TENSOR PRODUCT OF QUOTIENT HILBERT MODULES

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Abstract. In this paper, we present a unified approach to problems of tensor product of quotient modules of Hilbert modules over $\mathbb{C}[z]$ and corresponding submodules of reproducing kernel Hilbert modules over $\mathbb{C}[z_1, \ldots, z_n]$ and the doubly commutativity property of module multiplication operators by the coordinate functions. More precisely, for a reproducing kernel Hilbert module $\mathcal{H}$ over $\mathbb{C}[z_1, \ldots, z_n]$ of analytic functions on the polydisc in $\mathbb{C}^n$ which satisfies certain conditions, we characterize the quotient modules $\mathcal{Q}$ of $\mathcal{H}$ such that $\mathcal{Q}$ is of the form $\mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$, for some one variable quotient modules $\{\mathcal{Q}_1, \ldots, \mathcal{Q}_n\}$. For $\mathcal{H}$ the Hardy module over polydisc, $H^2(\mathbb{D}^n)$, this reduces to some recent results by Izuchi, Nakazi and Seto and the third author. This is used to obtain a classification of co-doubly commuting submodules for a class of reproducing kernel Hilbert modules over the unit polydisc. These results are applied to compute the cross commutators of co-doubly commuting submodules. Moreover, this provides further insight into the wandering subspaces and ranks of submodules of the Hardy module. Our results includes the case of weighted Bergman modules over the unit polydisc in $\mathbb{C}^n$.

1. Introduction

The question of describing the invariant and co-invariant subspaces of shift operators on various holomorphic functions spaces is an old subject that essentially began with the work of A. Beurling [7]. The analogous problems for holomorphic function spaces in several variables have been considered in the work by Ahern, Douglas, Clark, Yang, Guo, Nakazi, Izuchi, Seto and many more (see [1], [2], [10], [9], [13], [12], [15], [22], [27]).

In this paper, we will examine certain joint invariant and co-invariant subspaces of the multiplication operators by the coordinate functions defined on a class of reproducing kernel Hilbert spaces on the unit polydisc $\mathbb{D}^n = \{(z_1, \ldots, z_n) : |z_i| < 1, i = 1, \ldots, n\}$. More precisely, our main interest is the class of quotient Hilbert modules of reproducing kernel Hilbert modules over $\mathbb{C}[z_1, \ldots, z_n]$, the ring of polynomials of n commuting variables, that admit a simple tensor product representation of quotient modules of Hilbert modules over $\mathbb{C}[z]$. A related problem also arises in connection with the submodules and quotient modules of modules over $\mathbb{C}[z_1, \ldots, z_n]$ in commutative algebra:
Let $n \in \mathbb{N}$ be a fixed positive integer and $\{\mathcal{M}_i\}_{i=1}^n$ be a family of modules over the ring of one variable polynomials $\mathbb{C}[z]$. Then the vector space tensor product $\mathcal{M} := \mathcal{M}_1 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{M}_n$ is a module over $\mathbb{C}[z] \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}[z] \cong \mathbb{C}[z_1, \ldots, z_n]$. Here the module action on $\mathcal{M}$ is given by

$$(p_1 \otimes \cdots \otimes p_n) \cdot (f_1 \otimes \cdots \otimes f_n) \mapsto p_1 \cdot f_1 \otimes \cdots \otimes p_n \cdot f_n,$$

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where $\{p_i\}_{i=1}^n \subseteq \mathbb{C}[z]$ and $f_i \in \mathcal{M}_i$ $(1 \leq i \leq n)$. Furthermore, let $\mathcal{Q}_i \subseteq \mathcal{M}_i$ be a quotient module of $\mathcal{M}_i$ for each $1 \leq i \leq n$. Then

$$\mathcal{Q}_1 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{Q}_n, \tag{1}$$

is a quotient module of $\mathcal{M}$. Now let $\mathcal{Q}$ be a quotient module and $\mathcal{S}$ be a submodule of $\mathcal{M}$. The following question arises naturally in the context of tensor product of quotient modules.
(A) when is $\mathcal{Q}$ of the form (1).

The next natural question is:
(B) when is $\mathcal{M}/\mathcal{S}$ of the form (1).
To the best of our knowledge, this is a mostly unexplored area at the moment.

Our principal concern in this paper is to provide a complete answer to the above problems by considering a natural class of reproducing kernel Hilbert modules over $\mathbb{C}[z]$ replacing the modules in the algebraic set up. In particular, we prove that a quotient module $\mathcal{Q}$ of a *standard reproducing kernel Hilbert module* (see Definition 4.5) over $\mathbb{C}[z_1, \ldots, z_n]$ is of the form

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n,$$

for n "one-variable" quotient modules $\{\mathcal{Q}_i\}_{i=1}^n$ if and only if $\mathcal{Q}$ is *doubly commuting* (see Definition 2.1).

The study of the doubly commuting quotient modules, restricted to the Hardy module over the bidisc $H^2(\mathbb{D}^2)$, was initiated by Douglas and Yang in [10] (also see Berger, Coburn and Lebow [6]). Later in [15] Izuchi, Nakazi and Seto obtained the above classification result only for quotient modules of the Hardy module $H^2(\mathbb{D}^2)$. More recently, the third author extended this result to $H^2(\mathbb{D}^n)$ for any $n \geq 2$ (see [20], [19]).

One of the difficulties in extending the above classification result from the Hardy module to the setting of a reproducing kernel Hilbert module $\mathcal{H}$ is that the module maps $\{M_{z_1}, \ldots, M_{z_n}\}$ on $\mathcal{H}$, the multiplication operators by the coordinate functions, are not isometries. This paper overcomes such a difficulty by exploiting the precise geometric and algebraic structure of tensor product of reproducing kernel Hilbert modules. In what follows we develop methods which link the tensor product of Hilbert modules over $\mathbb{C}[z_1, \ldots, z_n]$ to Hilbert modules over $\mathbb{C}[z]$.

We also consider the issue of essentially doubly commutativity of co-doubly commuting submodules of analytic reproducing kernel Hilbert modules over $\mathbb{C}[z_1, \ldots, z_n]$. We also obtain an wandering subspace theorem for some co-doubly commuting submodules of weighed Bergman modules over $\mathbb{C}[z_1, \ldots, z_n]$ and compute the rank of co-doubly commuting submodules of $H^2(\mathbb{D}^n)$. Our results in this paper are new even in the case of weighted Bergman spaces over $\mathbb{D}^n$.

We now describe the contents of the paper. After recalling the notion of reproducing kernel Hilbert modules in Section 2, we introduce the class of standard Hilbert modules over $\mathbb{C}[z_1, \ldots, z_n]$ in Section 3. Furthermore, we obtain some basic properties and an useful classification result for the class of standard Hilbert modules. In Section 4, we obtain a characterization of doubly commuting quotient modules of an analytic Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$. In Section 4, we present a characterization result for co-doubly commuting submodules and compute the cross commutators of a co-doubly commuting submodule. In section 5, we prove an wandering subspace theorem for co-doubly commuting submodules of the weighted

Bergman modules over $\mathbb{D}^n$. We also compute the rank of a co-doubly commuting submodule of $H^2(\mathbb{D}^n)$. The final section is reserved for some concluding remarks.

Notations:

- Throughout this paper $n \geq 2$ is a fixed natural number.
- For a Hilbert space $\mathcal{H}$, the set of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$.
- We denote by $\otimes$ the Hilbert space tensor product and by $M\bar{\otimes}N$, the von-Neumann algebraic tensor product of von-Neumann algebras M and N.
- For a von-Neumann algebra $M \subseteq \mathcal{B}(\mathcal{H})$, we denote by M' the commutant of M that is the von-Neumann algebra of all operators in $\mathcal{B}(\mathcal{H})$ which commutes with all the operators in M.
- For Hilbert space operators $R, T \in \mathcal{B}(\mathcal{H})$, we write $[R,T] = RT - TR$, the commutator of R and T.
- For any set E, we denote by $\#E$ the cardinality of the set E.
- For a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, we denote by $P_{\mathcal{S}}$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{S}$.
- For a Hilbert space $\mathcal{E}$ we shall let $\mathcal{O}(\mathbb{D}^n, \mathcal{E})$ denote the space of $\mathcal{E}$-valued holomorphic functions on $\mathbb{D}^n$.
- $\mathbb{C}[\boldsymbol{z}] := \mathbb{C}[z_1, \ldots, z_n]$ denotes the polynomial ring over $\mathbb{C}$ in n commuting variables

2. Preliminaries

In this section we gather together some known results on reproducing kernel Hilbert spaces on product domains in $\mathbb{C}^n$. We start by recalling the notion of a Hilbert module over $\mathbb{C}[\boldsymbol{z}]$.

Let $\{T_1, \ldots, T_n\}$ be a set of n commuting bounded linear operators on a Hilbert space $\mathcal{H}$. Then the n-tuple $(T_1, \ldots, T_n)$ turns $\mathcal{H}$ into a module over $\mathbb{C}[\boldsymbol{z}]$ in the following sense:

$$\mathbb{C}[\boldsymbol{z}] \times \mathcal{H} \to \mathcal{H}, \qquad (p, h) \mapsto p(T_1, \ldots, T_n)h,$$

where $p \in \mathbb{C}[\boldsymbol{z}]$ and $h \in \mathcal{H}$. We say that the module $\mathcal{H}$ is a *Hilbert module* over $\mathbb{C}[\boldsymbol{z}]$ (see [11], [21]). Denote by $M_p : \mathcal{H} \to \mathcal{H}$ the bounded linear operator

$$M_p h = p \cdot h = p(T_1, \ldots, T_n)h, \qquad (h \in \mathcal{H})$$

for $p \in \mathbb{C}[\boldsymbol{z}]$. In particular, for $p = z_i \in \mathbb{C}[\boldsymbol{z}]$ we obtain the *module multiplication* operators as follows:

$$M_{z_i} h = z_i(T_1, \ldots, T_n)h = T_i h \qquad (h \in \mathcal{H},\, 1 \leq i \leq n).$$

In what follows, we will use the notion of a Hilbert module $\mathcal{H}$ over $\mathbb{C}[\boldsymbol{z}]$ in place of an n-tuple of commuting operators $\{T_1, \ldots, T_n\} \subseteq \mathcal{B}(\mathcal{H})$, where the operators are determined by module multiplication by the coordinate functions, and vice versa.

A function $K : \mathbb{D}^n \times \mathbb{D}^n \to \mathbb{C}$ is said to be *positive definite kernel* (cf. [5], [21]) if

$$\sum_{i,j=1}^{k} \overline{\lambda}_i \lambda_j K(\boldsymbol{z}_i, \boldsymbol{z}_j) > 0,$$

for all $\{\lambda_i\}_{i=1}^k \subseteq \mathbb{C}$, $\{\boldsymbol{z}_i\}_{i=1}^k \subseteq \mathbb{D}^n$ and $k \in \mathbb{N}$. Given a positive definite kernel K on $\mathbb{D}^n$, the scalar-valued reproducing kernel Hilbert space $\mathcal{H}_K$ is the Hilbert space completion of $\operatorname{span}\{K(\cdot, \boldsymbol{w}) : \boldsymbol{w} \in \mathbb{D}^n\}$ corresponding to the inner product

$$\langle K(\cdot, \boldsymbol{w}), K(\cdot, \boldsymbol{z})\rangle_{\mathcal{H}_K} = K(\boldsymbol{z}, \boldsymbol{w}) \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n).$$

The kernel function K has the reproducing property:

$$f(\boldsymbol{w}) = \langle f, K(\cdot, \boldsymbol{w})\rangle_{\mathcal{H}_K} \qquad (f \in \mathcal{H}_K, \boldsymbol{w} \in \mathbb{D}^n).$$

In particular, for each $\boldsymbol{w} \in \mathbb{D}^n$ the evaluation operator $\boldsymbol{ev_w} : \mathcal{H}_K \to \mathbb{C}$ defined by $\boldsymbol{ev_w}(f) = \langle f, K(\cdot, \boldsymbol{w})\rangle_{\mathcal{H}_K}$ $(f \in \mathcal{H}_K)$ is bounded. We say that $\mathcal{H}_K$ is the *reproducing kernel Hilbert space* over $\mathbb{D}^n$ with respect to the kernel function K.

We now assume that the function K is holomorphic in the first variable and anti-holomorphic in the second variable. Then $\mathcal{H}_K$ is a Hilbert space of holomorphic functions on $\mathbb{D}^n$ (cf. [21]). Moreover, $\mathcal{H}_K$ is said to be *reproducing kernel Hilbert module* over $\mathbb{C}[\boldsymbol{z}]$ if $1 \in \mathcal{H}_K \subseteq \mathcal{O}(\mathbb{D}^n, \mathbb{C})$ and the module multiplication operators $\{M_{z_i}\}_{i=1}^n$ are given by the multiplication by the coordinate functions, that is

$$M_{z_i} f = z_i f,$$

and

$$(z_i f)(\boldsymbol{w}) = w_i f(\boldsymbol{w}), \qquad (f \in \mathcal{H}_K, \boldsymbol{w} \in \mathbb{D}^n)$$

for $i = 1, \ldots, n$. It is easy to verify that

$$M_{z_i}^* K(\cdot, \boldsymbol{w}) = \bar{w}_i K(\cdot, \boldsymbol{w}), \qquad (\boldsymbol{w} \in \mathbb{D}^n)$$

for $i = 1, \ldots, n$.

Let $\{\mathcal{H}_{K_i}\}_{i=1}^n$ be a collection of reproducing kernel Hilbert modules over $\mathbb{D}$ corresponding to the positive definite kernel functions $K_i : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$, $i = 1, \ldots, n$. Thus

$$K(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^n K_i(z_i, w_i), \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n)$$

defines a positive definite kernel on $\mathbb{D}^n$ (cf. [26], [5]). Observe that $\mathcal{H}_{K_1} \otimes \cdots \otimes \mathcal{H}_{K_n}$ can be viewed as a reproducing kernel Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ in the following sense:

$$\mathbb{C}[\boldsymbol{z}] \times (\mathcal{H}_{K_1} \otimes \cdots \otimes \mathcal{H}_{K_n}) \to \mathcal{H}_{K_1} \otimes \cdots \otimes \mathcal{H}_{K_n}, \quad (p, f) \mapsto p(M_1, \ldots, M_n) f,$$

where $M_i \in B(\mathcal{H}_{K_1} \otimes \cdots \otimes \mathcal{H}_{K_n})$, and

$$M_i := I_{\mathcal{H}_{K_1}} \otimes \cdots \otimes \underbrace{M_z}_{\text{i-th place}} \otimes \cdots \otimes I_{\mathcal{H}_{K_n}} \qquad (1 \leq i \leq n).$$

Moreover, it also follows immediately from the definition of K that

$$\begin{aligned}\left\|\sum_{i=1}^{m} a_i K(\cdot,\boldsymbol{w}_i)\right\|^2 &= \sum_{i=1}^{m}\sum_{j=1}^{m} a_i\bar{a}_j K(\boldsymbol{w}_i,\boldsymbol{w}_j) = \sum_{i=1}^{m}\sum_{j=1}^{m} a_i\bar{a}_j\Big(\prod_{l=1}^{n} K_l\big((\boldsymbol{w}_i)_l,(\boldsymbol{w}_j)_l\big)\Big)\\ &= \sum_{i=1}^{m}\sum_{j=1}^{m} a_i\bar{a}_j\Big\langle K_1(\cdot,(\boldsymbol{w}_j)_1)\otimes\cdots\otimes K_n(\cdot,(\boldsymbol{w}_j)_n), K_1(\cdot,(\boldsymbol{w}_i)_1)\otimes\cdots\otimes K_n(\cdot,(\boldsymbol{w}_i)_n)\Big\rangle\\ &= \left\|\sum_{i=1}^{m} a_i K_1(\cdot,(\boldsymbol{w}_i)_1)\otimes\cdots\otimes K_n(\cdot,(\boldsymbol{w}_i)_n)\right\|^2,\end{aligned}$$

where $\{\boldsymbol{w}_i = \big((\boldsymbol{w}_i)_1,\dots,(\boldsymbol{w}_i)_n\big) : 1\le i\le m\}\subseteq\mathbb{D}^n$ and $\{a_i\}_{i=1}^m\subseteq\mathbb{C}$ and $m\in\mathbb{N}$. Therefore, the map

$$U:\operatorname{span}\{K(\cdot,\boldsymbol{w}):\boldsymbol{w}\in\mathbb{D}^n\}\longrightarrow\operatorname{span}\{K_1(\cdot,w_1)\otimes\cdots\otimes K_n(\cdot,w_n):\boldsymbol{w}\in\mathbb{D}^n\}$$

defined by

$$UK(\cdot,\boldsymbol{w}) = K_1(\cdot,w_1)\otimes\cdots\otimes K_n(\cdot,w_n),\qquad (\boldsymbol{w}\in\mathbb{D}^n)$$

extends to a unitary operator from $\mathcal{H}_K$ onto $\mathcal{H}_{K_1}\otimes\cdots\otimes\mathcal{H}_{K_n}$. We also have

$$M_{z_i} = U^* M_i U \qquad (1\le i\le n).$$

This implies that $\mathcal{H}_K\cong\mathcal{H}_{K_1}\otimes\cdots\otimes\mathcal{H}_{K_n}$ is a reproducing kernel Hilbert module over $\mathbb{C}[\boldsymbol{z}]$.

In what follows we identify the Hilbert tensor product of Hilbert modules $\mathcal{H}_{K_1}\otimes\cdots\otimes\mathcal{H}_{K_n}$ with the Hilbert module $\mathcal{H}_K$ over $\mathbb{C}[\boldsymbol{z}]$. It also enables us to identify $z^{k_1}\otimes\cdots\otimes z^{k_n}$ with $z^{\boldsymbol{k}}$ for all $\boldsymbol{k}=(k_1,\cdots,k_n)\in\mathbb{N}^n$.

We now recall the definitions of submodules and quotient modules of reproducing kernel Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$ to be used in this paper:
Let $\mathcal{S}$ and $\mathcal{Q}$ be a pair of closed subspaces of $\mathcal{H}_K$. Then $\mathcal{S}$ is a *submodule* of $\mathcal{H}_K$ if $M_{z_i}\mathcal{S}\subseteq\mathcal{S}$ for all $i=1,\dots,n$ and $\mathcal{Q}$ is a *quotient module* if $\mathcal{Q}^\perp(\cong\mathcal{H}_K/\mathcal{Q})$ is a submodule of $\mathcal{H}_K$. The module multiplication operators on the submodule $\mathcal{S}$ and the quotient module $\mathcal{Q}$ of $\mathcal{H}_K$ are given by restrictions $(R_{z_1},\dots,R_{z_n})$ and compressions $(C_{z_1},\dots,C_{z_n})$ of the module multiplication operators $(M_{z_1},\dots,M_{z_n})$ on $\mathcal{H}_K$:

$$R_{z_i} := M_{z_i}|_{\mathcal{S}} \quad\text{and}\quad C_{z_i} := P_{\mathcal{Q}}M_{z_i}|_{\mathcal{Q}}, \tag{2}$$

for $i=1,\dots,n$.

Definition 2.1. A quotient module $\mathcal{Q}$ of $\mathcal{H}_K$ is *doubly commuting* if for $1\le i<j\le n$,

$$C_{z_i}C_{z_j}^* = C_{z_j}^*C_{z_i}.$$

A submodule $\mathcal{S}$ of $\mathcal{H}_K$ is *doubly commuting* if for $1\le i<j\le n$,

$$R_{z_i}R_{z_j}^* = R_{z_j}^*R_{z_i},$$

and it is *co-doubly commuting* if the quotient module $\mathcal{S}^\perp(\cong\mathcal{H}_K/\mathcal{S})$ is doubly commuting.

The notion of a co-doubly commuting submodule was introduced in [20] and [19] in the context of Hardy module over $\mathbb{D}^n$. However, the interplay between the doubly commuting quotient modules and the co-doubly commuting submodules has also been previously used by Izuchi, Nakazi and Seto and Yang [16], [15], [28], [27].

We end this preliminary section by recalling a result concerning commutant of von-Neumann algebras (cf. Theorem 5.9, Chapter-IV of [25]) which will be used in later sections.

Theorem 2.2. *Let M and N be two von-Neumann algebras. Then $(M\bar{\otimes}N)' = M'\bar{\otimes}N'$.*

3. Standard Hilbert modules

In this section we introduce the notion of a standard reproducing kernel Hilbert module and establish some basic properties. A characterization of this class is also obtained which we use throughout this note.

Definition 3.1. A reproducing kernel Hilbert module $\mathcal{H} \subseteq \mathcal{O}(\mathbb{D}, \mathbb{C})$ over $\mathbb{C}[z]$ is said to be standard Hilbert module over $\mathbb{C}[z]$ if there does not exist two non-zero quotient modules of $\mathcal{H}$ which are orthogonal to each other.

It follows immediately that a standard Hilbert module $\mathcal{H}$ over $\mathbb{C}[z]$ is always *irreducible*, that is, the module multiplication operator M_z does not have any non-trivial reducing subspace.

One of the pleasant features of working with a standard Hilbert module over $\mathbb{C}[z]$ is that the quotient modules of this space have the following useful characterization.

Proposition 3.2. *Let $\mathcal{H}$ be a reproducing kernel Hilbert module over $\mathbb{C}[z]$. Then $\mathcal{H}$ is a standard Hilbert module over $\mathbb{C}[z]$ if and only if for any non-zero quotient module $\mathcal{Q}$ of $\mathcal{H}$, the smallest submodule containing $\mathcal{Q}$ is $\mathcal{H}$, that is,*

$$\bigvee_{l=0}^{\infty} z^l \mathcal{Q} = \mathcal{H}.$$

Proof. Let $\mathcal{H}$ be a standard Hilbert module over $\mathbb{C}[z]$. Let $\mathcal{Q}$ be a quotient module of $\mathcal{H}$ such that

$$\tilde{\mathcal{Q}} := \bigvee_{l=0}^{\infty} z^l \mathcal{Q} \neq \mathcal{H}.$$

It follows that the quotient module $\tilde{\mathcal{Q}}^{\perp}$ is non-trivial and $\mathcal{Q} \perp \tilde{\mathcal{Q}}^{\perp}$. This contradicts the assumption that $\mathcal{H}$ is a standard Hilbert module.

We now turn our attention to the converse part. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two non-zero quotient modules of $\mathcal{H}$, and $\mathcal{Q}_1 \perp \mathcal{Q}_2$. For all $f_1 \in \mathcal{Q}_1$ and $f_2 \in \mathcal{Q}_2$ and $l \in \mathbb{N}$,

$$\langle z^l f_1, f_2 \rangle = \langle M_z^l f_1, f_2 \rangle = \langle f_1, M_z^{*l} f_2 \rangle = 0.$$

This shows that

$$\bigvee_{l=0}^{\infty} z^l \mathcal{Q}_1 \perp \mathcal{Q}_2.$$

On the other hand, $\vee_{l=0}^{\infty} z^l \mathcal{Q}_1 = \mathcal{H}$ implies that $\mathcal{Q}_2 = \{0\}$. This is a contradiction. Therefore, $\mathcal{Q}_1$ is not orthogonal to $\mathcal{Q}_2$ as desired. □

Our next result shows that if $K^{-1} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ is a polynomial in z and $\bar{w}$, then $\mathcal{H}_K$ can be realized as a standard Hilbert module over $\mathbb{C}[z]$.

Theorem 3.3. *Let $\mathcal{H}_K$ be a reproducing kernel Hilbert module over $\mathbb{C}[z]$ with reproducing kernel $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ such that $K^{-1}(z, w) = \sum\limits_{i,j=0}^{k} a_{ij} z^i \bar{w}^j$ is a polynomial in z and $\bar{w}$. Then $\mathcal{H}_K$ is a standard Hilbert module over $\mathbb{C}[z]$.*

Proof. Let $K^{-1}(z, w) = \sum\limits_{i,j=0}^{k} a_{ij} z^i \bar{w}^j$ and set $K^{-1}(M_z, M_z^*) := \sum\limits_{i,j=0}^{k} a_{ij} M_z^i M_z^{*j}$. For $z, w \in \mathbb{D}$ we notice that

$$\begin{aligned} \langle K^{-1}(M_z, M_z^*) K(\cdot, w), K(\cdot, z) \rangle &= \sum_{i,j=0}^{k} \langle a_{ij} M_z^i M_z^{*j} K(\cdot, w), K(\cdot, z) \rangle \\ &= \sum_{i,j=0}^{k} \langle a_{ij} M_z^{*j} K(\cdot, w), M_z^{*i} K(\cdot, z) \rangle \\ &= \sum_{i,j=0}^{k} z^i \bar{w}^j a_{ij} \langle K(\cdot, w), K(\cdot, z) \rangle \\ &= K^{-1}(z, w) K(z, w) \\ &= \langle P_{\mathbb{C}} K(\cdot, w), K(\cdot, z) \rangle, \end{aligned}$$

where $P_{\mathbb{C}}$ is the orthogonal projection of $\mathcal{H}_K$ onto the subspace of all constant functions. Consequently, it follows that

$$K^{-1}(M_z, M_z^*) = P_{\mathbb{C}}.$$

We now assume that $\mathcal{Q}$ is a non-zero quotient module of $\mathcal{H}$ and $\widetilde{\mathcal{Q}} = \vee_{l=0}^{\infty} z^l \mathcal{Q}$. It readily follows that

$$P_{\mathbb{C}}(\mathcal{Q}) = K^{-1}(M_z, M_z^*)(\mathcal{Q}) \subseteq \widetilde{\mathcal{Q}}.$$

Now if $P_{\mathbb{C}}(\mathcal{Q}) = \{0\}$, then $\mathcal{Q}^{\perp}$ contains the constant function 1 and so $\mathcal{Q} = \{0\}$ contradicting the fact that $\mathcal{Q} \neq \{0\}$.
On the other hand, if $P_{\mathbb{C}}(\mathcal{Q}) \neq \{0\}$, then $1 \in \widetilde{\mathcal{Q}}$ and hence $\widetilde{\mathcal{Q}} = \mathcal{H}$. The theorem now follows from Proposition 3.2. $\square$

Remark. We remark that the assumptions of the above theorem includes implicitly the additional hypothesis that one can define a functional calculus so that $\frac{1}{K}(M_z, M_z^*)$ make sense for the kernel function K. It was pointed out in the paper by Arazy and Englis [4] that for many reproducing kernel Hilbert spaces, one can define such a $\frac{1}{K}$-calculus. In particular, examples of standard Hilbert modules over $\mathbb{C}[z]$ includes the weighted Bergman spaces $L^2_{a,\alpha}(\mathbb{D})$, $\alpha > -1$, with kernel functions

$$K_{a,\alpha}(z, w) = \frac{1}{(1 - z\bar{w})^{\alpha+2}}. \qquad (z, w \in \mathbb{D})$$

We will make repeated use of the following lemma concerning commutant of $C_z = P_{\mathcal{Q}} M_z|_{\mathcal{Q}}$ in $B(\mathcal{Q})$ where $\mathcal{Q}$ is a quotient module of a standard Hilbert module $\mathcal{H}_K$.

Lemma 3.4. *Let $\mathcal{H}$ be a standard Hilbert module over $\mathbb{C}[z]$ and $\mathcal{Q}$ be a non-trivial quotient module of $\mathcal{H}$. Let P be a non-zero orthogonal projection in $B(\mathcal{Q})$. Then*

$$PC_z = C_zP,$$

if and only if $P = I_{\mathcal{Q}}$.

Proof. Let $\mathcal{S}$ be a non-zero closed subspace of $\mathcal{Q}$ such that

$$P_{\mathcal{S}}C_z = C_zP_{\mathcal{S}},$$

or equivalently, $P_{\mathcal{S}}C_z^* = C_z^*P_{\mathcal{S}}$. Hence

$$P_{\mathcal{S}}M_z^*|_{\mathcal{Q}} = M_z^*|_{\mathcal{Q}}P_{\mathcal{S}} = M_z^*P_{\mathcal{S}}.$$

By multiplying both sides of

$$P_{\mathcal{S}}M_z^*|_{\mathcal{Q}} = M_z^*P_{\mathcal{S}},$$

to the right with $P_{\mathcal{S}}$ we get $P_{\mathcal{S}}M_z^*P_{\mathcal{S}} = M_z^*P_{\mathcal{S}}$. Hence $\mathcal{S}$ is a quotient module of $\mathcal{H}$.

On the other hand, using $P_{\mathcal{S}}M_z^*P_{\mathcal{S}} = P_{\mathcal{S}}M_z^*P_{\mathcal{Q}}$ along with the fact that $\mathcal{Q}$ is a quotient module we have

$$\begin{aligned} P_{\mathcal{Q}\ominus\mathcal{S}}M_z^*P_{\mathcal{Q}\ominus\mathcal{S}} &= P_{\mathcal{Q}}M_z^*P_{\mathcal{Q}} - P_{\mathcal{Q}}M_z^*P_{\mathcal{S}} - P_{\mathcal{S}}M_z^*P_{\mathcal{Q}} + P_{\mathcal{S}}M_z^*P_{\mathcal{S}} \\ &= M_z^*P_{\mathcal{Q}} - M_z^*P_{\mathcal{S}} = M_z^*P_{\mathcal{Q}\ominus\mathcal{S}}. \end{aligned}$$

Thus $\mathcal{Q}$ and $\mathcal{Q}\ominus\mathcal{S}$ are two orthogonal quotient modules of $\mathcal{H}$. This contradicts the fact that $\mathcal{H}$ is a standard Hilbert module over $\mathbb{C}[z]$. Consequently, $\mathcal{Q}\ominus\mathcal{S} = \{0\}$, that is, $\mathcal{Q} = \mathcal{S}$. This completes the proof. ☐

Let $\mathcal{Q}$ be a quotient module of a Hilbert module $\mathcal{H}$ over $\mathbb{C}[z]$ and $\mathcal{S}$ be a non-trivial closed subspace of $\mathcal{Q}$. Let

$$P_{\mathcal{S}}C_z = C_zP_{\mathcal{S}}.$$

The above proof shows that both $\mathcal{S}$ and $\mathcal{Q}\ominus\mathcal{S}$ are quotient modules of $\mathcal{H}$. One can show that the converse is also true. Hence this is an equivalent condition.

It is of interest to know whether an irreducible reproducing kernel Hilbert module over $\mathbb{C}[z]$ is necessarily standard Hilbert module over $\mathbb{C}[z]$. However, this question is not relevant in the context of the present paper.

4. Doubly commuting quotient module

In this section we introduce the notion of a standard Hilbert module in several variables. We present a characterization result for quotient modules of standard Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$, which are doubly commuting as well as satisfy an additional natural condition. We also obtain a characterization result for doubly commuting quotient modules of the weighted Bergman modules over $\mathbb{D}^n$.

We begin by defining the notion of a standard Hilbert module over $\mathbb{C}[\boldsymbol{z}]$.

Definition 4.1. A reproducing kernel Hilbert module $\mathcal{H} \subseteq \mathcal{O}(\mathbb{D}^n, \mathbb{C})$ over $\mathbb{C}[\boldsymbol{z}]$ is said to be a *standard Hilbert module over* $\mathbb{C}[\boldsymbol{z}]$ if

$$\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n,$$

for some standard Hilbert modules $\{\mathcal{H}_i\}_{i=1}^n$ over $\mathbb{C}[z]$.

Here, as well as in the rest of this paper we specialize to the class of standard Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$.

The following illuminating example makes clear the connection between the tensor product of quotient modules of standard Hilbert modules over $\mathbb{C}[z]$ and doubly commuting quotient modules of standard Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$:
Let $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ be a standard Hilbert module over $\mathbb{C}[\boldsymbol{z}]$, and let $\mathcal{Q}_j \subseteq \mathcal{H}_j$ be a quotient module for each $j = 1, \ldots, n$. Then

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n,$$

is a doubly commuting quotient module of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ with the module multiplication operators

$$I_{\mathcal{Q}_1} \otimes \cdots \otimes \underbrace{P_{\mathcal{Q}_i} M_z|_{\mathcal{Q}_i}}_{i\text{-th}} \otimes \cdots \otimes I_{\mathcal{Q}_n} \quad (i = 1, \ldots, n).$$

The purpose of this section is to prove that under a rather natural condition a doubly commuting quotient module of a standard Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ is always represented in the above form.

The key ingredient in our approach will be the following propositions concerning reducing subspaces of standard Hilbert modules.

Proposition 4.2. *Let $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ be a standard Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. Let $\mathcal{Q}$ be a closed subspace of $\mathcal{H}$ and let $k \in \{1, \ldots, n\}$. Then $\mathcal{Q}$ is M_{z_i}-reducing for $i = k, k+1, \ldots, n$, if and only if*

$$\mathcal{Q} = \mathcal{E} \otimes \mathcal{H}_k \otimes \cdots \otimes \mathcal{H}_n,$$

for some closed subspace $\mathcal{E} \subseteq \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}$.

Proof. For $k \leq i \leq n$, let $\mathcal{N}_i$ be the von-Neumann algebra generated by $\{I_{\mathcal{H}_i}, M_z\}$, where M_z is the module multiplication operator on $\mathcal{H}_i$. It follows immediately that the von-Neumann algebra generated by

$$\{I_{\mathcal{H}}, M_{z_i} : i = k, k+1, \ldots, n\} \subseteq \mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n),$$

is given by

$$\mathbb{C}I_{\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}} \bar{\otimes} \mathcal{N}_k \bar{\otimes} \cdots \bar{\otimes} \mathcal{N}_n.$$

By virtue of Lemma 3.4 we have

$$\mathcal{N}_i' = \mathbb{C}I_{\mathcal{H}_i} \qquad (k \leq i \leq n).$$

On account of Theorem 2.2 we have then

$$\left(\mathbb{C}I_{\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}} \bar{\otimes} \mathcal{N}_k \bar{\otimes} \cdots \bar{\otimes} \mathcal{N}_n\right)' = \mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}) \bar{\otimes} \mathbb{C}I_{\mathcal{H}_k \otimes \cdots \otimes \mathcal{H}_n},$$

and hence $\mathcal{Q}$ is M_{z_i}-reducing subspace for all $i = k, k+1, \ldots, n$, if and only if

$$P_{\mathcal{Q}} \in \left(\mathbb{C}I_{\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_{k-1}}\bar{\otimes}\mathcal{N}_k\bar{\otimes}\cdots\bar{\otimes}\mathcal{N}_n\right)' = \mathcal{B}(\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_{k-1})\bar{\otimes}\mathbb{C}I_{\mathcal{H}_k\otimes\cdots\otimes\mathcal{H}_n}.$$

On the other hand, since $P_{\mathcal{Q}}$ is a projection in $\mathcal{B}(\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_{k-1})\bar{\otimes}\mathbb{C}I_{\mathcal{H}_k\otimes\cdots\otimes\mathcal{H}_n}$, there exists a closed subspace $\mathcal{E}$ of $\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_{k-1}$ such that

$$P_{\mathcal{Q}} = P_{\mathcal{E}}\otimes I_{\mathcal{H}_k\otimes\cdots\otimes\mathcal{H}_n}.$$

Hence it follows that

$$\mathcal{Q} = \mathcal{E}\otimes\mathcal{H}_k\otimes\cdots\otimes\mathcal{H}_n.$$

This completes the proof. ☐

Proposition 4.3. *Let $\mathcal{H} = \mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_n$ be a standard Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ and let $\mathcal{Q}_1$ be a quotient module of $\mathcal{H}_1$. Then a closed subspace $\mathcal{M}$ of $\mathcal{Q} := \mathcal{Q}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n$ is $P_{\mathcal{Q}}M_{z_1}|_{\mathcal{Q}}$-reducing if and only if there exists a closed subspace $\mathcal{E}$ of $\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n$ such that*

$$\mathcal{M} = \mathcal{Q}_1\otimes\mathcal{E}.$$

Proof. Suppose $\mathcal{Q}_1$ is a quotient module of $\mathcal{H}_1$. We observe that

$$P_{\mathcal{Q}}M_{z_1}|_{\mathcal{Q}} = \left(P_{\mathcal{Q}_1}M_z|_{\mathcal{Q}_1}\otimes I_{\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n}\right).$$

We also note that a closed subspace $\mathcal{M}$ of $\mathcal{Q}$ is $P_{\mathcal{Q}}M_{z_1}|_{\mathcal{Q}}$-reducing if and only if

$$P_{\mathcal{M}} \in \left(\mathcal{N}\bar{\otimes}I_{\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n}\right)',$$

where $\mathcal{N}\subseteq\mathcal{B}(\mathcal{Q}_1)$ is the von-Neumann algebra generated by $\{I_{\mathcal{Q}_1}, P_{\mathcal{Q}_1}M_z|_{\mathcal{Q}_1}\}$. Now

$$(\mathcal{N}\bar{\otimes}I_{\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n})' = \mathcal{N}'\bar{\otimes}\mathcal{B}(\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n).$$

By Lemma 3.4 we have $\mathcal{N}' = \mathbb{C}I_{\mathcal{Q}_1}$ and hence

$$(\mathcal{N}\bar{\otimes}I_{\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n})' = \mathbb{C}I_{\mathcal{Q}_1}\ \bar{\otimes}\mathcal{B}(\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n).$$

Therefore, $P_{\mathcal{M}}\in(\mathcal{N}\bar{\otimes}I_{\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n})'$ if and only if

$$P_{\mathcal{M}} = I_{\mathcal{Q}_1}\otimes P_{\mathcal{E}},$$

that is, $\mathcal{M} = \mathcal{Q}_1\otimes\mathcal{E}$, for some closed subspace $\mathcal{E}$ of $\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_n$. This completes the proof. ☐

Let $\mathcal{Q}$ be a quotient module of a standard Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. For $1\le k\le n$, let $[\mathcal{Q}]_{z_k,z_{k+1},\ldots,z_n}$ denote the smallest joint $(M_{z_k},\ldots,M_{z_n})$-invariant subspace containing $\mathcal{Q}$. That is,

$$[\mathcal{Q}]_{z_k,z_{k+1},\ldots,z_n} := \bigvee_{(l_k,l_{k+1},\ldots,l_n)\in\mathbb{N}^{(n-k+1)}} M_{z_k}^{l_k}\cdot M_{z_{k+1}}^{l_{k+1}}\cdots M_{z_n}^{l_n}\mathcal{Q}. \tag{3}$$

We are now ready to prove the characterization result concerning tensor product of quotient modules of standard Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$.

Theorem 4.4. *Let $\mathcal{Q}$ be a quotient module of a standard Hilbert module $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ over $\mathbb{C}[\boldsymbol{z}]$. Then*

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n,$$

for some quotient module $\mathcal{Q}_i$ of $\mathcal{H}_i$, $i = 1, \ldots, n$, if and only if
(i) *$\mathcal{Q}$ is doubly commuting, and*
(ii) *$[\mathcal{Q}]_{z_k, z_{k+1}, \ldots, z_n}$ is a joint $\big(M_{z_k}, M_{z_{k+1}}, \ldots, M_{z_n}\big)$-reducing subspace of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ for $k = 1, \ldots, n$.*

Proof. Let $\mathcal{Q}$ be a doubly commuting quotient module of $\mathcal{H}$ and $[\mathcal{Q}]_{z_k, z_{k+1}, \ldots, z_n}$ be a joint $\big(M_{z_k}, M_{z_{k+1}}, \ldots, M_{z_n}\big)$-reducing subspace for $k = 1, \ldots, n$. In particular for $k = 2$,

$$\widetilde{\mathcal{Q}} := [\mathcal{Q}]_{z_2, z_3, \ldots, z_n},$$

is a joint $(M_{z_2}, M_{z_3}, \ldots, M_{z_n})$-reducing subspace of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. By virtue of Proposition 4.2 we have

$$\widetilde{\mathcal{Q}} = \mathcal{Q}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n,$$

for some closed subspace $\mathcal{Q}_1$ of $\mathcal{H}_1$. Also since $\mathcal{Q}$ is a quotient module and $M_{z_i}^*$ commutes with M_{z_j} for $i \neq j$, it follows that $\widetilde{\mathcal{Q}}$ is a $M_{z_1}^*$-invariant subspace. Hence $\mathcal{Q}_1$ is a quotient module of $\mathcal{H}_1$. Now we claim that $\mathcal{Q}$ is a $P_{\widetilde{\mathcal{Q}}} M_{z_1}|_{\widetilde{\mathcal{Q}}}$-reducing subspace of $\widetilde{\mathcal{Q}}$. To this end, since $\mathcal{Q} \subseteq \widetilde{\mathcal{Q}}$, it is enough to show that

$$P_{\mathcal{Q}} M_{z_1}^*|_{\widetilde{\mathcal{Q}}} = M_{z_1}^*|_{\mathcal{Q}}.$$

Using the fact that $\mathcal{Q}$ is doubly commuting it follows that

$$C_{z_1}^* C_{z_i}^l = C_{z_i}^l C_{z_1}^*,$$

for $l \geq 0$ and $2 \leq i \leq n$, and hence

$$C_{z_1}^* C_{z_2}^{l_2} \cdots C_{z_n}^{l_n} = C_{z_2}^{l_2} \cdots C_{z_n}^{l_n} C_{z_1}^*,$$

for $l_2, l_3, \ldots, l_n \geq 0$. Therefore

$$M_{z_1}^* P_{\mathcal{Q}} M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} P_{\mathcal{Q}} = P_{\mathcal{Q}} M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} M_{z_1}^* P_{\mathcal{Q}} \qquad (l_2, l_3, \ldots, l_n \geq 0).$$

This implies that

$$M_{z_1}^* P_{\mathcal{Q}} (M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} P_{\mathcal{Q}}) = P_{\mathcal{Q}} M_{z_1}^* (M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} P_{\mathcal{Q}}),$$

for $l_2, l_3, \ldots, l_n \geq 0$. This proves the claim.
Now applying Proposition 4.3, we obtain a closed subspace $\mathcal{E}_1$ of $\mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ such that

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \mathcal{E}_1.$$

Finally note that since $\mathcal{Q}$ is doubly commuting, $\mathcal{E}_1$ is also doubly commuting quotient module of $\mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ and it satisfies the condition (ii) in the statement of this theorem. Repeating the argument above for $\mathcal{E}_1$, we conclude that

$$\mathcal{E}_1 = \mathcal{Q}_2 \otimes \mathcal{E}_2,$$

for some quotient module $\mathcal{Q}_2$ of $\mathcal{H}_2$ and doubly commuting quotient module $\mathcal{E}_2$ of $\mathcal{H}_3 \otimes \cdots \otimes \mathcal{H}_n$. Continuing in this way we obtain quotient modules $\mathcal{Q}_i \subseteq \mathcal{H}_i$, for $i = 1, \ldots, n$, such that

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \mathcal{Q}_2 \otimes \cdots \otimes \mathcal{Q}_n.$$

This proves the sufficient part.
To prove the necessary part, let $\mathcal{Q} = \mathcal{Q}_1 \otimes \mathcal{Q}_2 \otimes \cdots \otimes \mathcal{Q}_n$ be a quotient module of $\mathcal{H}$. Clearly

$$(I_{\mathcal{Q}_1} \otimes I_{\mathcal{Q}_2} \otimes \cdots \otimes \underbrace{P_{\mathcal{Q}_i} M_z|_{\mathcal{Q}_i}}_{i-\text{th place}} \otimes \cdots \otimes I_{\mathcal{Q}_n})_{i=1}^n,$$

is a doubly commuting tuple, that is, $\mathcal{Q}$ is doubly commuting. Finally, using the fact that $\mathcal{H}_i$ is a standard Hilbert module over $\mathbb{C}[z]$ for all $i = 1, \ldots, n$, we have

$$\begin{aligned} [\mathcal{Q}]_{z_k,\ldots,z_n} &= \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_{k-1} \otimes [\mathcal{Q}_k]_z \otimes \cdots \otimes [\mathcal{Q}_n]_z \\ &= \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_{k-1} \otimes \mathcal{H}_k \otimes \cdots \otimes \mathcal{H}_n, \end{aligned}$$

for $1 \leq k \leq n$. Now the result follows from Proposition 4.2. $\square$

Remark. Let $\mathcal{H}_i$ be a Hilbert module over $\mathbb{C}[z]$ with module multiplication operator T_i, $i = 1, \ldots, n$. Moreover, assume that $\mathcal{H}_i$ is a standard Hilbert module over $\mathbb{C}[z]$, that is, there does not exists a pair of non-zero quotient modules $\mathcal{Q}_1$ and $\mathcal{Q}_2$ such that $\mathcal{Q}_1 \perp \mathcal{Q}_2$. In this case, the above theorem still remains true for the Hilbert module $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ over $\mathbb{C}[\boldsymbol{z}]$ with module multiplication operators

$$\{I_{\mathcal{H}_1} \otimes \cdots \otimes I_{\mathcal{H}_{i-1}} \otimes T_i \otimes I_{\mathcal{H}_{i+1}} \otimes \cdots \otimes I_{\mathcal{H}_n}\}_{i=1}^n.$$

Let $\mathcal{H}_i$ be a reproducing kernel Hilbert module over $\mathbb{C}[z]$ with kernel K_i such that K_i^{-1} is a polynomial for all $i = 1, \ldots, n$. Then by Theorem 3.3 we know that $\mathcal{H}_i$'s are standard Hilbert modules over $\mathbb{C}[z]$ (see also the remark following Theorem 3.3). Thus $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ is a standard Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. This subclass of standard Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$ plays the central role in the rest of this paper. So we make the following definition to refer this subclass.

Definition 4.5. A standard Hilbert module $\mathcal{H} = \mathcal{H}_{K_1} \otimes \cdots \otimes \mathcal{H}_{K_n}$ over $\mathbb{C}[\boldsymbol{z}]$ is said to be *analytic Hilbert module* if K_i^{-1} is a polynomial in two variables z and $\bar{w}$ for all $i = 1, \ldots, n$.

The notion of analytic Hilbert module is closely related to the $\frac{1}{K}$-calculus introduced by Arazy and Englis [4]. Our result is true in the generality of Arazy-Englis. However, to avoid technical complications we restrict our attention to the analytic Hilbert modules.

Let $\mathcal{H}$ be a standard Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. Then $\mathcal{H}$ is an analytic Hilbert module if and only if $K^{-1}(\boldsymbol{z}, \boldsymbol{w})$ is a polynomial in $z_1, \ldots, z_n, \bar{w}_1, \ldots, \bar{w}_n$.

We show now that the condition (ii) in Theorem 4.4 holds for any quotient module of an analytic Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. After the proof of the proposition we will give some examples in order.

Proposition 4.6. *Let $\mathcal{Q}$ be a non-zero quotient module of an analytic Hilbert module $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ over $\mathbb{C}[\boldsymbol{z}]$. Then $[\mathcal{Q}]_{z_k,\ldots,z_n}$ is $(M_{z_k}, M_{z_{k+1}}, \ldots, M_{z_n})$-reducing subspace for $k = 1, \ldots, n$.*

Proof. Let $1 \leq k \leq n$ be fixed. Set

$$\prod_{i=k}^{n} K_i^{-1}(z_i, w_i) = \sum_{\mathbf{l},\mathbf{m} \in \mathbb{N}^{(n-k+1)}} a_{\mathbf{l},\mathbf{m}} z^{\boldsymbol{l}} \bar{w}^{\boldsymbol{m}},$$

where $z^{\boldsymbol{l}} = z_k^{l_k} \cdots z_n^{l_n}$ and $\bar{w}^{\boldsymbol{m}} = \bar{w}_k^{m_k} \cdots \bar{w}_n^{m_n}$ and $\boldsymbol{l} = (l_k, \ldots, l_n)$ and $\boldsymbol{m} = (m_k, \ldots, m_n)$ are in $\mathbb{N}^{(n-k+1)}$. Likewise, if $\mathbf{l} = (l_k, \ldots, l_n) \in \mathbb{N}^{(n-k+1)}$, then define $M_z^{\mathbf{l}} = M_{z_k}^{l_k} \cdots M_{z_n}^{l_n}$. Notice first that

$$I_{\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}} \otimes P_{\mathbb{C}}^{\otimes(n-k+1)} = \prod_{i=k}^{n} K_i^{-1}(M_{z_i}, M_{z_i}^*) = \sum_{\mathbf{l},\mathbf{m} \in \mathbb{N}^{(n-k+1)}} a_{\mathbf{l},\mathbf{m}} M_z^{\mathbf{l}} M_z^{*\mathbf{m}}. \tag{4}$$

In the last equality we used the fact that $M_{z_i} M_{z_j}^* = M_{z_j}^* M_{z_i}$ for $i \neq j$. This implies

$$\big(I_{\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}} \otimes P_{\mathbb{C}^{\otimes(n-k+1)}}\big)(\mathcal{Q}) \subseteq [\mathcal{Q}]_{z_k, \ldots, z_n}.$$

By a similar argument as the in the proof of Theorem 3.3, we have

$$\big(I_{\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}} \otimes P_{\mathbb{C}^{\otimes(n-k+1)}}\big)(\mathcal{Q}) \neq \{0\}.$$

Setting

$$\big(I_{\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}} \otimes P_{\mathbb{C}^{\otimes(n-k+1)}}\big)(\mathcal{Q}) := \mathcal{Q}_1 \otimes \mathbb{C}^{\otimes(n-k+1)},$$

for some closed subspace $\mathcal{Q}_1$ of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}$, we obtain

$$\mathcal{Q}_1 \otimes \mathcal{H}_k \otimes \cdots \otimes \mathcal{H}_n \subseteq [\mathcal{Q}]_{z_k, \ldots, z_n}.$$

To see $[\mathcal{Q}]_{z_k, \ldots, z_n} \subseteq \mathcal{Q}_1 \otimes \mathcal{H}_k \otimes \cdots \otimes \mathcal{H}_n$, it is enough to prove that $\mathcal{Q} \subseteq \mathcal{Q}_1 \otimes \mathcal{H}_k \otimes \cdots \otimes \mathcal{H}_n$, or equivalently,

$$\mathcal{Q}_1^{\perp} \otimes \mathcal{H}_k \otimes \cdots \otimes \mathcal{H}_n \subseteq \mathcal{Q}^{\perp}.$$

Since $\mathcal{Q}^{\perp}$ is a submodule the last containment will follow if we show that $f \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(n-k+1)-\text{times}} \in \mathcal{Q}^{\perp}$ for any $f \in \mathcal{Q}_1^{\perp}$. Now for $f \in \mathcal{Q}_1^{\perp}$ and $g \in \mathcal{Q}$, we have

$$\begin{aligned} \langle f \otimes 1 \otimes \cdots \otimes 1, g \rangle &= \langle (I_{\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}} \otimes P_{\mathbb{C}^{\otimes(n-k+1)}})(f \otimes 1 \otimes \cdots \otimes 1), g \rangle \\ &= \langle f \otimes 1 \otimes \cdots \otimes 1, (I_{\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}} \otimes P_{\mathbb{C}^{\otimes(n-k+1)}}) g \rangle \\ &= 0, \end{aligned}$$

where the last equality follows from the fact that $(I_{\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}} \otimes P_{\mathbb{C}^{\otimes(n-k+1)}}) g \in \mathcal{Q}_1 \otimes \mathbb{C}^{\otimes(n-k+1)}$. Therefore for any $1 \leq k \leq n$,

$$[\mathcal{Q}]_{z_k, \ldots, z_n} = \mathcal{Q}_1 \otimes \mathcal{H}_k \otimes \cdots \otimes \mathcal{H}_n,$$

for some closed subspace $\mathcal{Q}_1$ of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k-1}$. The result now follows from Proposition 4.2. $\square$

Combining above proposition, Theorems 3.3 and 4.4 we have the following result.

Theorem 4.7. *Let $\mathcal{Q}$ be a quotient module of an analytic Hilbert module $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ over $\mathbb{C}[\boldsymbol{z}]$. Then the following conditions are equivalent:*
(i) *$\mathcal{Q}$ is doubly commuting.*
(ii) *$\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ for some quotient module $\mathcal{Q}_i$ of $\mathcal{H}_i$, $i = 1, \ldots, n$.*

Now we pass to discuss some examples of analytic Hilbert modules and applications of Theorem 4.7. First consider the case of the Hardy module $H^2(\mathbb{D}^n)$ over the unit polydisc $\mathbb{D}^n$. The kernel function of $H^2(\mathbb{D})$ is given by

$$\mathbb{S}(z, w) = \frac{1}{1 - z\bar{w}} \qquad (z, w \in \mathbb{D}).$$

In particular, $\mathbb{S}^{-1}(z, w)$ is a polynomial. On account of the Hilbert module isomorphism

$$H^2(\mathbb{D}^n) \cong \underbrace{H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})}_{n\text{-times}},$$

we recover the following result of [19] (Theorem 3.2) and [16].

Theorem 4.8. *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{D}^n)$. Then $\mathcal{Q}$ is doubly commuting if and only if $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ for some quotient modules $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ of $H^2(\mathbb{D})$.*

Next we consider the case of *weighted Bergman spaces* over $\mathbb{D}^n$. The weighted Bergman space over the unit disc is denoted by $L^2_{a,\alpha}(\mathbb{D})$, with $\alpha > -1$, and is defined by

$$L^2_{a,\alpha}(\mathbb{D}) := \{f \in \mathcal{O}(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^2 \, dA_\alpha(z) < \infty\},$$

where $dA_\alpha(z) = (a+1)(1-|z|^2)^a dA(z)$ and dA refers the normalized area measure on $\mathbb{D}$. The weighted Bergman modules are reproducing kernel Hilbert modules with kernel functions

$$K_\alpha(z, w) = \frac{1}{(1 - z\bar{w})^{\alpha+2}} \qquad (z, w \in \mathbb{D}).$$

It is evident that K_α^{-1} is a polynomial if $\alpha \in \mathbb{N}$. Let $\boldsymbol{\alpha} \in \mathbb{Z}^n$ with $\alpha_i > -1$ for $i = 1, \ldots, n$. The weighted Bergman space $L^2_{a,\boldsymbol{\alpha}}(\mathbb{D}^n)$ over $\mathbb{D}^n$ with weight $\boldsymbol{\alpha}$ is a standard Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ with kernel function

$$K_{\boldsymbol{\alpha}}(\boldsymbol{z}, \boldsymbol{w}) := \prod_{i=1}^{n} K_{\alpha_i}(z_i, w_i) = \prod_{i=1}^{n} \frac{1}{(1 - z_i\bar{w}_i)^{\alpha_i+2}} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n).$$

Thus we have the following theorem.

Theorem 4.9. *Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ with $\alpha_i > -1$ for $i = 1, \ldots, n$. Then a quotient module $\mathcal{Q}$ of $L^2_{a,\boldsymbol{\alpha}}(\mathbb{D}^n)$ is doubly commuting if and only if $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ for some quotient modules $\mathcal{Q}_i$ of $L^2_{a,\alpha_i}(\mathbb{D})$, $i = 1, \ldots, n$.*

Note that by the remark after Theorem 3.3 the above characterization result also holds for $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_i > -1$, $i = 1, \ldots, n$.

5. Co-doubly commuting submodules

The purpose of this section is twofold. First, we explicitly compute the cross commutators of a co-doubly commuting submodule (see Definition 2.1) of analytic Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$. Second, we investigate a variety of issues related to essential doubly commutativity of co-doubly commuting submodules. In particular, we completely classify the class of co-doubly commuting submodules which are essentially doubly commuting for $n \geq 3$.

We start with a well known result (cf. [19]) concerning sum of a family of commuting orthogonal projections on Hilbert spaces.

Lemma 5.1. *Let $\{P_i\}_{i=1}^n$ be a collection of commuting orthogonal projections on a Hilbert space $\mathcal{H}$. Then $\mathcal{L} := \sum_{i=1}^n ranP_i$ is closed and the orthogonal projection of $\mathcal{H}$ onto $\mathcal{L}$ is given by*

$$P_{\mathcal{L}} = I_{\mathcal{H}} - \prod_{i=1}^n (I_{\mathcal{H}} - P_i).$$

Now we are ready to present a characterization of co-doubly commuting submodules of an analytic Hilbert module $\mathbb{C}[\boldsymbol{z}]$. Recall that a submodule $\mathcal{S}$ of an analytic Hilbert module $\mathcal{H}$ over $\mathbb{C}[\boldsymbol{z}]$ is co-doubly commuting if $\mathcal{Q} = \mathcal{S}^{\perp} (\cong \mathcal{H}/\mathcal{S})$ is doubly commuting.

Theorem 5.2. *Let $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ be an analytic Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ and $\mathcal{S}$ be a submodule of $\mathcal{H}$. Then $\mathcal{S}$ is co-doubly commuting if and only if*

$$\mathcal{S} = (\mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n)^{\perp} = \sum_{i=1}^n \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{i-1} \otimes \mathcal{Q}_i^{\perp} \otimes \mathcal{H}_{i+1} \otimes \cdots \otimes \mathcal{H}_n,$$

for some quotient module $\mathcal{Q}_i$ of $\mathcal{H}_i$ and $i = 1, \ldots, n$.

Proof. Let $\mathcal{S}$ be a co-doubly commuting submodule of $\mathcal{H}$. Applying Theorem 4.7 to $\mathcal{S}$ we have

$$\mathcal{S} = (\mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n)^{\perp},$$

for some quotient module $\mathcal{Q}_i$ of $\mathcal{H}_i$ and $i = 1, \ldots, n$. Now let P_i be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{i-1} \otimes \mathcal{Q}_i^{\perp} \otimes \mathcal{H}_{i+1} \otimes \cdots \otimes \mathcal{H}_n$. Then $\{P_i\}_{i=1}^n$ satisfies the hypothesis of Lemma 5.1. Also note that $\mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ is the range of the orthogonal projection of $\prod_{i=1}^n (I_{\mathcal{H}} - P_i)$, that is,

$$P_{\mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n} = \prod_{i=1}^n (I_{\mathcal{H}} - P_i).$$

From this and Lemma 5.1 we readily obtain

$$\mathcal{S} = \sum_{i=1}^n \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{i-1} \otimes \mathcal{Q}_i^{\perp} \otimes \mathcal{H}_{i+1} \otimes \cdots \otimes \mathcal{H}_n.$$

This completes the proof. ☐

In the sequel we will make use of the following notation.
Let $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ be a doubly commuting quotient module of an analytic Hilbert module $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ over $\mathbb{C}[\boldsymbol{z}]$, where $\mathcal{Q}_i$ is a quotient module of $\mathcal{H}_i$, $i = 1, \ldots, n$. Let $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_k\}$ be a non-empty subset of $\{1, \ldots, n\}$. The subspace $\mathcal{Q}_{\boldsymbol{\lambda}}^{\perp}$ of $\mathcal{H}$ is defined by

$$\mathcal{Q}_{\boldsymbol{\lambda}}^{\perp} := \mathcal{Q}_1 \otimes \cdots \otimes \underbrace{\mathcal{Q}_{\lambda_1}^{\perp}}_{\lambda_1\text{-th}} \otimes \cdots \otimes \underbrace{\mathcal{Q}_{\lambda_k}^{\perp}}_{\lambda_k\text{-th}} \otimes \cdots \otimes \mathcal{Q}_n. \tag{5}$$

Notice that

$$\mathcal{Q}_{\boldsymbol{\lambda}}^{\perp} \perp \mathcal{Q}_{\boldsymbol{\lambda}'}^{\perp},$$

for each non-empty $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \subseteq \{1, \ldots, n\}$ and $\boldsymbol{\lambda} \neq \boldsymbol{\lambda}'$. This implies that

$$(\mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n)^{\perp} = \bigoplus_{\emptyset \neq \boldsymbol{\lambda} \subseteq \{1,\ldots,n\}} \mathcal{Q}_{\boldsymbol{\lambda}}^{\perp}.$$

The following theorem provides us with an easy way to calculate the cross commutators of co-doubly commuting submodules of analytic Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$.

Theorem 5.3. *Let $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ be an analytic Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ and $\mathcal{S} = (\mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n)^{\perp}$ be a co-doubly commuting submodule of $\mathcal{H}$. Then for all $1 \leq i < j \leq n$,*

$$[R_{z_i}^*, R_{z_j}] = P_{\mathcal{Q}_1} \otimes \cdots \otimes \underbrace{P_{\mathcal{Q}_i} M_z^* P_{\mathcal{Q}_i^{\perp}}}_{i\text{-th}} \otimes \cdots \otimes \underbrace{P_{\mathcal{Q}_j^{\perp}} M_z P_{\mathcal{Q}_j}}_{j\text{-th}} \otimes \cdots \otimes P_{\mathcal{Q}_n},$$

where $R_{z_j} = M_{z_j}|_{\mathcal{S}}$ for $1 \leq j \leq n$.

Proof. Let $\mathcal{S} = (\mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n)^{\perp}$ be a co-doubly commuting submodule of $\mathcal{H}$. By definition $R_{z_l} = M_{z_l}|_{\mathcal{S}}$ and hence $R_{z_l}^* = P_{\mathcal{S}} M_{z_l}^*|_{\mathcal{S}}$ for $l = 1, \ldots, n$. Let $1 \leq i < j \leq n$. Then

$$\begin{aligned}
[R_{z_i}^*, R_{z_j}] &= R_{z_i}^* R_{z_j} - R_{z_j} R_{z_i}^* \\
&= P_{\mathcal{S}} M_{z_i}^* M_{z_j}|_{\mathcal{S}} - P_{\mathcal{S}} M_{z_j} P_{\mathcal{S}} M_{z_i}^*|_{\mathcal{S}} \\
&= P_{\mathcal{S}} M_{z_i}^* M_{z_j}|_{\mathcal{S}} - P_{\mathcal{S}} M_{z_j} (I - P_{\mathcal{S}^{\perp}}) M_{z_i}^*|_{\mathcal{S}} \\
&= P_{\mathcal{S}} M_{z_j} P_{\mathcal{S}^{\perp}} M_{z_i}^*|_{\mathcal{S}} \\
&= P_{\mathcal{S}} M_{z_j} P_{\mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n} M_{z_i}^* P_{\mathcal{S}}.
\end{aligned}$$

Combining this with (5), we have

$$[R_{z_i}^*, R_{z_j}] = \Big(\sum_{\emptyset \neq \boldsymbol{\lambda} \subseteq \{1,\ldots,n\}} P_{\mathcal{Q}_{\boldsymbol{\lambda}}^{\perp}} \Big) M_{z_j} P_{\mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n} M_{z_i}^* \Big(\sum_{\emptyset \neq \boldsymbol{\lambda}' \subseteq \{1,\ldots,n\}} P_{\mathcal{Q}_{\boldsymbol{\lambda}'}^{\perp}} \Big).$$

Observe that for each $\boldsymbol{\lambda} \neq \{l\}$ and $l \in \{1, \ldots, n\}$,

$$P_{\mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n} M_{z_l}^* P_{\mathcal{Q}_{\boldsymbol{\lambda}}^{\perp}} = 0,$$

and therefore

$$\begin{aligned}
[R^*_{z_i}, R_{z_j}] &= \Big(\sum_{\emptyset\neq\boldsymbol{\lambda}\subseteq\{1,\ldots,n\}} P_{\mathcal{Q}^\perp_{\boldsymbol{\lambda}}}\Big) M_{z_j} P_{\mathcal{Q}_1\otimes\cdots\otimes\mathcal{Q}_n} M^*_{z_i} \Big(\sum_{\emptyset\neq\boldsymbol{\lambda}'\subseteq\{1,\ldots,n\}} P_{\mathcal{Q}^\perp_{\boldsymbol{\lambda}'}}\Big)\\
&= \sum_{\emptyset\neq\boldsymbol{\lambda},\boldsymbol{\lambda}'\subseteq\{1,\ldots,n\}} P_{\mathcal{Q}^\perp_{\boldsymbol{\lambda}}} M_{z_j} P_{\mathcal{Q}_1\otimes\cdots\otimes\mathcal{Q}_n} M^*_{z_i} P_{\mathcal{Q}^\perp_{\boldsymbol{\lambda}'}}\\
&= P_{\mathcal{Q}^\perp_{\{j\}}} M_{z_j} P_{\mathcal{Q}_1\otimes\cdots\otimes\mathcal{Q}_n} M^*_{z_i} P_{\mathcal{Q}^\perp_{\{i\}}}\\
&= P_{\mathcal{Q}_1}\otimes\cdots\otimes \underbrace{\big(P_{\mathcal{Q}_i} M^*_z P_{\mathcal{Q}^\perp_i}\big)}_{i\text{-th}} \otimes\cdots\otimes \underbrace{\big(P_{\mathcal{Q}^\perp_j} M_z P_{\mathcal{Q}_j}\big)}_{j\text{-th}} \otimes\cdots\otimes P_{\mathcal{Q}_n}.
\end{aligned}$$

This completes the proof. □

We still need a few more definitions about "small commutators" on Hilbert spaces.

Let $\mathcal{H}$ be a Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. Let $\mathcal{S}$ and $\mathcal{Q}$ be submodule and quotient module of $\mathcal{H}$, respectively. Then $\mathcal{S}$ is said to be *essentially doubly commuting* if

$$[R^*_{z_i}, R_{z_j}] \in \mathcal{K}(\mathcal{S}),$$

for $1 \leq i < j \leq n$. Here $\mathcal{K}(\mathcal{S})$ denotes the algebra of all compact operators on $\mathcal{S}$. Moreover, it is *essentially normal* if $[R^*_{z_i}, R_{z_j}] \in \mathcal{K}(\mathcal{S})$ for $1 \leq i, j \leq n$. Similarly $\mathcal{Q}$ is *essentially doubly commuting* if

$$[C^*_{z_i}, C_{z_j}] \in \mathcal{K}(\mathcal{Q}),$$

for all $1 \leq i < j \leq n$, and it is *essentially normal* if $[C^*_{z_i}, C_{z_j}] \in \mathcal{K}(\mathcal{Q})$ for $1 \leq i, j \leq n$ (see [20]). Here R_{z_i} and C_{z_i} are as in (2).

Now we provide a characterization of essentially doubly commuting co-doubly commuting submodules of an analytic Hilbert module over $\mathbb{C}[\boldsymbol{z}]$.

Theorem 5.4. *Let $\mathcal{S} = (\mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n)^\perp$ be a co-doubly commuting submodule of an analytic Hilbert module $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ over $\mathbb{C}[\boldsymbol{z}]$, where $\mathcal{Q}_i$ is a quotient module of $\mathcal{H}_i$, $i = 1, \ldots, n$. Then:*
(i) *For $n = 2$, $\mathcal{S}$ is essentially doubly commuting if and only if $P_{\mathcal{Q}_j} M^*_z P_{\mathcal{Q}^\perp_j}$ is compact for all $j = 1, 2$.*
(ii) *For $n > 2$, $\mathcal{S}$ is essentially doubly commuting if and only if $\mathcal{S}$ is of finite co-dimension.*

Proof. The proof follows from the above lemma. □

If the analytic Hilbert module $\mathcal{H}$ in the above theorem is $H^2(\mathbb{D}^n)$, then $P_{\mathcal{Q}_j} M^*_z P_{\mathcal{Q}^\perp_j}$ is a rank one operator for all quotient modules $\mathcal{Q}_i$ of $H^2(\mathbb{D})$ and $i = 1, \ldots, n$ (see Proposition 2.3 in [20]). In particular, for $\mathcal{H} = H^2(\mathbb{D}^2)$, the submodule $\mathcal{S} = (\mathcal{Q}_1 \otimes \mathcal{Q}_2)^\perp$ is always essentially doubly commuting. This result is due to Yang [28]. For the Hardy space $H^2(\mathbb{D}^n)$, Part (ii) was obtained by the third author in [20].

The next two results becomes a useful variant of the above theorem.

Corollary 5.5. *For $n > 2$, let $\mathcal{S}$ be a co-doubly commuting submodule of an analytic Hilbert module $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ and $\mathcal{Q} = \mathcal{S}^\perp (\cong \mathcal{H}/\mathcal{S})$. Then the following are equivalent.*
(i) *$\mathcal{S}$ is essentially doubly commuting.*

(ii) $\mathcal{S}$ *is of finite co-dimension.*
(iii) $\mathcal{Q}$ *is essentially normal.*

Corollary 5.6. *Let $\mathcal{S}$ be an essentially normal co-doubly commuting submodule of an analytic Hilbert module $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. If $\mathcal{S}$ is of infinite co-dimension, then $n = 2$.*

In the case $\mathcal{H} = H^2(\mathbb{D}^n)$, both the Corollaries 5.5 and 5.6 were obtained by the third author in [20].

We end the section by the follwoing remark.

Remark. In view of Theorems 5.3 and 5.4 one is tempted to consider the issue of compactness (or Hilbert-Schmidt class, trace class) of products of cross-commutators of co-doubly commuting submodules. However, we observe as an easy consequence of Theorems 5.3 that

$$[R_{z_i}^*, R_{z_j}][R_{z_l}^*, R_{z_m}] = 0,$$

for any $1 \leq i < j \leq n$ and $1 \leq l < m \leq n$.
Nevertheless, it is interesting to observe that for a co-doubly commuting submodule $\mathcal{S} = (\mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n)^\perp$, the "repeated commutator"

$$[R_{z_n}, [R_{z_{n-1}}, \cdots [R_{z_3}, [R_{z_1}^*, R_{z_2}]] \cdots]] = P_{\mathcal{Q}_1} M_z^* P_{\mathcal{Q}_1^\perp} \otimes P_{\mathcal{Q}_2^\perp} M_z P_{\mathcal{Q}_2} \otimes \cdots \otimes P_{\mathcal{Q}_n^\perp} M_z P_{\mathcal{Q}_n},$$

is a non-zero rank-one operator. Let us verify this fact in case $n = 3$ (the general case follows by induction on n): Let $\mathcal{S} = (\mathcal{Q}_1 \otimes \mathcal{Q}_2 \otimes \mathcal{Q}_3)^\perp$ be a co-doubly commuting submodule of $H^2(\mathbb{D}^3)$. Then by Theorem 5.3,

$$\begin{aligned}
&[R_{z_3}, [R_{z_1}^*, R_{z_2}]] \\
&= M_{z_3} P_{\mathcal{S}} (P_{\mathcal{Q}_1} M_z^* P_{\mathcal{Q}_1^\perp} \otimes P_{\mathcal{Q}_2^\perp} M_z P_{\mathcal{Q}_2} \otimes P_{\mathcal{Q}_3}) - (P_{\mathcal{Q}_1} M_z^* P_{\mathcal{Q}_1^\perp} \otimes P_{\mathcal{Q}_2^\perp} M_z P_{\mathcal{Q}_2} \otimes P_{\mathcal{Q}_3}) M_{z_3} P_{\mathcal{S}} \\
&= (P_{\mathcal{Q}_1} \otimes P_{\mathcal{Q}_2^\perp} \otimes M_z P_{\mathcal{Q}_3})(P_{\mathcal{Q}_1} M_z^* P_{\mathcal{Q}_1^\perp} \otimes P_{\mathcal{Q}_2^\perp} M_z P_{\mathcal{Q}_2} \otimes P_{\mathcal{Q}_3}) \\
&\quad -(P_{\mathcal{Q}_1} M_z^* P_{\mathcal{Q}_1^\perp} \otimes P_{\mathcal{Q}_2^\perp} M_z P_{\mathcal{Q}_2} \otimes P_{\mathcal{Q}_3})(P_{\mathcal{Q}_1^\perp} \otimes P_{\mathcal{Q}_2} \otimes M_z P_{\mathcal{Q}_3}) \\
&= P_{\mathcal{Q}_1} M_z^* P_{\mathcal{Q}_1^\perp} \otimes P_{\mathcal{Q}_2^\perp} M_z P_{\mathcal{Q}_2} \otimes P_{\mathcal{Q}_3^\perp} M_z P_{\mathcal{Q}_3}.
\end{aligned}$$

We do not know any module theoretic interpretations of the above fact. These issues will be addressed in a future paper.

6. Wandering subspaces and ranks of submodules

In this section we investigate the existence of wandering subspace, in the sense of Halmos [14], of a co-doubly commuting submodule of $L^2_{a,\boldsymbol{\alpha}}(\mathbb{D}^n)$, and compute the rank of a co-doubly commuting submodules of $H^2(\mathbb{D}^n)$. In particular, we explicitly compute the rank of a co-doubly commuting submodule $\mathcal{S}$ of $H^2(\mathbb{D}^n)$ and prove that the rank of $\mathcal{S}$ is not greater than n.

We begin with the definition of wandering subspaces for submodules of analytic Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$.

Let $\mathcal{S}$ be a submodule of an analytic Hilbert module $\mathcal{H}$ over $\mathbb{C}[\boldsymbol{z}]$ and $\mathcal{W} \subseteq \mathcal{S}$ be a closed subspace. Then $\mathcal{W}$ is a *wandering subspace* of $\mathcal{S}$ if

$$\mathcal{W} \perp M_{\boldsymbol{z}}^{\boldsymbol{k}} \mathcal{W},$$

for all $\boldsymbol{k} \in \mathbb{N}^n \setminus \{0\}$ and

$$\mathcal{S} = \bigvee_{\boldsymbol{k} \in \mathbb{N}^n} M_z^{\boldsymbol{k}} \mathcal{W}.$$

Let $\mathcal{S}$ be a submodule of $H^2(\mathbb{D})$, or $L^2_a(\mathbb{D})$. Then $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$ is the wandering subspace of $\mathcal{S}$. Moreover, the dimension of $\mathcal{W}$ is always one for $\mathcal{H} = H^2(\mathbb{D})$ [7], and any value in the range $1, 2, \ldots, \infty$, for $\mathcal{H} = L^2_a(\mathbb{D})$ [3]. For a general n, the existence of wandering subspaces of doubly commuting submodules of $L^2_a(\mathbb{D}^n)$ is obtained in [17] and [8].

Now let $\mathcal{S} = (\mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n)^\perp$ be a co-doubly commuting submodule of $L^2_{a,\boldsymbol{\alpha}}(\mathbb{D}^n)$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, and $\mathcal{Q}_i$ is a quotient module of $L^2_{a,\alpha_i}(\mathbb{D})$, $i = 1, \ldots, n$. Let

$$\mathcal{W}_i = (\mathcal{Q}_i^\perp \ominus z\mathcal{Q}_i^\perp),$$

be the wandering subspace of $\mathcal{Q}_i^\perp$ for $i = 1, \ldots, n$. Consider the set

$$\mathcal{W} = \bigvee_{i=1}^n 1 \otimes \cdots \otimes 1 \otimes \mathcal{W}_i \otimes 1 \cdots \otimes 1 \subseteq \mathcal{S}.$$

By virtue of Theorem 5.2, it then follows easily that

$$\mathcal{S} = \bigvee_{\boldsymbol{k} \in \mathbb{N}^n} M_z^{\boldsymbol{k}} \mathcal{W}.$$

There is, however, no guarantee that $\mathcal{W} \perp M_z^{\boldsymbol{k}} \mathcal{W}$ for all $\boldsymbol{k} \in \mathbb{N}^n \setminus \{\mathbf{0}\}$. For instance, it is not necessarily true that

$$\langle 1 \otimes f_2 \otimes 1 \otimes \cdots \otimes 1, f_1 \otimes M_z 1 \otimes 1 \otimes \cdots \otimes 1 \rangle = 0,$$

for all $f_1 \in \mathcal{W}_1$ and $f_2 \in \mathcal{W}_2$.
However, if we further assume that $1 \in \mathcal{Q}_i$ for all $i = 1, \ldots, n$, it then easily follows that $\mathcal{W}$ is a wandering subspace of $\mathcal{S}$. Thus we have the following result on the existence of wandering subspaces of a class of co-doubly commuting submodules of $L^2_{a,\boldsymbol{\alpha}}(\mathbb{D}^n)$.

Theorem 6.1. *Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $\mathcal{Q}_i$ be a quotient module of $L^2_{a,\alpha_i}(\mathbb{D})$ and $1 \in \mathcal{Q}_i$, $i = 1, \ldots, n$. Then*

$$\mathcal{W} = \bigvee_{i=1}^n 1 \otimes \cdots \otimes 1 \otimes \mathcal{W}_i \otimes 1 \cdots \otimes 1$$

is a wandering subspace of the co-doubly commuting submodule $\mathcal{S} = (\mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n)^\perp$, where

$$\mathcal{W}_i = (\mathcal{Q}_i^\perp \ominus z\mathcal{Q}_i^\perp),$$

for $i = 1, \ldots, n$.

We now study the rank of a co-doubly commuting submodule of an analytic Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. Recall that the *rank* of a Hilbert module $\mathcal{H}$ over $\mathbb{C}[\boldsymbol{z}]$ is the smallest cardinality of its generating sets [11]. More precisely,

$$\operatorname{rank}(\mathcal{H}) = \min_{E \in \mathcal{G}(\mathcal{H})} \# E,$$

where

$$\mathcal{G}(\mathcal{H}) = \{E \subseteq \mathcal{H} : \bigvee_{\boldsymbol{k}\in\mathbb{N}^n} M_z^{\boldsymbol{k}} E = \mathcal{H}\}.$$

Let $\mathcal{S} = \theta H^2(\mathbb{D})$ be a submodule of $H^2(\mathbb{D})$ for some inner function $\theta \in H^\infty(\mathbb{D})$ [7]. Then

$$\mathcal{S} = \theta H^2(\mathbb{D}) = \bigvee_{m\geq 0} z^m E,$$

where $E = \{\theta\}$. Consequently, $\mathcal{S}$ is of rank one. This is no longer true for Hardy space over $\mathbb{D}^n$ and $n \geq 2$. As pointed out by Rudin [18], there exists a submodule $\mathcal{S}$ of $H^2(\mathbb{D})$ such that the rank of $\mathcal{S}$ is not finite (see also [16], [23] and [24]). We now consider the class of co-doubly commuting submodules of $H^2(\mathbb{D}^n)$.

Let $\mathcal{S}$ be a non-trivial proper co-doubly commuting submodule of $H^2(\mathbb{D}^n)$. Theorem 5.2 implies that there exists non-zero quotient modules $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ of $H^2(\mathbb{D})$ such that

$$\mathcal{S} = (\mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n)^\perp = \sum_{i=1}^n H^2(\mathbb{D}^{i-1}) \otimes \mathcal{Q}_i^\perp \otimes H^2(\mathbb{D}^{n-i}).$$

Then there exists a natural number $1 \leq m \leq n$ such that

$$\mathcal{Q}_{l_j} \neq H^2(\mathbb{D}) \qquad (j = 1, \ldots, m).$$

Let θ_{l_j} be the inner function corresponding to the non-zero submodule $\mathcal{Q}_{l_j}^\perp$, that is,

$$\mathcal{Q}_{l_j}^\perp = \theta_{l_j} H^2(\mathbb{D}) \qquad (j = 1, \ldots, m).$$

Let E be the set of one variable inner functions corresponding to $\{\theta_{l_j}\}_{j=1}^m$ over $\mathbb{D}^n$, that is,

$$E := \{\Theta_{l_j} \in \mathcal{S} : \Theta_{l_j} = 1 \otimes \cdots \otimes 1 \otimes \underbrace{\theta_{l_j}}_{l_j\text{-th}} \otimes 1 \otimes \ldots \otimes 1, j = 1, \ldots, m\}.$$

Then invoking again Theorem 5.2 we conclude that

$$\bigvee_{\boldsymbol{k}\in\mathbb{N}^n} M_z^{\boldsymbol{k}} E = \mathcal{S}.$$

Consequently,

$$\text{rank}(\mathcal{S}) \leq m.$$

If, in addition, we assume that $1 \in \mathcal{Q}_i$, for $1 \leq i \leq n$ then

$$\Theta_{l_j} \in \ker P_{\mathcal{S}} M_z^{*\boldsymbol{k}}, \qquad (1 \leq j \leq m)$$

for any non-zero $\boldsymbol{k} \in \mathbb{N}^n$. Then with a standard argument we obtain $\text{rank}(\mathcal{S}) \geq m$ and hence $\text{rank}(\mathcal{S}) = m$.

We summarize the results given above as follows.

Theorem 6.2. *Let $\mathcal{S} = (\mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n)^\perp$ be a co-doubly commuting submodule of $H^2(\mathbb{D}^n)$. Then the rank of $\mathcal{S}$ is less than or equal to the number of quotient modules $\mathcal{Q}_i$ which are different from $H^2(\mathbb{D})$. Moreover, equality holds if $1 \in \mathcal{Q}_i$ for all $1 \leq i \leq n$.*

7. Concluding Remarks

It is worth stressing here that the results of this paper are based on three essential assumptions on the Hilbert module $\mathcal{H}$:

(1) $\mathcal{H}$ is a reproducing kernel Hilbert module over $\mathbb{D}^n$. Moreover, the kernel function $K_{\mathcal{H}}$ of $\mathcal{H}$ is a product of one variable kernel functions over the unit disk $\mathbb{D}$. That is,

$$K_{\mathcal{H}}(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^{n} K_i(z_i, w_i) \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n).$$

(2) $\mathcal{H}$ is a standard reproducing kernel Hilbert module, that is, there does not exists a pair of non-zero orthogonal quotient modules of $\mathcal{H}_{K_i} \subseteq \mathcal{O}(\mathbb{D}, \mathbb{C})$, where $\mathcal{H}_{K_i}$ is the reproducing kernel Hilbert module corresponding to the kernel K_i and $i = 1, \ldots, n$.

(3) $K_{\mathcal{H}}^{-1}$ is a polynomial, or, that $\mathcal{H}$ admits a $\frac{1}{K}$-calculus, in the sense of Arazy and Englis.

The purpose of the following example is to show that the conclusion of Theorem 4.7 is false if we drop the assumption that $\mathcal{H}$ is standard.

Let $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ be a reproducing kernel Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ such that $\mathcal{H}_1$ is not a standard reproducing kernel Hilbert module over $\mathbb{C}[z]$. This implies $\mathcal{H}_1$ has two orthogonal quotient modules $\mathcal{Q}_1$ and $\mathcal{Q}_1'$. Now consider the following quotient module of $\mathcal{H}$

$$\mathcal{Q} = (\mathcal{Q}_1 \otimes \mathcal{Q}_2 \otimes \mathcal{Q}_3 \otimes \cdots \otimes \mathcal{Q}_n) \oplus (\mathcal{Q}_1' \otimes \mathcal{Q}_2' \otimes \mathcal{Q}_3 \otimes \cdots \otimes \mathcal{Q}_n),$$

for two different quotient modules $\mathcal{Q}_2$ and $\mathcal{Q}_2'$ of $\mathcal{H}_2$ and some quotient modules $\mathcal{Q}_i$ of $\mathcal{H}_i$, $i = 3, \ldots, n$. Then it is evident that $\mathcal{Q}$ is a doubly commuting quotient module of $\mathcal{H}$ but it can not be represented in the form of tensor product of n one variable quotient modules. Therefore one may ask the following general question.

Is every doubly commuting quotient module of a Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ orthogonal sum of quotient modules each of which is Hilbert tensor product of one variable quotient modules?

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