We remind the reader that the invariant subspaces of Brownian shifts are also described by inner functions. In view of this, we now investigate the unitary equivalence of the invariant subspaces. At this point, it is convenient to recall the definition of the canonical representations of invariant subspaces of $B^E_{\sigma, e^{i\theta}}$, as given in Definition 1.2.

Theorem 3.1. *Fix angles $\theta_1, \theta_2 \in [0, 2\pi)$ and covariances $\sigma_1, \sigma_2 > 0$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be nonzero closed invariant subspaces of the Brownian shifts $B^E_{\sigma_1, e^{i\theta_1}}$ and $B^E_{\sigma_2, e^{i\theta_2}}$, respectively. Then*

$$B^E_{\sigma_1, e^{i\theta_1}}\big|_{\mathcal{M}_1} \cong B^E_{\sigma_2, e^{i\theta_2}}\big|_{\mathcal{M}_2},$$

if and only if any one of the following conditions is true:

1. *Both $\mathcal{M}_1$ and $\mathcal{M}_2$ are Type I, and if $\mathcal{M}_i = \Phi_i H^2_{E_i}(\mathbb{T}) \oplus \{0\}$ for $i = 1, 2$, then $\dim E_1 = \dim E_2$.*
2. *Both $\mathcal{M}_1$ and $\mathcal{M}_2$ are Type II. Furthermore, $\theta_1 = \theta_2$, and if $\mathcal{M}_j = \langle \mathcal{G}_j \rangle \oplus (\Phi_j H^2_{E_j}(\mathbb{T}) \oplus \{0\})$ is the canonical representation of $\mathcal{M}_j$, $j = 1, 2$, then there exist a pair of unitaries $U_{\mathcal{G}} : \langle \mathcal{G}_1 \rangle \to \langle \mathcal{G}_2 \rangle$ and $U_E : E_1 \to E_2$, such that*

$$U_E x_1' = x_2',$$

whenever

$$U_{\mathcal{G}} \begin{bmatrix} g_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} g_2 \\ y_2 \end{bmatrix},$$

where $\begin{bmatrix} g_j \\ y_j \end{bmatrix} \in \langle \mathcal{G}_j \rangle$ with

$$g_j = \frac{\Phi_j x_j' - \sigma_j y_j}{z - e^{i\theta_j}},$$

for some unique $x_j' \in E_j$, $j = 1, 2$.

Proof. Let us start with the proof of the "if" part. Provided that condition (1) holds, we assume $\mathcal{M}_i = \Phi_i H^2_{E_i}(\mathbb{T}) \oplus \{0\}$ for some inner functions $\Phi_i \in H^\infty_{\mathcal{B}(E_i, E)}(\mathbb{T})$, $i = 1, 2$, with $\dim E_1 = \dim E_2$. Then there exists a unitary operator $\hat{U}$ between E_1 and E_2. Consider the operator $U : \mathcal{M}_1 \to \mathcal{M}_2$, defined by

$$U \begin{bmatrix} \Phi_1 h \\ 0 \end{bmatrix} = \begin{bmatrix} \Phi_2 \tilde{U} h \\ 0 \end{bmatrix}$$

for all $h \in H^2_{E_1}(\mathbb{T})$, where $\tilde{U} : H^2_{E_1}(\mathbb{T}) \to H^2_{E_2}(\mathbb{T})$ is the unitary operator induced by $\hat{U}$, acting on h as follows:

$$\tilde{U}\tilde{h} = \sum_{n=0}^{\infty} z^n \hat{U} \tilde{x}_n, \tag{3.1}$$

for all $\tilde{h} = \sum_{n=0}^{\infty} \tilde{x}_n z^n \in H^2_{E_1}(\mathbb{T})$. Note that

$$z^k \tilde{U} h = \tilde{U} z^k h,$$

for all $k \geq 0$. It is clear from the construction that U is a surjective isometry between $\mathcal{M}_1$ and $\mathcal{M}_2$ and therefore, is unitary. Moreover,

$$B^E_{\sigma_2, e^{i\theta_2}} U \begin{bmatrix} \Phi_1 h \\ 0 \end{bmatrix} = \begin{bmatrix} z\Phi_2 \tilde{U} h \\ 0 \end{bmatrix} = \begin{bmatrix} \Phi_2 \tilde{U}(zh) \\ 0 \end{bmatrix} = U \begin{bmatrix} \Phi_1 zh \\ 0 \end{bmatrix} = U B^E_{\sigma_1, e^{i\theta_1}} \begin{bmatrix} \Phi_1 h \\ 0 \end{bmatrix},$$

which implies $B^E_{\sigma_1, e^{i\theta_1}}\big|_{\mathcal{M}_1} \cong B^E_{\sigma_2, e^{i\theta_2}}\big|_{\mathcal{M}_2}$. Next, we assume that (2) is true. Set

$$\theta = \theta_1 = \theta_2.$$

Define a linear operator $U : \mathcal{M}_1 \to \mathcal{M}_2$ by

$$U\left(\begin{bmatrix} \Phi_1 h \\ 0 \end{bmatrix} + \begin{bmatrix} g_1 \\ y_1 \end{bmatrix} \right) = \begin{bmatrix} \Phi_2 \tilde{U}_E h \\ 0 \end{bmatrix} + U_{\mathcal{G}} \begin{bmatrix} g_1 \\ y_1 \end{bmatrix},$$

for $h \in H^2_{E_1}(\mathbb{T})$ and $\begin{bmatrix} g_1 \\ y_1 \end{bmatrix} \in \langle \mathcal{G}_1 \rangle$, where $\tilde{U}_E$ is the unitary operator from $H^2_{E_1}(\mathbb{T})$ to $H^2_{E_2}(\mathbb{T})$, induced by U_E in the same way as in (3.1). It is evident from the construction that U is surjective. In addition, since $U_{\mathcal{G}} \begin{bmatrix} g_1 \\ y_1 \end{bmatrix} \in \langle \mathcal{G}_2 \rangle$, we have

$$\left\| \begin{bmatrix} \Phi_2 \tilde{U}_E h \\ 0 \end{bmatrix} + U_{\mathcal{G}} \begin{bmatrix} g_1 \\ y_1 \end{bmatrix} \right\|^2 = \|h\|^2 + \|g_1\|^2 + \|y_1\|^2 = \left\| \begin{bmatrix} \Phi_1 h \\ 0 \end{bmatrix} + \begin{bmatrix} g_1 \\ y_1 \end{bmatrix} \right\|^2.$$

This implies that U is an isometry, and therefore, is a unitary operator as well. For notational simplicity, set

$$F = \begin{bmatrix} \Phi_1 h \\ 0 \end{bmatrix} + \begin{bmatrix} g_1 \\ y_1 \end{bmatrix}.$$

Assuming $U_{\mathcal{G}} \begin{bmatrix} g_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} g_2 \\ y_2 \end{bmatrix}$, we observe that

$$B^E_{\sigma_2, e^{i\theta}} U F = B^E_{\sigma_2, e^{i\theta}} \left(\begin{bmatrix} \Phi_2 \tilde{U}_E h \\ 0 \end{bmatrix} + \begin{bmatrix} g_2 \\ y_2 \end{bmatrix} \right) = \begin{bmatrix} z\Phi_2 \tilde{U}_E h \\ 0 \end{bmatrix} + \begin{bmatrix} zg_2 + \sigma_2 y_2 \\ e^{i\theta} y_2 \end{bmatrix}.$$

At this point, we recall that $g_j(z) = \frac{\Phi_j x'_j - \sigma_j y_j}{z - e^{i\theta_j}}$ for a unique $x'_j \in E_j$, $j = 1, 2$. This yields

$$z g_j + \sigma_j y_j = e^{i\theta} g_j + \Phi_j x'_j,$$

for $j = 1, 2$. Therefore, recalling that $\tilde{U}_E x'_1 = U_E x'_1 = x'_2$ whenever $U_{\mathcal{G}} \begin{bmatrix} g_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} g_2 \\ y_2 \end{bmatrix}$, and that $z^k \tilde{U}_E h = \tilde{U}_E z^k h$ for any $k \geq 0$, we have

$$\begin{aligned} B^E_{\sigma_2, e^{i\theta}} UF &= \begin{bmatrix} z\Phi_2 \tilde{U}_E h \\ 0 \end{bmatrix} + \begin{bmatrix} zg_2 + \sigma_2 y_2 \\ e^{i\theta} y_2 \end{bmatrix} \\ &= U\left(\begin{bmatrix} \Phi_1 zh \\ 0 \end{bmatrix} + e^{i\theta} \begin{bmatrix} g_1 \\ y_1 \end{bmatrix} \right) + \begin{bmatrix} \Phi_2 \tilde{U}_E x'_1 \\ 0 \end{bmatrix} \\ &= U\left(\begin{bmatrix} \Phi_1 zh \\ 0 \end{bmatrix} + e^{i\theta} \begin{bmatrix} g_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} \Phi_1 x'_1 \\ 0 \end{bmatrix} \right) \\ &= U\left(\begin{bmatrix} z\Phi_1 h \\ 0 \end{bmatrix} + \begin{bmatrix} zg_1 + \sigma_1 y_1 \\ e^{i\theta} y_1 \end{bmatrix} \right) \\ &= UB^E_{\sigma_1, e^{i\theta}} \left(\begin{bmatrix} \Phi_1 h \\ 0 \end{bmatrix} + \begin{bmatrix} g_1 \\ y_1 \end{bmatrix} \right), \end{aligned}$$

that is,

$$B^E_{\sigma_2, e^{i\theta}} UF = UB^E_{\sigma_1, e^{i\theta}} F,$$

which again ensures that $B^E_{\sigma_1, e^{i\theta_1}}\big|_{\mathcal{M}_1} \cong B^E_{\sigma_2, e^{i\theta_2}}\big|_{\mathcal{M}_2}$. We now turn to the converse. For simplicity, we divide the proof into three parts:

Step 1: We claim that if we assume, without loss of generality, that $\mathcal{M}_1 = \Phi H^2_{E'}(\mathbb{T}) \oplus \{0\}$ is of Type I, and $\mathcal{M}_2 = \langle \mathcal{G} \rangle \oplus (\Psi H^2_{E''}(\mathbb{T}) \oplus \{0\})$ is of Type II, for certain inner functions Φ and Ψ, then $B^E_{\sigma_1, e^{i\theta_1}}\big|_{\mathcal{M}_1}$ and $B^E_{\sigma_2, e^{i\theta_2}}\big|_{\mathcal{M}_2}$ are not unitarily equivalent. Indeed, if it is not true, then, in particular, we will have the norm identity

$$\left\| B^E_{\sigma_1, e^{i\theta_1}}\big|_{\mathcal{M}_1} \right\| = \left\| B^E_{\sigma_2, e^{i\theta_2}}\big|_{\mathcal{M}_2} \right\|.$$

However, for any $h \in H^2_{E'}(\mathbb{T})$, we have

$$\left\| B^E_{\sigma_1, e^{i\theta_1}} \begin{bmatrix} \Phi h \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} z\Phi h \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} \Phi h \\ 0 \end{bmatrix} \right\|,$$

that is, $\left\| B^E_{\sigma_1, e^{i\theta_1}}\big|_{\mathcal{M}_1} \right\| = 1$. On the other hand, we have

$$\left\| B^E_{\sigma_2, e^{i\theta_2}} \begin{bmatrix} g \\ y \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} zg + \sigma_2 y \\ e^{i\theta_2} y \end{bmatrix} \right\|^2 = \|y\|^2 + \|g\|^2 + \sigma_2^2 \|y\|^2 = \left\| \begin{bmatrix} g \\ y \end{bmatrix} \right\|^2 + \sigma_2^2 \|y\|^2 > \left\| \begin{bmatrix} g \\ y \end{bmatrix} \right\|^2,$$

for any nonzero $\begin{bmatrix} g \\ y \end{bmatrix} \in \langle \mathcal{G} \rangle$ (recall that $y \neq 0$ in this case), and hence

$$\left\| B^E_{\sigma_2, e^{i\theta_2}}\big|_{\mathcal{M}_2} \right\| > 1,$$

which leads to a contradiction. Therefore, $\mathcal{M}_1$ and $\mathcal{M}_2$ must be of same type.

Step 2: Suppose both of them are Type I with the canonical decomposition

$$\mathcal{M}_i = \Phi_i H^2_{E_i}(\mathbb{T}) \oplus \{0\},$$

for some inner function $\Phi_i \in H^\infty_{\mathcal{B}(E_i, E)}(\mathbb{T})$, $i = 1, 2$. Now according to our hypothesis, there exists a unitary operator $U : \mathcal{M}_1 \to \mathcal{M}_2$ such that $UB^E_{\sigma_1, e^{i\theta_1}}\big|_{\mathcal{M}_1} = B^E_{\sigma_2, e^{i\theta_2}}\big|_{\mathcal{M}_2} U$. First, let us observe that for $h_1 \in H^2_{E_1}(\mathbb{T})$, if

$$U \begin{bmatrix} \Phi_1 h_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \Phi_2 h_2 \\ 0 \end{bmatrix},$$

for some $h_2 \in H^2_{E_2}(\mathbb{T})$, then

$$U \begin{bmatrix} z^n \Phi_1 h_1 \\ 0 \end{bmatrix} = U \left(B^E_{\sigma_1, e^{i\theta_1}} \right)^n \begin{bmatrix} \Phi_1 h_1 \\ 0 \end{bmatrix} = \left(B^E_{\sigma_2, e^{i\theta_2}} \right)^n U \begin{bmatrix} \Phi_1 h_1 \\ 0 \end{bmatrix} = \left(B^E_{\sigma_2, e^{i\theta_2}} \right)^n \begin{bmatrix} \Phi_2 h_2 \\ 0 \end{bmatrix},$$

that is,

$$U \begin{bmatrix} z^n \Phi_1 h_1 \\ 0 \end{bmatrix} = \begin{bmatrix} z^n \Phi_2 h_2 \\ 0 \end{bmatrix},$$

for all $n \geq 0$. Now, we take any $x_1'' \in E_1$, and assume that

$$U \begin{bmatrix} \Phi_1 x_1'' \\ 0 \end{bmatrix} = \begin{bmatrix} \Phi_2 h' \\ 0 \end{bmatrix},$$

for some $h'(z) = \sum_{n=0}^{\infty} z^n \tilde{x}_n' \in H^2_{E_2}(\mathbb{T})$. Suppose $\tilde{x}_k' \neq 0$ for some $k \geq 1$, and also suppose $U \begin{bmatrix} \Phi_1 g_k \\ 0 \end{bmatrix} = \begin{bmatrix} \Phi_2 \tilde{x}_k' \\ 0 \end{bmatrix}$ for some $g_k \in H^2_{E_1}(\mathbb{T})$. Therefore, $\|\tilde{x}_k'\|^2 = \langle \Phi_2 h', \Phi_2 z^k \tilde{x}_k' \rangle$ implies

$$\|\tilde{x}_k'\|^2 = \left\langle \begin{bmatrix} \Phi_2 h' \\ 0 \end{bmatrix}, \begin{bmatrix} z^k \Phi_2 \tilde{x}_k' \\ 0 \end{bmatrix} \right\rangle = \left\langle U \begin{bmatrix} \Phi_1 x_1'' \\ 0 \end{bmatrix}, U \begin{bmatrix} z^k \Phi_1 g_k \\ 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} \Phi_1 x_1'' \\ 0 \end{bmatrix}, \begin{bmatrix} \Phi_1 z^k g_k \\ 0 \end{bmatrix} \right\rangle,$$

and hence $\|\tilde{x}_k'\|^2 = \langle x_1'', z^k g_k \rangle = 0$ for any $k \geq 1$, which ensures that $h' \in E_2$. On the other hand, given $x_2'' \in E_2$, there exists $h(z) = \sum_{n=0}^{\infty} z^n \tilde{x}_n \in H^2_{E_1}(\mathbb{T})$ such that

$$U \begin{bmatrix} \Phi_1 h \\ 0 \end{bmatrix} = \begin{bmatrix} \Phi_2 x_2'' \\ 0 \end{bmatrix}.$$

If $\tilde{x}_k \neq 0$ for some $k \geq 1$, and if $U \begin{bmatrix} \Phi_1 \tilde{x}_k \\ 0 \end{bmatrix} = \begin{bmatrix} \Phi_2 h'' \\ 0 \end{bmatrix}$ for some $h'' \in H^2_{E_2}(\mathbb{T})$, then $\|\tilde{x}_k\|^2 = \langle \Phi_1 h, \Phi_1 z^k \tilde{x}_k \rangle$ implies

$$\|\tilde{x}_k\|^2 = \left\langle U \begin{bmatrix} \Phi_1 h \\ 0 \end{bmatrix}, U \begin{bmatrix} z^k \Phi_1 \tilde{x}_k \\ 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} \Phi_2 x_2'' \\ 0 \end{bmatrix}, \begin{bmatrix} z^k \Phi_2 h'' \\ 0 \end{bmatrix} \right\rangle = \langle x_2'', z^k h'' \rangle = 0,$$

thereby implying $h \in E_1$. The above information allows us to define $U_1 : E_1 \to E_2$ by

$$U_1 x_1 = x_2,$$

whenever

$$U \begin{bmatrix} \Phi_1 x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \Phi_2 x_2 \\ 0 \end{bmatrix}.$$

It is easy to check that U_1 is a well-defined, surjective as well as isometric linear map between E_1 and E_2, and therefore, is unitary. Thus, $\dim E_1 = \dim E_2$.

Step 3: Let us now assume that both $\mathcal{M}_1$ and $\mathcal{M}_2$ are Type II, with the canonical representations $\mathcal{M}_j = \langle \mathcal{G}_j \rangle \oplus (\Phi_j H^2_{E_j}(\mathbb{T}) \oplus \{0\})$, $j = 1, 2$, as described in the statement of this theorem, and, like in the previous part of the proof, $U : \mathcal{M}_1 \to \mathcal{M}_2$ is the unitary operator satisfying the intertwining relation $UB^E_{\sigma_1,e^{i\theta_1}}\big|_{\mathcal{M}_1} = B^E_{\sigma_2,e^{i\theta_2}}\big|_{\mathcal{M}_2}U$. Suppose now for a given $h_1 \in H^2_{E_1}(\mathbb{T})$,

$$U\begin{bmatrix}\Phi_1 h_1\\ 0\end{bmatrix} = \begin{bmatrix}\Phi_2 h_2\\ 0\end{bmatrix} + \begin{bmatrix}g_2'\\ y_2'\end{bmatrix}$$

for some $h_2 \in H^2_{E_2}(\mathbb{T})$ and $\begin{bmatrix}g_2'\\ y_2'\end{bmatrix} \in \langle \mathcal{G}_2\rangle$. In particular, we have

$$\|h_1\|^2 = \|h_2\|^2 + \|g_2'\|^2 + \|y_2'\|^2.$$

At the same time, we have

$$U\begin{bmatrix}z\Phi_1 h_1\\ 0\end{bmatrix} = UB^E_{\sigma_1,e^{i\theta_1}}\begin{bmatrix}\Phi_1 h_1\\ 0\end{bmatrix} = B^E_{\sigma_2,e^{i\theta_2}}U\begin{bmatrix}\Phi_1 h_1\\ 0\end{bmatrix} = \begin{bmatrix}z\Phi_2 h_2\\ 0\end{bmatrix} + \begin{bmatrix}zg_2' + \sigma_2 y_2'\\ e^{i\theta_2}y_2'\end{bmatrix},$$

and consequently,

$$\|h_1\|^2 = \|h_2\|^2 + \|g_2'\|^2 + \|y_2'\|^2 + \sigma_2^2\|y_2'\|^2.$$

Therefore, $y_2' = 0$, and then

$$\begin{bmatrix}g_2'\\ y_2'\end{bmatrix} \in \mathcal{M}_2 \cap (H^2_{E_2}(\mathbb{T}) \oplus \{0\}) = \Phi_2 H^2_{E_2}(\mathbb{T}) \oplus \{0\},$$

forcing $g_2' = 0$ as well. Therefore,

$$U(\Phi_1 H^2_{E_1}(\mathbb{T}) \oplus \{0\}) \subseteq \Phi_2 H^2_{E_2}(\mathbb{T}) \oplus \{0\}.$$

Using exactly similar lines of argument for U^*, we get

$$U^*(\Phi_2 H^2_{E_2}(\mathbb{T}) \oplus \{0\}) \subseteq \Phi_1 H^2_{E_1}(\mathbb{T}) \oplus \{0\},$$

and hence U is a unitary mapping between $\Phi_1 H^2_{E_1}(\mathbb{T}) \oplus \{0\}$ and $\Phi_2 H^2_{E_2}(\mathbb{T}) \oplus \{0\}$, such that

$$UB^E_{\sigma_1,e^{i\theta_1}}\big|_{\Phi_1 H^2_{E_1}(\mathbb{T})\oplus\{0\}} = B^E_{\sigma_2,e^{i\theta_2}}\big|_{\Phi_2 H^2_{E_2}(\mathbb{T})\oplus\{0\}}U.$$

Looking at the Step 2, where $\mathcal{M}_1$ and $\mathcal{M}_2$ both are taken to be Type I, it readily follows that for any $x_1 \in E_1$, $U\begin{bmatrix}\Phi_1 x_1\\ 0\end{bmatrix} = \begin{bmatrix}\Phi_2 x_2\\ 0\end{bmatrix}$ for some $x_2 \in E_2$, and conversely, for any $x_2 \in E_2$ there exists $x_1 \in E_1$ such that the above equality holds. This now guarantees the existence of a unitary map $U_E : E_1 \to E_2$, defined as follows:

$$U_E(x_1) = x_2,$$

whenever

$$U\begin{bmatrix}\Phi_1 x_1\\ 0\end{bmatrix} = \begin{bmatrix}\Phi_2 x_2\\ 0\end{bmatrix},$$

for $x_1 \in E_1, x_2 \in E_2$. For a given $\begin{bmatrix}g\\ y\end{bmatrix} \in \langle \mathcal{G}_1\rangle$, suppose now

$$U\begin{bmatrix}g\\ y\end{bmatrix} = \begin{bmatrix}\Phi_2 h'\\ 0\end{bmatrix} + \begin{bmatrix}g'\\ y'\end{bmatrix},$$

for some $h' \in H^2_{E_2}(\mathbb{T})$ and $\begin{bmatrix}g'\\ y'\end{bmatrix} \in \langle \mathcal{G}_2\rangle$. This implies

$$\|g\|^2 + \|y\|^2 = \|h'\|^2 + \|g'\|^2 + \|y'\|^2. \tag{3.2}$$

Now there exists $h \in H^2_{E_1}(\mathbb{T})$ such that $U \begin{bmatrix} \Phi_1 h \\ 0 \end{bmatrix} = \begin{bmatrix} \Phi_2 h' \\ 0 \end{bmatrix}$. As a result,

$$U \begin{bmatrix} g \\ y \end{bmatrix} = U \begin{bmatrix} \Phi_1 h \\ 0 \end{bmatrix} + \begin{bmatrix} g' \\ y' \end{bmatrix},$$

that is,

$$U \begin{bmatrix} g - \Phi_1 h \\ y \end{bmatrix} = \begin{bmatrix} g' \\ y' \end{bmatrix}.$$

Consequently,

$$\|g\|^2 + \|h\|^2 + \|y\|^2 = \left\| U \begin{bmatrix} g - \Phi_1 h \\ y \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} g' \\ y' \end{bmatrix} \right\|^2 = \|g'\|^2 + \|y'\|^2,$$

which, combined with (3.2) yields $h = h' = 0$. In other words, $U(\langle \mathcal{G}_1 \rangle) \subseteq \langle \mathcal{G}_2 \rangle$. Imitating this argument line by line with U replaced by U^*, it immediately follows that $U^*(\langle \mathcal{G}_2 \rangle) \subseteq \langle \mathcal{G}_1 \rangle$. We therefore conclude that there exists a unitary map $U_{\mathcal{G}} = U\big|_{\langle \mathcal{G}_1 \rangle}$ between $\langle \mathcal{G}_1 \rangle$ and $\langle \mathcal{G}_2 \rangle$. Now consider any $\begin{bmatrix} g_j \\ y_j \end{bmatrix} \in \langle \mathcal{G}_j \rangle$, $j = 1, 2$, such that $U \begin{bmatrix} g_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} g_2 \\ y_2 \end{bmatrix}$, and recall that

$$z g_j + \sigma_j y_j = e^{i\theta_j} g_j + \Phi_j x_j', \tag{3.3}$$

for a unique $x_j' \in E_j$. Then

$$\left\langle B^E_{\sigma_1, e^{i\theta_1}} \begin{bmatrix} g_1 \\ y_1 \end{bmatrix}, U^* \begin{bmatrix} g_2 \\ y_2 \end{bmatrix} \right\rangle = \left\langle B^E_{\sigma_2, e^{i\theta_2}} U \begin{bmatrix} g_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} g_2 \\ y_2 \end{bmatrix} \right\rangle.$$

The above identity implies

$$\left\langle \begin{bmatrix} z g_1 + \sigma_1 y_1 \\ e^{i\theta_1} y_1 \end{bmatrix}, \begin{bmatrix} g_1 \\ y_1 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} z g_2 + \sigma_2 y_2 \\ e^{i\theta_2} y_2 \end{bmatrix}, \begin{bmatrix} g_2 \\ y_2 \end{bmatrix} \right\rangle,$$

or, equivalently,

$$\left\langle e^{i\theta_1} \begin{bmatrix} g_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} \Phi_1 x_1' \\ 0 \end{bmatrix}, \begin{bmatrix} g_1 \\ y_1 \end{bmatrix} \right\rangle = \left\langle e^{i\theta_2} \begin{bmatrix} g_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} \Phi_2 x_2' \\ 0 \end{bmatrix}, \begin{bmatrix} g_2 \\ y_2 \end{bmatrix} \right\rangle,$$

and so

$$e^{i\theta_1} \left\| \begin{bmatrix} g_1 \\ y_1 \end{bmatrix} \right\|^2 = e^{i\theta_1} \left\| U \begin{bmatrix} g_1 \\ y_1 \end{bmatrix} \right\|^2 = e^{i\theta_2} \left\| \begin{bmatrix} g_2 \\ y_2 \end{bmatrix} \right\|^2.$$

It follows that $e^{i\theta_1} = e^{i\theta_2}$. Since $\theta_1, \theta_2 \in [0, 2\pi)$, we conclude that $\theta_1 = \theta_2$. Let us set again $\theta = \theta_1 = \theta_2$. Using this information and (3.3), we finally observe that

$$U \begin{bmatrix} \Phi_1 x_1' \\ 0 \end{bmatrix} + e^{i\theta} U \begin{bmatrix} g_1 \\ y_1 \end{bmatrix} = U \begin{bmatrix} z g_1 + \sigma_1 y_1 \\ e^{i\theta} y_1 \end{bmatrix} = U B^E_{\sigma_1, e^{i\theta}} \begin{bmatrix} g_1 \\ y_1 \end{bmatrix} = B^E_{\sigma_2, e^{i\theta}} \begin{bmatrix} g_2 \\ y_2 \end{bmatrix}.$$

As

$$B^E_{\sigma_2, e^{i\theta}} \begin{bmatrix} g_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} z g_2 + \sigma_2 y_2 \\ e^{i\theta} y_2 \end{bmatrix} = \begin{bmatrix} \Phi_2 x_2' \\ 0 \end{bmatrix} + e^{i\theta} \begin{bmatrix} g_2 \\ y_2 \end{bmatrix},$$

it follows that $U \begin{bmatrix} \Phi_1 x_1' \\ 0 \end{bmatrix} = \begin{bmatrix} \Phi_2 x_2' \\ 0 \end{bmatrix}$. As a consequence, $U_E x_1' = x_2'$. This completes the proof. □

This result becomes more concrete in the scalar case, which we will explain in Section 5.

4. $\frac{1}{\sqrt{1+\sigma^2}} B^E_{\sigma,e^{i\theta}} \in C_{00}$

The aim of this section is to prove that Brownian shifts, when scaled by their reciprocal norms, belong to C_{00}. Scaling an operator by its reciprocal norm does turn it into a contraction, but does not necessarily place it in the class C_{00} (nor even $C_{\cdot 0}$). Simply consider a unitary operator or the shift operator. This is where Brownian shifts exhibit different behavior.

We first prove that Brownian shifts are not even similar to contractions, and we establish this by showing that they are not power bounded. Recall that a bounded linear operator A acting on a Hilbert space $\mathcal{H}$ is said to be *power bounded* if the sequence of real numbers

$$\{\|A^n\|\}_{n=1}^{\infty},$$

is bounded. It is evident that any bounded linear operator similar to a contraction is power bounded; however, the converse is far from being true. Let us fix a Brownian shift $B^E_{\sigma,e^{i\theta}}$ on $H^2_E(\mathbb{T}) \oplus E$. Observe that for any nonzero $y \in E$ with $\|y\| = 1$:

$$B^E_{\sigma,e^{i\theta}} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} S_E & \sigma i_E \\ 0 & e^{i\theta} I_E \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} \sigma y \\ e^{i\theta} y \end{bmatrix}.$$

In general, by the principle of mathematical induction, we conclude that

$$\left(B^E_{\sigma,e^{i\theta}}\right)^m \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} \sigma \sum_{k=0}^{m-1} e^{ik\theta} z^{m-k-1} y \\ e^{im\theta} y \end{bmatrix}, \tag{4.1}$$

for all $m \geq 1$. Using the norm of functions in $H^2_E(\mathbb{T})$, we conclude that

$$\left\| \begin{bmatrix} \sigma \sum_{k=0}^{m-1} e^{ik\theta} z^{m-k-1} y \\ e^{im\theta} y \end{bmatrix} \right\|^2 = 1 + m\sigma^2.$$

As $\left\| \begin{bmatrix} 0 \\ y \end{bmatrix} \right\| = 1$, it follows that

$$\left\| \left(B^E_{\sigma,e^{i\theta}}\right)^m \right\|^2 \geq \left\| \left(B^E_{\sigma,e^{i\theta}}\right)^m \begin{bmatrix} 0 \\ y \end{bmatrix} \right\|^2 = 1 + m\sigma^2 \to \infty,$$

as $m \to \infty$. This proves the following:

Proposition 4.1. *$B^E_{\sigma,e^{i\theta}}$ on $H^2_E(\mathbb{T}) \oplus E$ is not power bounded for all covariance $\sigma > 0$ and angle $\theta \in [0, 2\pi)$.*

This in particular shows that $B^E_{\sigma,e^{i\theta}}$ is not similar to contractions. However, the following is true:

Theorem 4.2. *$\frac{1}{\left\|B^E_{\sigma,e^{i\theta}}\right\|} B^E_{\sigma,e^{i\theta}} \in C_{00}$ for all covariance $\sigma > 0$ and angle $\theta \in [0, 2\pi)$.*

Proof. For simplicity of notation, we set $B = B^E_{\sigma,e^{i\theta}}$. Let us begin by computing the norm of B. For any $\begin{bmatrix} f \\ y \end{bmatrix} \in H^2_E(\mathbb{T}) \oplus E$ with unit norm $\|f\|^2 + \|y\|^2 = 1$, we have, in particular, that $\|y\| \leq 1$. Moreover,

$$\left\| B \begin{bmatrix} f \\ y \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} zf + \sigma y \\ e^{i\theta} y \end{bmatrix} \right\|^2 = \|zf + \sigma y\|^2 + \|y\|^2 = 1 + \sigma^2 \|y\|^2 \leq 1 + \sigma^2,$$

and equality occurs for $f = 0$ and any $y \in E$ with $\|y\| = 1$. Therefore,

$$\|B\| = \sqrt{1 + \sigma^2},$$

and consequently, the operator

$$\tilde{B} := \frac{1}{\sqrt{1+\sigma^2}} B,$$

becomes a contraction on $H^2_E(\mathbb{T}) \oplus E$. Pick $u \in H^2_E(\mathbb{T}) \oplus E$ and write

$$u = \begin{bmatrix} 0 \\ x_0 \end{bmatrix} + \sum_{k=0}^{\infty} \begin{bmatrix} z^k x_{k+1} \\ 0 \end{bmatrix} \in H^2_E(\mathbb{T}) \oplus E,$$

for some $x_k \in E$ for $k \geq 0$. Making use of (4.1), a little computation reveals, for each $n \geq 1$, that

$$\begin{aligned} \left\| \tilde{B}^n u \right\|^2 &= \frac{1}{(1+\sigma^2)^n} \left\| B^n \begin{bmatrix} 0 \\ x_0 \end{bmatrix} + \sum_{k=0}^{\infty} B^n \begin{bmatrix} z^k x_{k+1} \\ 0 \end{bmatrix} \right\|^2 \\ &= \frac{1}{(1+\sigma^2)^n} \left\| \begin{bmatrix} \sigma \sum_{k=0}^{n-1} e^{ik\theta} z^{n-k-1} x_0 \\ e^{in\theta} x_0 \end{bmatrix} + \sum_{k=0}^{\infty} \begin{bmatrix} z^{n+k} x_{k+1} \\ 0 \end{bmatrix} \right\|^2 \\ &= \frac{1}{(1+\sigma^2)^n} \left((1 + n\sigma^2)\|x_0\|^2 + \sum_{k=0}^{\infty} \|x_{k+1}\|^2 \right), \end{aligned}$$

and hence

$$\left\| \tilde{B}^n u \right\|^2 = \frac{\|u\|^2 + n\|x_0\|^2 \sigma^2}{(1+\sigma^2)^n} \leq \frac{2}{\sigma^4} \left(\frac{\|u\|^2}{n(n-1)} + \frac{\sigma^2 \|x_0\|^2}{n-1} \right) \longrightarrow 0,$$

as $n \to \infty$. This implies that $\tilde{B}^* \in C_{\cdot 0}$. On the other hand, we know that

$$B^* = \begin{bmatrix} S_E^* & 0 \\ \sigma i_E^* & e^{-i\theta} I_E \end{bmatrix},$$

on $H^2_E(\mathbb{T}) \oplus E$. Therefore, we have

$$\tilde{B}^{*n} \begin{bmatrix} 0 \\ x_0 \end{bmatrix} = \frac{1}{(1+\sigma^2)^{\frac{n}{2}}} B^{*n} \begin{bmatrix} 0 \\ x_0 \end{bmatrix} = \frac{e^{-in\theta}}{(1+\sigma^2)^{\frac{n}{2}}} \begin{bmatrix} 0 \\ x_0 \end{bmatrix},$$

for all $n \geq 1$. Moreover, if $0 \leq k < n$, then

$$\begin{aligned} \tilde{B}^{*n} \begin{bmatrix} z^k x_{k+1} \\ 0 \end{bmatrix} &= \frac{1}{(1+\sigma^2)^{\frac{n}{2}}} B^{*n} \begin{bmatrix} z^k x_{k+1} \\ 0 \end{bmatrix} \\ &= \frac{1}{(1+\sigma^2)^{\frac{n}{2}}} B^{*n-k} \begin{bmatrix} x_{k+1} \\ 0 \end{bmatrix} \\ &= \frac{\sigma}{(1+\sigma^2)^{\frac{n}{2}}} B^{*n-k-1} \begin{bmatrix} 0 \\ x_{k+1} \end{bmatrix}, \end{aligned}$$

that is,

$$\tilde{B}^{*n}\begin{bmatrix} z^k x_{k+1} \\ 0 \end{bmatrix} = \frac{\sigma e^{-i(n-k-1)\theta}}{(1+\sigma^2)^{\frac{n}{2}}}\begin{bmatrix} 0 \\ x_{k+1} \end{bmatrix}.$$

Finally, for $k \geq n$, we have

$$\tilde{B}^{*n}\begin{bmatrix} z^k x_{k+1} \\ 0 \end{bmatrix} = \frac{1}{(1+\sigma^2)^{\frac{n}{2}}} B^{*n}\begin{bmatrix} z^k x_{k+1} \\ 0 \end{bmatrix} = \frac{1}{(1+\sigma^2)^{\frac{n}{2}}}\begin{bmatrix} z^{k-n} x_{k+1} \\ 0 \end{bmatrix}.$$

Therefore, we have

$$\begin{aligned}
\left\|\tilde{B}^{*n}u\right\|^2 &= \left\|\tilde{B}^{*n}\begin{bmatrix} 0 \\ x_0 \end{bmatrix} + \sum_{k=0}^{n-1}\tilde{B}^{*n}\begin{bmatrix} z^k x_{k+1} \\ 0 \end{bmatrix} + \sum_{k=n}^{\infty}\tilde{B}^{*n}\begin{bmatrix} z^k x_{k+1} \\ 0 \end{bmatrix}\right\|^2 \\
&= \frac{1}{(1+\sigma^2)^n}\left\| e^{-in\theta}\begin{bmatrix} 0 \\ x_0 \end{bmatrix} + \sigma\sum_{k=0}^{n-1} e^{-i(n-k-1)\theta}\begin{bmatrix} 0 \\ x_{k+1} \end{bmatrix} + \sum_{k=n}^{\infty}\begin{bmatrix} z^{k-n} x_{k+1} \\ 0 \end{bmatrix}\right\|^2 \\
&= \frac{1}{(1+\sigma^2)^n}\left(\left\| e^{-in\theta}x_0 + \sigma\sum_{k=0}^{n-1} e^{-i(n-k-1)\theta}x_{k+1}\right\|^2 + \sum_{k=n}^{\infty}\|x_{k+1}\|^2\right) \\
&\leq \frac{1}{(1+\sigma^2)^n}\left(\left(\|x_0\| + \sigma\sum_{k=0}^{n-1}\|x_{k+1}\|\right)^2 + \sum_{k=n}^{\infty}\|x_{k+1}\|^2\right) \\
&\leq \frac{1}{(1+\sigma^2)^n}\left((1+n\sigma^2)\left(\sum_{k=0}^{n}\|x_k\|^2\right) + \sum_{k=n}^{\infty}\|x_{k+1}\|^2\right) \\
&\leq \|u\|^2\frac{2+n\sigma^2}{(1+\sigma^2)^n} \\
&\leq \frac{2\|u\|^2}{\sigma^4}\left(\frac{2}{n(n-1)} + \frac{\sigma^2}{n-1}\right).
\end{aligned}$$

But

$$\frac{2}{n(n-1)} + \frac{\sigma^2}{n-1} \longrightarrow 0,$$

as $n \to \infty$. As a result, $\tilde{B} \in C_{\cdot 0}$, which completes the proof of the theorem. □

As we have proved in the theorem above that $\left\|B^E_{\sigma,e^{i\theta}}\right\| = \sqrt{1+\sigma^2}$, it follows that $\frac{1}{\sqrt{1+\sigma^2}}B^E_{\sigma,e^{i\theta}} \in C_{00}$. In particular, if $\mathcal{M}$ is an invariant subspace of $B^E_{\sigma,e^{i\theta}}$, then the compression operator

$$\frac{1}{\sqrt{1+\sigma^2}}P_{\mathcal{M}^\perp}B^E_{\sigma,e^{i\theta}}|_{\mathcal{M}^\perp} \in C_{\cdot 0}.$$

Compressions of this type, in the context of shifts on vector-valued Hardy spaces, are fundamental in the theory of linear operators. We refer the reader to the classic reference [9] for further details.

5. The scalar case

The purpose of this section is to recover the representations of invariant subspaces of Brownian shifts on $H^2(\mathbb{T})$, as obtained by Agler and Stankus (see [1, pp. 21-24]). More specifically, in the particular case of $E = \mathbb{C}$, by using Theorem 2.1 and combining Theorems 2.3 and 2.5 with Remark 2.4, we retrieve the representations of invariant subspaces of Agler and Stankus:

Theorem 5.1. *Let $\mathcal{M}$ be a nonzero closed subspace of $H^2(\mathbb{T}) \oplus \mathbb{C}$. Then $\mathcal{M}$ is invariant under $B_{\sigma,e^{i\theta}}$ if and only if it admits one of the following representations:*

$$\mathcal{M} = \varphi H^2(\mathbb{T}) \oplus \{0\},$$

for some inner function $\varphi \in H^\infty(\mathbb{T})$, or

$$\mathcal{M} = \mathbb{C} \begin{bmatrix} g \\ 1 \end{bmatrix} \oplus \left(\psi H^2(\mathbb{T}) \oplus \{0\}\right),$$

where $\psi \in H^\infty(\mathbb{T})$ is an inner function with the condition that $\psi(e^{i\theta})$ exists, and

$$g = \sigma \left(\frac{\overline{\psi(e^{i\theta})}\psi - 1}{z - e^{i\theta}} \right) \in H^2(\mathbb{T}).$$

Proof. We start with the proof of "only if" part. Suppose $\mathcal{M} \subseteq H^2(\mathbb{T}) \oplus \{0\}$. If $B_{\sigma,e^{i\theta}}(\mathcal{M}) \subseteq \mathcal{M}$, then setting $E = \mathbb{C}$ in Theorem 2.1, we find that $E_1 = \mathbb{C}$ (note that $\mathcal{M} \neq \{0\}$), and

$$\mathcal{M} = \varphi H^2(\mathbb{T}) \oplus \{0\},$$

for some inner function $\varphi \in H^\infty(\mathbb{T})$. Next, assume that $\mathcal{M} \nsubseteq H^2(\mathbb{T}) \oplus \{0\}$. Under the assumption $E = \mathbb{C}$, Theorem 2.3 asserts that

$$\mathcal{M} = \mathcal{D}_{\mathcal{M}} \oplus (\psi H^2(\mathbb{T}) \oplus \{0\}),$$

for some inner function $\psi \in H^\infty(\mathbb{T})$, where

$$\mathcal{D}_{\mathcal{M}} = \mathcal{M} \ominus (\mathcal{M} \cap (H^2(\mathbb{T}) \oplus \{0\})) \neq \{0\}.$$

It now remains to show that any element of $\mathcal{D}_{\mathcal{M}}$ is a scalar multiple of $\begin{bmatrix} g \\ 1 \end{bmatrix}$, g as given in the statement of the theorem. Making use of Theorem 2.3 and Remark 2.4, we see that for any nonzero $\begin{bmatrix} g_1 \\ \beta \end{bmatrix} \in \mathcal{D}_{\mathcal{M}}$, there exists unique $\alpha \in \mathbb{C}$ with $|\alpha| = \sigma|\beta| > 0$ such that

$$g_1 = \frac{\psi\alpha - \sigma\beta}{z - e^{i\theta}}, \tag{5.1}$$

and

$$\lim_{r \to 1-} \psi(re^{i\theta})\alpha = \sigma\beta.$$

Since $|\alpha| = \sigma|\beta| > 0$, $|\psi(e^{i\theta})| = 1$. As a result,

$$\alpha = \sigma\overline{\psi(e^{i\theta})}\beta.$$

Using this value of α in (5.1), we deduce that

$$g_1 = \beta\sigma \left(\frac{\overline{\psi(e^{i\theta})}\psi - 1}{z - e^{i\theta}} \right),$$

which means $\begin{bmatrix} g_1 \\ \beta \end{bmatrix} = \beta \begin{bmatrix} g \\ 1 \end{bmatrix}$. The "if" part follows trivially from Theorems 2.1 and 2.5. □

In the scalar case, the unitary equivalence of invariant subspaces established in Theorem 3.1 takes the following form:

Theorem 5.2. *Fix angles $\theta_1, \theta_2 \in [0, 2\pi)$ and covariances $\sigma_1, \sigma_2 > 0$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be nonzero closed invariant subspaces of the Brownian shifts $B_{\sigma_1, e^{i\theta_1}}$ and $B_{\sigma_2, e^{i\theta_2}}$ in $H^2(\mathbb{T}) \oplus \mathbb{C}$, respectively. Then*

$$B_{\sigma_1, e^{i\theta_1}}\big|_{\mathcal{M}_1} \cong B_{\sigma_2, e^{i\theta_2}}\big|_{\mathcal{M}_2},$$

if and only if any one of the following conditions is true:

(1) *Both $\mathcal{M}_1$ and $\mathcal{M}_2$ are Type I.*
(2) *Both $\mathcal{M}_1$ and $\mathcal{M}_2$ are Type II, along with the facts that*

$$\theta_1 = \theta_2,$$

and

$$\sigma_2^2(1 + \|g_1\|^2) = \sigma_1^2(1 + \|g_2\|^2),$$

where $\mathcal{M}_j = \mathbb{C}\begin{bmatrix} g_j \\ 1 \end{bmatrix} \oplus (\varphi_j H^2(\mathbb{T}) \oplus \{0\})$ is the canonical representation of $\mathcal{M}_j$, and $g_j = \sigma_j \left(\frac{\overline{\varphi_j(e^{i\theta_j})}\varphi_j - 1}{z - e^{i\theta_j}} \right)$ for $j = 1, 2$.

Proof. It is clear that we can now assume $E_1 = E_2 = \mathbb{C}$ and $\mathcal{G}_1 = \mathbb{C}\begin{bmatrix} g_1 \\ 1 \end{bmatrix}$, $\mathcal{G}_2 = \mathbb{C}\begin{bmatrix} g_2 \\ 1 \end{bmatrix}$ in the statement of Theorem 3.1. All we have to verify is that condition (2) of Theorem 3.1 becomes equivalent to condition (2) of this theorem for $E = \mathbb{C}$. In both cases $\theta_1 = \theta_2$ is common in condition (2), so we only need to concentrate on the rest. Suppose

$$U_{\mathcal{G}} \begin{bmatrix} g_1 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} g_2 \\ 1 \end{bmatrix}$$

for some $\alpha \in \mathbb{C}$. It is evident that $1 + \|g_1\|^2 = |\alpha|^2(1 + \|g_2\|^2)$, and at the same time, according to condition (2) of Theorem 3.1

$$U_E \left(\sigma_1 \overline{\varphi_1(e^{i\theta})} \right) = \alpha \sigma_2 \overline{\varphi_2(e^{i\theta})},$$

which gives $|\alpha| = \sigma_1/\sigma_2$. As a result,

$$\sigma_2^2(1 + \|g_1\|^2) = \sigma_1^2(1 + \|g_2\|^2). \tag{5.2}$$

Conversely, if we start by assuming (5.2), then it is immediately seen that

$$U_{\mathcal{G}} \begin{bmatrix} g_1 \\ 1 \end{bmatrix} = \frac{\sigma_1}{\sigma_2} \begin{bmatrix} g_2 \\ 1 \end{bmatrix},$$

and

$$U_E\left(\overline{\varphi_1(e^{i\theta})}\right) = \overline{\varphi_2(e^{i\theta})}$$

define surjective isometries $\langle \mathcal{G}_1 \rangle \to \langle \mathcal{G}_2 \rangle$ and $E_1 \to E_2$, respectively, and therefore, they are unitaries. Our proof is therefore done. $\square$

Setting $\varphi_2 = 1$, $g_2 = 0$ in the above theorem, we get that for any Type II invariant subspace $\mathcal{M}_1$ of $B_{\sigma_1, e^{i\theta_1}}$,

$$B_{\sigma_1, e^{i\theta_1}}\big|_{\mathcal{M}_1} \cong B_{\frac{\sigma_1}{\sqrt{1+\|g_1\|^2}}, e^{i\theta_1}}.$$

This was previously observed by Agler and Stankus in [1, Proposition 5.75].

Therefore, in contrast to the shift operator on $H^2(\mathbb{T})$, the invariant subspaces of the Brownian shifts have the potential to lead to different operators. In the final section of this paper, we illustrate these results with concrete examples.

6. Reducing subspaces

Given a bounded linear operator A on a Hilbert space $\mathcal{H}$, a closed subspace $\mathcal{S} \subseteq \mathcal{H}$ is said to be *reducing* for A (or A-reducing) if $\mathcal{S}$ is invariant under both A and A^*. Our goal here is to classify all reducing subspaces of Brownian shifts.

Recall that a closed subspace $\mathcal{M}$ of $H^2_E(\mathbb{T})$ is reducing for S_E if and only if there exists a closed subspace $F \subseteq E$ such that $\mathcal{M} = H^2_F(\mathbb{T})$ (cf. [10, Theorem 3.22]). For the question of reducing subspaces of a Brownian shift $B^E_{\sigma, e^{i\theta}}$ on $H^2_E(\mathbb{T}) \oplus E$, the answer is as follows:

Theorem 6.1. *Let $\mathcal{M}$ be a closed subspace of $H^2_E(\mathbb{T}) \oplus E$. Then $\mathcal{M}$ is a reducing subspace for $B^E_{\sigma, e^{i\theta}}$ if and only if there exists a closed subspace G of E such that*

$$\mathcal{M} = H^2_G(\mathbb{T}) \oplus G.$$

Proof. Given the adjoint operator $\left(B^E_{\sigma, e^{i\theta}}\right)^* = \begin{bmatrix} S_E^* & 0 \\ \sigma i_E^* & e^{-i\theta} I_E \end{bmatrix}$, the sufficiency is straightforward. Suppose $\mathcal{M}$ is a reducing subspace for $B^E_{\sigma, e^{i\theta}}$. Let $\begin{bmatrix} f \\ x \end{bmatrix} \in \mathcal{M}$. Then

$$\begin{bmatrix} f \\ \sigma^2 x + x \end{bmatrix} = \left(B^E_{\sigma, e^{i\theta}}\right)^* B^E_{\sigma, e^{i\theta}} \begin{bmatrix} f \\ x \end{bmatrix} \in \mathcal{M},$$

and hence

$$\begin{bmatrix} 0 \\ \sigma^2 x \end{bmatrix} = \begin{bmatrix} f \\ \sigma^2 x + x \end{bmatrix} - \begin{bmatrix} f \\ x \end{bmatrix} \in \mathcal{M}.$$

As $\sigma > 0$, this implies $\begin{bmatrix} 0 \\ x \end{bmatrix} \in \mathcal{M}$. On the other hand, since $B^E_{\sigma, e^{i\theta}} \begin{bmatrix} 0 \\ x \end{bmatrix} = \begin{bmatrix} \sigma x \\ e^{i\theta} x \end{bmatrix} \in \mathcal{M}$, it follows that

$$\begin{bmatrix} \sigma x \\ e^{i\theta} x \end{bmatrix} - e^{i\theta} \begin{bmatrix} 0 \\ x \end{bmatrix} = \sigma \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{M},$$

and hence, $\begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{M}$, and also, $\begin{bmatrix} f \\ 0 \end{bmatrix} = \begin{bmatrix} f \\ x \end{bmatrix} - \begin{bmatrix} 0 \\ x \end{bmatrix} \in \mathcal{M}$. Therefore, for each $\begin{bmatrix} f \\ x \end{bmatrix} \in \mathcal{M}$, we have

$$\left\{ \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x \end{bmatrix}, \begin{bmatrix} f \\ 0 \end{bmatrix} \right\} \subseteq \mathcal{M}. \tag{6.1}$$

Write $f = \sum_{n=0}^{\infty} \hat{f}(n) z^n$, where $\hat{f}(n) \in E$ for all $n \geq 0$. Then $\left(B^E_{\sigma, e^{i\theta}}\right)^* \begin{bmatrix} f \\ 0 \end{bmatrix} = \begin{bmatrix} S_E^* f \\ \sigma \hat{f}(0) \end{bmatrix} \in \mathcal{M}$, and hence (6.1) implies $\left\{ \begin{bmatrix} \hat{f}(0) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \hat{f}(0) \end{bmatrix}, \begin{bmatrix} S_E^* f \\ 0 \end{bmatrix} \right\} \subseteq \mathcal{M}$. Assuming that

$$\left\{ \begin{bmatrix} \hat{f}(m-1) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \hat{f}(m-1) \end{bmatrix}, \begin{bmatrix} {S_E^*}^m f \\ 0 \end{bmatrix} \right\} \subseteq \mathcal{M}$$

for any $m \geq 1$, we observe that $\left(B^E_{\sigma, e^{i\theta}}\right)^* \begin{bmatrix} {S_E^*}^m f \\ 0 \end{bmatrix} = \begin{bmatrix} {S_E^*}^{m+1} f \\ \sigma \hat{f}(m) \end{bmatrix} \in \mathcal{M}$, and again from (6.1), it follows that

$$\left\{ \begin{bmatrix} \hat{f}(m) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \hat{f}(m) \end{bmatrix}, \begin{bmatrix} {S_E^*}^{m+1} f \\ 0 \end{bmatrix} \right\} \subseteq \mathcal{M}.$$

The principle of mathematical induction now yields

$$\left\{ \begin{bmatrix} \hat{f}(n) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \hat{f}(n) \end{bmatrix} : n \geq 0 \right\} \subseteq \mathcal{M}.$$

Set $G = \langle G_0 \rangle$, where

$$G_0 = \left\{ x, \hat{f}(n) : \begin{bmatrix} f \\ x \end{bmatrix} \in \mathcal{M}, n \geq 0 \right\}.$$

Therefore, G is a closed subspace of E. Moreover, we let

$$G_1 = \left\{ \begin{bmatrix} 0 \\ x \end{bmatrix}, \begin{bmatrix} 0 \\ \hat{f}(n) \end{bmatrix} : \begin{bmatrix} f \\ x \end{bmatrix} \in \mathcal{M}, n \geq 0 \right\},$$

and

$$G_2 = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} \hat{f}(n) \\ 0 \end{bmatrix} : \begin{bmatrix} f \\ x \end{bmatrix} \in \mathcal{M}, n \geq 0 \right\}.$$

Then $\langle G_1 \rangle = \{0\} \oplus G$ and $\langle G_2 \rangle = G \oplus \{0\}$. Now $\langle G_2 \rangle \subseteq \mathcal{M}$ implies $H^2_G(\mathbb{T}) \oplus \{0\} \subseteq \mathcal{M}$. This and $\{0\} \oplus G = \langle G_1 \rangle \subseteq \mathcal{M}$ yields

$$H^2_G(\mathbb{T}) \oplus G \subseteq \mathcal{M}.$$

For the reverse inclusion, we pick $\begin{bmatrix} f \\ x \end{bmatrix} \in \mathcal{M}$, and assume $\begin{bmatrix} f \\ x \end{bmatrix} \perp (H^2_G(\mathbb{T}) \oplus G)$. In particular, $\begin{bmatrix} f \\ x \end{bmatrix} \perp \{0\} \oplus G$ implies $x = 0$. Similarly, $\begin{bmatrix} \hat{f}(n) \\ 0 \end{bmatrix} \perp G \oplus \{0\}$ for all $n \geq 0$ implies that $f = 0$. This proves $H^2_G(\mathbb{T}) \oplus G = \mathcal{M}$. □

Recall that a bounded linear operator A on $\mathcal{H}$ is irreducible if there is no nontrivial closed subspace of $\mathcal{H}$ that reduces A. The following is now straightforward:

Corollary 6.2. *$B_{\sigma, e^{i\theta}}$ on $H^2(\mathbb{T}) \oplus \mathbb{C}$ is irreducible for all angle $\theta \in [0, 2\pi)$ and covariance $\sigma > 0$.*

7. Examples

The purpose of this section is to illustrate the classification results, Theorem 3.1 and Theorem 5.2, using concrete examples. We aim to specifically show that Theorem 5.2, which is the scalar-valued reformulation of Theorem 3.1, indeed provides examples of unitarily as well as non-unitarily equivalent invariant subspaces of Brownian shifts.

We will present two examples, and Blaschke factors will play a role in both of them. For each $\alpha \in \mathbb{D}$, the *Blaschke factor* b_α corresponding to α is defined by

$$b_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z},$$

for all $z \in \mathbb{D}$. Blaschke factors are the simplest examples of inner functions.

Example 7.1. For $\{\alpha_1, \alpha_2\} \subseteq (0,1)$, consider inner functions $\varphi_j = b_{\alpha_j}$, $j = 1, 2$. Also, for each $\theta_1, \theta_2 \in [0, 2\pi)$ and $\sigma_1, \sigma_2 > 0$, define $g_j \in H^2(\mathbb{T})$ by

$$g_j(z) = \sigma_j \frac{\varphi_j(z)\overline{\varphi_j(e^{i\theta_j})} - 1}{z - e^{i\theta_j}},$$

for $j = 1, 2$. It is easy to see that

$$\|g_j\|^2 = \sigma_j^2 \left\| \frac{1-\alpha_j^2}{(1-\alpha_j e^{i\theta_j})(1-\alpha_j z)} \right\|^2 = \sigma_j^2 \frac{1-\alpha_j^2}{1+\alpha_j^2 - 2\alpha_j \cos\theta_j},$$

for $j = 1, 2$. In particular, for the choice $\theta_1 = \theta_2 = 0$, we have

$$\|g_j\|^2 = \sigma_j^2 \frac{1+\alpha_j}{1-\alpha_j},$$

and therefore, a little computation reveals that $\sigma_2^2(1 + \|g_1\|^2) = \sigma_1^2(1 + \|g_2\|^2)$ is satisfied, provided we have

$$\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2} = \frac{2(\alpha_2 - \alpha_1)}{(1-\alpha_1)(1-\alpha_2)}.$$

In particular, in view of Theorem 5.2, if the pairs $\{\alpha_1, \alpha_2\}$ and $\{\sigma_1, \sigma_2\}$ fail to satisfy the above identity, then $B_{\sigma_1,1}\big|_{\mathcal{M}_1}$ and $B_{\sigma_2,1}\big|_{\mathcal{M}_2}$ are not unitarily equivalent, where

$$\mathcal{M}_j = \mathbb{C}\begin{bmatrix} g_j \\ 1 \end{bmatrix} \oplus (\varphi_j H^2(\mathbb{T}) \oplus \{0\}),$$

for $j = 1, 2$.

In the following example, we bring a singular inner function with a single atom.

Example 7.2. For $\alpha \in (0,1)$, consider the inner function $\varphi_1 = b_\alpha$, and the other inner function as

$$\varphi_2(z) = \exp\left(\frac{z+1}{z-1}\right),$$

for all $z \in \mathbb{D}$. As usual, for each $\theta_1, \theta_2 \in [0, 2\pi)$ and $\sigma_1, \sigma_2 > 0$, define $g_j : \mathbb{D} \to \mathbb{C}$ by

$$g_j(z) = \sigma_j \frac{\varphi_j(z)\overline{\varphi_j(e^{i\theta_j})} - 1}{z - e^{i\theta_j}} \qquad (z \in \mathbb{D}),$$

for $j = 1, 2$. Clearly, $g_1 \in H^2(\mathbb{T})$. Let us first assume $\theta_1 = \theta_2 = \pi$. From the calculations of our previous example, we know that

$$\|g_1\|^2 = \sigma_1^2 \frac{1-\alpha^2}{1+\alpha^2 - 2\alpha\cos\pi} = \sigma_1^2 \frac{1-\alpha}{1+\alpha}.$$

Now, we prove that $g_2 \in H^2(\mathbb{T})$. We do so by first proving that g_2 is indeed in $L^2(\mathbb{T})$-the space of complex-valued Lebesgue square integrable functions on $\mathbb{T}$. First, we note that

$$g_2 = \sigma_2 \frac{\exp\left(\frac{z+1}{z-1}\right) - 1}{z+1},$$

is analytic on $\mathbb{D}$ and its radial limits exist a.e. on $\mathbb{T}$. A straightforward calculation gives

$$\left| \frac{\exp\left(\frac{e^{i\theta}+1}{e^{i\theta}-1}\right) - 1}{e^{i\theta}+1} \right|^2 = \frac{1 - \cos\left(\frac{\sin\theta}{1-\cos\theta}\right)}{1+\cos\theta} = \frac{\sin^2\left(\frac{1}{2}\cot\frac{\theta}{2}\right)}{\cos^2\frac{\theta}{2}}.$$

We put $\xi = \frac{1}{2}\cot\frac{\theta}{2}$, and change the variable from θ to ξ to see

$$\int_0^{2\pi} \left| \frac{\exp\left(\frac{e^{i\theta}+1}{e^{i\theta}-1}\right) - 1}{e^{i\theta}+1} \right|^2 d\theta = \int_0^{2\pi} \frac{\sin^2\left(\frac{1}{2}\cot\frac{\theta}{2}\right)}{\cos^2\frac{\theta}{2}} d\theta = \int_{-\infty}^{\infty} \frac{\sin^2\xi}{\xi^2} d\xi = \pi < \infty.$$

Hence, $g_2 \in L^2(\mathbb{T})$. Consider the Fourier series expansion of g_2 on $\mathbb{T}$ as

$$g_2 = \sum_{n=-\infty}^{\infty} \alpha_n z^n,$$

where α_n, $n \in \mathbb{Z}$, are the Fourier coefficients. Now using the fact that

$$(z+1)g_2 = \sigma_2(\varphi_2 - 1) \in H^2(\mathbb{T}),$$

we have, for any $n \geq 1$, that

$$\langle (z+1)g_2, \bar{z}^n \rangle = \langle \sigma_2(\varphi_2 - 1), \bar{z}^n \rangle = 0,$$

and consequently

$$\alpha_{-(n+1)} = -\alpha_{-n}.$$

In particular, if $\alpha_{-m} \neq 0$, for some $m \geq 1$, then

$$\alpha_{-n} = (-1)^{n-m}\alpha_{-m},$$

for all $n \geq m$. This implies that the series $\sum_{n=m}^{\infty} |\alpha_{-n}|^2$ diverges, contradicting the fact that $g_2 \in L^2(\mathbb{T})$. Therefore, we have

$$\alpha_{-n} = 0,$$

for all $n \geq 1$, and hence $g_2 \in H^2(\mathbb{T})$. Now, we compute the norm of g_2 as

$$\|g_2\|^2 = \sigma_2^2 \left\| \frac{\exp\left(\frac{z+1}{z-1}\right) - 1}{z+1} \right\|^2 = \frac{\sigma_2^2}{2\pi} \int_0^{2\pi} \left| \frac{\exp\left(\frac{e^{i\theta}+1}{e^{i\theta}-1}\right) - 1}{e^{i\theta}+1} \right|^2 d\theta = \frac{\sigma_2^2}{2}.$$

Hence, the relation $\sigma_2^2(1 + \|g_1\|^2) = \sigma_1^2(1 + \|g_2\|^2)$ is satisfied under the condition

$$\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2} = \frac{3\alpha - 1}{2(1+\alpha)}.$$

Therefore, in this case also, there is an abundance of examples of invariant subspaces of Brownian shifts, both unitarily equivalent and non-equivalent.

The argument used to prove that g_2 belongs to $H^2(\mathbb{T})$ in the above proof is perhaps a standard method. In a more general context, this conclusion follows from [6, Corollary 4.28], which is a more involved result.

In closing, we remark that unitary equivalence or nonequivalence of operators arising from natural operators defined on Hilbert spaces is a fundamental and decades-old problem (cf. [3, 4]). On one hand, it raises the question of defining new classes of operators from invariant subspaces, and on the other, it analyzes the characteristics of these operators under consideration. For instance, as already pointed out, among known operators, the shift on $H^2(\mathbb{T})$ always yields unitarily equivalent invariant subspaces. At the other extreme, the Bergman shift and the Dirichlet shift never yield unitarily equivalent invariant subspaces [11]. We have now enlarged this list by observing that the Brownian shift sometimes yields unitarily equivalent invariant subspaces and sometimes does not. This is particularly intriguing, as we have pointed out in (1.2) that a Brownian shift on $H^2(\mathbb{T})$ is a rank-one perturbation of an isometry.

Acknowledgement: We sincerely thank Professor Jan Stochel for his insightful discussions on the topic of this paper. The first named author is supported by a post-doctoral fellowship provided by the National Board for Higher Mathematics (NBHM), India (Order No.: 0204/10/(18)/2023/R&D-II/2791 dated 28 February, 2023). The research of the second named author is supported by a post-doctoral fellowship provided by the National Board for Higher Mathematics (NBHM), India (Order No: 0204/16(8)/2024/R&D-II/6760, dated May 09, 2024). The research of the third named author is supported in part by TARE (TAR/2022/000063) by SERB, Department of Science & Technology (DST), Government of India.

References

[1] J. Agler and M. Stankus, *m-isometric transformations of Hilbert space, II*, Integr. Equ. Oper. Theory 23, (1995), 1–48.

[2] A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. 81 (1949), 239–255.

[3] R. Douglas, *Variations on a theme of Beurling*, New York J. Math. 17A (2011), 1–10.

[4] R. Douglas and J. Sarkar, *On unitarily equivalent submodules*, Indiana Univ. Math. J. 57 (2008), 2729–2743.

[5] C. Foiaş, A. E. Frazho, *The commutant lifting approach to interpolation problems*, Oper. Theory Adv. Appl., 44 Birkhäuser Verlag, Basel, 1990, xxiv+632 pp.

[6] E. Fricain and J. Mashreghi, *The theory of $\mathcal{H}(b)$ spaces. Vol. 1*, New Mathematical Monographs, 20. Cambridge University Press, Cambridge, 2016, xix+681 pp.

[7] W. Helton, *Infinite dimensional Jordan operators and Sturm-Liouville conjugate point theory*, Trans. Am. Math. Soc. 170 (1972), 305–331.

[8] Z. Jabłoński, I. Jung, and J. Stochel, *m-isometric operators and their local properties*, Linear Algebra Appl. 596 (2020), 49–70.

[9] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, translated from the French and revised North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York; Akadémiai Kiadó, Budapest 1970 xiii+389 pp.

[10] H. Radjavi, P. Rosenthal, *Invariant subspaces*, Dover Publications, Inc., Mineola, NY, 2003, xii+248 pp.

[11] S. Richter, *Unitary equivalence of invariant subspaces of Bergman and Dirichlet spaces*, Pac. J. Math. 133 (1988), 151–156.

Indian Statistical Institute, Statistics and Mathematics Unit, 8th Mile, Mysore Road, Bangalore, 560059, India
Email address: nilanjand7@gmail.com

Indian Statistical Institute, Statistics and Mathematics Unit, 8th Mile, Mysore Road, Bangalore, 560059, India
Email address: dsoma994@gmail.com

Indian Statistical Institute, Statistics and Mathematics Unit, 8th Mile, Mysore Road, Bangalore, 560059, India
Email address: jay@isibang.ac.in, jaydeb@gmail.com