RANK OF A CO-DOUBLY COMMUTING SUBMODULE IS 2

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Dedicated to the memory of our friend and colleague Sudipta Dutta

Abstract. We prove that the rank of a non-trivial co-doubly commuting submodule is 2. More precisely, let $\varphi, \psi \in H^\infty(\mathbb{D})$ be two inner functions. If $\mathcal{Q}_\varphi = H^2(\mathbb{D})/\varphi H^2(\mathbb{D})$ and $\mathcal{Q}_\psi = H^2(\mathbb{D})/\psi H^2(\mathbb{D})$, then

$$\text{rank } (\mathcal{Q}_\varphi \otimes \mathcal{Q}_\psi)^\perp = 2.$$

An immediate consequence is the following: Let $\mathcal{S}$ be a co-doubly commuting submodule of $H^2(\mathbb{D}^2)$. Then rank $\mathcal{S} = 1$ if and only if $\mathcal{S} = \Phi H^2(\mathbb{D}^2)$ for some one variable inner function $\Phi \in H^\infty(\mathbb{D}^2)$. This answers a question posed by R. G. Douglas and R. Yang [4].

1. Introduction

Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting bounded linear operators on a Hilbert space $\mathcal{H}$. For a subset $E \subseteq \mathcal{H}$ we denote $[E]_T$ by the close subspace $\overline{\text{span}}\{T_1^{k_1} \cdots T_n^{k_n} E : k_j \in \mathbb{N}, j = 1, \ldots, n\}$ of $\mathcal{H}$. Then the *rank* of T [3] is the unique number

$$\text{rank}(T) = \min\{\#E : [E]_T = \mathcal{H}, E \subseteq \mathcal{H}\}.$$

A closed subspace $\mathcal{S}$ of $H^2(\mathbb{D}^n)$, the Hardy space over the unit polydisc $\mathbb{D}^n$, is said to be shift invariant if $M_{z_i}(\mathcal{S}) \subseteq \mathcal{S}$ for $i = 1, 2, \ldots, n$, where M_{z_i} is the co-ordinate multiplication operator on $H^2(\mathbb{D}^n)$. The *rank* of a shift invariant subspace $\mathcal{S}$ of $H^2(\mathbb{D}^n)$ is the rank of the corresponding n-tuple of restricted co-ordinate shift operators, that is

$$\text{rank } \mathcal{S} = \text{rank } (M_{z_1}|_\mathcal{S}, \ldots, M_{z_n}|_\mathcal{S}).$$

The rank of a bounded linear operator (or, of a commuting tuple of bounded linear operators) on a Hilbert space is an important numerical invariant. Very briefly, the rank of a bounded linear operator is the cardinality of a minimal generating set (see the definition below). One of the most intriguing and important problems in operator theory and function theory is the existence of a finite generating set for a commuting tuple of operators. Alternatively, one may ask when the rank of a commuting tuple of operators is finite.

Prototype examples of rank one operators are the co-ordinate multiplication operator tuple $(M_{z_1}, \ldots, M_{z_n})$ on the Hardy space, the (weighted) Bergman space over the unit ball and the polydisc in $\mathbb{C}^n$, $n \geq 1$, and the Drury-Arveson space over the unit ball in $\mathbb{C}^n$. Moreover, a particular version of the celebrated invariant subspace theorem of Beurling says: A shift invariant (or, shift co-invariant) subspace of the one variable Hardy space is of rank one.

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Computation of ranks of shift invariant as well as shift co-invariant subspaces beyond the case of the one variable Hardy space is an excruciatingly difficult problem, even if one considers only shift invariant (as well as co-invariant) subspaces of the Hardy space over the unit polydisc in $\mathbb{C}^n$, $n > 1$ (see however [2, 6, 7, 8, 14]).

The purpose of this paper is to compute the rank of a tractable class of shift invariant subspaces of the two variables Hardy space, $H^2(\mathbb{D}^2)$, over the bidisc $\mathbb{D}^2$ in $\mathbb{C}^2$. In order to state the precise contribution of this paper, we need to introduce first some definitions and notations.

We denote the open unit disc of $\mathbb{C}$ by $\mathbb{D}$, and the unit circle by $\mathbb{T}$. The Hardy space over the unit disc $\mathbb{D}$ (bidisc $\mathbb{D}^2$), denoted by $H^2(\mathbb{D})$ ($H^2(\mathbb{D}^2)$), is the Hilbert space of all square summable holomorphic functions on $\mathbb{D}$ (on $\mathbb{D}^2$). Also we will denote by M_z and M_w the multiplication operators on $H^2(\mathbb{D}^2)$ by the coordinate functions z and w, respectively. It is easy to see that (M_z, M_w) is a pair of commuting isometries, that is,

$$M_z M_w = M_w M_z, \quad M_z^* M_z = M_w^* M_w = I_{H^2(\mathbb{D}^2)}.$$

Identifying $H^2(\mathbb{D}^2)$ with the 2-fold Hilbert space tensor product $H^2(\mathbb{D}) \otimes H^2(\mathbb{D})$, one can represent (M_z, M_w) as $(M_z \otimes I_{H^2(\mathbb{D})}, I_{H^2(\mathbb{D})} \otimes M_w)$.

Let $\mathcal{S}$ and $\mathcal{Q}$ be closed subspaces of $H^2(\mathbb{D}^2)$. Then $\mathcal{S}$ is said to be a *submodule* if $M_z(\mathcal{S}) \subseteq \mathcal{S}$ and $M_w(\mathcal{S}) \subseteq \mathcal{S}$. We say that $\mathcal{Q}$ is a *quotient module* if $\mathcal{Q}^\perp$ is a submodule.

A well-known result due to Beurling states that if $\mathcal{S}$ is a submodule of $H^2(\mathbb{D})$ (that is, $\mathcal{S}$ is closed subspace of $H^2(\mathbb{D})$ and $M_z\mathcal{S} \subseteq \mathcal{S}$), then $\mathcal{S}$ can be represented as

$$\mathcal{S} = \mathcal{S}_\varphi := \varphi H^2(\mathbb{D}),$$

where $\varphi \in H^\infty(\mathbb{D})$ is an inner function (that is, φ is a bounded holomorphic function on $\mathbb{D}$ and $|\varphi| = 1$ a.e. on $\mathbb{T}$). Consequently, a quotient module $\mathcal{Q}$ (that is, $\mathcal{Q}$ is a closed subspace of $H^2(\mathbb{D})$ and $M_z^*\mathcal{Q} \subseteq \mathcal{Q}$) of $H^2(\mathbb{D})$ can be represented as

$$\mathcal{Q} = \mathcal{Q}_\varphi := (\mathcal{S}_\varphi)^\perp = H^2(\mathbb{D})/\varphi H^2(\mathbb{D}).$$

It readily follows that

$$\text{rank } (M_z|_{\mathcal{S}_\varphi}) = \text{rank } (P_{\mathcal{Q}_\varphi} M_z|_{\mathcal{Q}_\varphi}) = 1.$$

Rudin [10], however, pointed out that there exists a submodule $\mathcal{S}$ of $H^2(\mathbb{D}^2)$ such that the rank of $\mathcal{S}$ is not finite (see also [7], [12] and [13]).

A quotient module $\mathcal{Q}$ of $H^2(\mathbb{D}^2)$ is *doubly commuting* if $C_z C_w^* = C_w^* C_z$, where $C_z = P_{\mathcal{Q}} M_z|_{\mathcal{Q}}$ and $C_w = P_{\mathcal{Q}} M_w|_{\mathcal{Q}}$. A submodule $\mathcal{S}$ of $H^2(\mathbb{D}^2)$ is *co-doubly commuting* if the quotient module $\mathcal{S}^\perp (\cong H^2(\mathbb{D}^2)/\mathcal{S})$ is doubly commuting.

The following useful characterization of co-doubly commuting submodules is essential for our study (see [9, 11]): If $\mathcal{Q}$ is a quotient module of $H^2(\mathbb{D}^2)$, then $\mathcal{Q}$ is a doubly commuting quotient module if and only if

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \mathcal{Q}_2,$$

for some quotient modules $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of $H^2(\mathbb{D})$.

Let $\mathcal{S} = (\mathcal{Q}_1 \otimes \mathcal{Q}_2)^\perp$ be a non-zero co-doubly commuting submodule. If $\mathcal{Q}_j = H^2(\mathbb{D})$, for some $j = 1, 2$, then it is easy to see that

$$\text{rank } \mathcal{S} = 1.$$

Now let both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be non-trivial quotient modules of $H^2(\mathbb{D})$, that is, $\mathcal{Q}_j \neq \{0\}, H^2(\mathbb{D})$, $j = 1, 2$. Then there exist inner functions $\varphi, \psi \in H^\infty(\mathbb{D})$ such that $\mathcal{Q}_1 = \mathcal{Q}_\varphi$ and $\mathcal{Q}_2 = \mathcal{Q}_\psi$. The main purpose of the present paper is to prove that (see Theorem 2.1)

$$\text{rank } (\mathcal{Q}_\varphi \otimes \mathcal{Q}_\psi)^\perp = 2.$$

As a consequence of this, we give a complete and affirmative answer to a conjecture of Douglas and Yang (see page 220 [4]): If $\mathcal{S}$ is a rank one co-doubly commuting submodule, then $\mathcal{S} = \Phi H^2(\mathbb{D}^2)$ for some one variable inner function $\Phi \in H^\infty(\mathbb{D})$.

2. Proof of the main result

We begin with a simple but crucial observation on the rank of a joint semi-invariant subspace of a commuting tuple of operators.

Lemma 2.1. Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting operators on a Hilbert space $\mathcal{H}$. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two joint T-invariant subspaces of $\mathcal{H}$ and $\mathcal{S}_2 \subseteq \mathcal{S}_1$. If $\mathcal{S} = \mathcal{S}_1 \ominus \mathcal{S}_2$, then

$$\text{rank } (P_\mathcal{S} T_1|_\mathcal{S}, \ldots, P_\mathcal{S} T_n|_\mathcal{S}) \leq \text{rank } (T_1|_{\mathcal{S}_1}, \ldots, T_n|_{\mathcal{S}_1}).$$

Proof. Let $m \in \mathbb{N}$ be the right side of the above inequality. Let $\{f_j\}_{j=1}^m \subseteq \mathcal{S}_1$ be a generating set for $(T_1|_{\mathcal{S}_1}, \ldots, T_n|_{\mathcal{S}_1})$. Clearly, $P_\mathcal{S} T_j P_\mathcal{S} = P_\mathcal{S} T_j|_{\mathcal{S}_1}$ for all $j = 1, \ldots, n$. This yields

$$(P_\mathcal{S} T_i P_\mathcal{S})(P_\mathcal{S} T_j P_\mathcal{S}) = P_\mathcal{S}(T_i T_j)|_{\mathcal{S}_1} \qquad (i, j = 1, \ldots, n).$$

It hence follows that $\{P_\mathcal{S} f_j\}_{j=1}^m$ is a generating set for $(P_\mathcal{S} T_1|_\mathcal{S}, \ldots, P_\mathcal{S} T_n|_\mathcal{S})$. This completes the proof. $\square$

We now prove the main result of this paper.

Theorem 2.1. *Let $\varphi, \psi \in H^\infty(\mathbb{D})$ be two inner functions. If*

$$\mathcal{S} = (\mathcal{Q}_\varphi \otimes \mathcal{Q}_\psi)^\perp,$$

then rank $\mathcal{S} = 2$.

Proof. Let $X = I_{H^2(\mathbb{D}^2)} - (I_{H^2(\mathbb{D}^2)} - M_\varphi M_\varphi^* \otimes I_{H^2(\mathbb{D})})(I_{H^2(\mathbb{D}^2)} - I_{H^2(\mathbb{D})} \otimes M_\psi M_\psi^*)$. Since

$$\mathcal{S} = \text{ran} X,$$

and

$$X = ((M_\varphi M_\varphi^*) \otimes (I_{H^2(\mathbb{D})} - M_\psi M_\psi^*)) \oplus (I_{H^2(\mathbb{D})} \otimes M_\psi M_\psi^*),$$

it follows that

$$\mathcal{S} = (\mathcal{S}_\varphi \otimes \mathcal{Q}_\psi) \oplus \left(H^2(\mathbb{D}) \otimes \mathcal{S}_\psi\right).$$

Since by Theorem 6.2 of [1], rank $\mathcal{S} \leq 2$, we only need to show that rank $\mathcal{S} \geq 2$. Set

$$\mathcal{E} = \mathcal{S} \ominus (\mathcal{S}_\varphi \otimes \mathcal{S}_\psi).$$

It follows that

$$\mathcal{E} = (\mathcal{S}_\varphi \otimes \mathcal{Q}_\psi) \oplus (\mathcal{Q}_\varphi \otimes \mathcal{S}_\psi).$$

Since $\mathcal{S}_\varphi \otimes \mathcal{S}_\psi \subseteq \mathcal{S}$ is a submodule of $H^2(\mathbb{D}^2)$, by Lemma 2.1, it follows that

$$\text{rank}(P_\mathcal{E} M_z|_\mathcal{E}, P_\mathcal{E} M_w|_\mathcal{E}) \leq \text{rank}(M_z|_\mathcal{S}, M_w|_\mathcal{S}) = \text{rank}(\mathcal{S}). \tag{2.1}$$

Note that

$$P_{\mathcal{E}} = (P_{\mathcal{S}_\varphi} \otimes P_{\mathcal{Q}_\psi}) \oplus (P_{\mathcal{Q}_\varphi} \otimes P_{\mathcal{S}_\psi}).$$

and hence, an easy calculation yields

$$P_{\mathcal{E}} M_z|_{\mathcal{E}} = (M_z|_{\mathcal{S}_\varphi} \otimes P_{\mathcal{Q}_\psi}) \oplus (P_{\mathcal{Q}_\varphi} M_z|_{\mathcal{Q}_\varphi} \otimes P_{\mathcal{S}_\psi}),$$

and

$$P_{\mathcal{E}} M_w|_{\mathcal{E}} = (P_{\mathcal{S}_\varphi} \otimes P_{\mathcal{Q}_\psi} M_w|_{\mathcal{Q}_\psi}) \oplus (P_{\mathcal{Q}_\varphi} \otimes M_w|_{\mathcal{S}_\psi}).$$

Therefore it follows from the above equalities that $(\mathcal{S}_{\varphi^2} \otimes \mathcal{Q}_\psi) \oplus (\mathcal{Q}_\varphi \otimes \mathcal{S}_{\psi^2})$ is a joint $(P_{\mathcal{E}} M_z|_{\mathcal{E}}, P_{\mathcal{E}} M_w|_{\mathcal{E}})$ invariant subspace of $\mathcal{E}$. Set

$$\tilde{\mathcal{E}} = \mathcal{E} \ominus ((\mathcal{S}_{\varphi^2} \otimes \mathcal{Q}_\psi) \oplus (\mathcal{Q}_\varphi \otimes \mathcal{S}_{\psi^2})).$$

Notice that for any inner function $\theta \in H^\infty(\mathbb{D})$, we have

$$\mathcal{S}_\theta \ominus \mathcal{S}_{\theta^2} = \theta \mathcal{Q}_\theta.$$

From this and the representation of $\mathcal{E} = (\mathcal{S}_\varphi \otimes \mathcal{Q}_\psi) \oplus (\mathcal{Q}_\varphi \otimes \mathcal{S}_\psi)$ it follows that

$$\begin{aligned}\tilde{\mathcal{E}} &= ((\mathcal{S}_\varphi \otimes \mathcal{Q}_\psi) \oplus (\mathcal{Q}_\varphi \otimes \mathcal{S}_\psi)) \ominus ((\mathcal{S}_{\varphi^2} \otimes \mathcal{Q}_\psi) \oplus (\mathcal{Q}_\varphi \otimes \mathcal{S}_{\psi^2})) \\ &= (\varphi \mathcal{Q}_\varphi \otimes \mathcal{Q}_\psi) \oplus (\mathcal{Q}_\varphi \otimes \psi \mathcal{Q}_\psi).\end{aligned}$$

Then Lemma 2.1 and (2.1) implies that

$$\operatorname{rank}(P_{\tilde{\mathcal{E}}} M_z|_{\tilde{\mathcal{E}}}, P_{\tilde{\mathcal{E}}} M_w|_{\tilde{\mathcal{E}}}) \leq \operatorname{rank}(P_{\mathcal{E}} M_z|_{\mathcal{E}}, P_{\mathcal{E}} M_w|_{\mathcal{E}}) \leq \operatorname{rank}(\mathcal{S}) \leq 2.$$

To finish the proof of the theorem it is now enough to prove the following:

$$\operatorname{rank}(P_{\tilde{\mathcal{E}}} M_z|_{\tilde{\mathcal{E}}}, P_{\tilde{\mathcal{E}}} M_w|_{\tilde{\mathcal{E}}}) > 1.$$

Equivalently, it is enough to prove that the set $\{\xi\}$, for any $\xi \in \tilde{\mathcal{E}}$, is not a generating set corresponding to $(P_{\tilde{\mathcal{E}}} M_z|_{\tilde{\mathcal{E}}}, P_{\tilde{\mathcal{E}}} M_w|_{\tilde{\mathcal{E}}})$. Equivalently, given $\xi \in \tilde{\mathcal{E}}$, we show that there exists $\eta_\xi (\neq 0) \in \tilde{\mathcal{E}}$ such that

$$\langle (z^p \otimes w^q)\xi, \eta_\xi \rangle = 0 \qquad (p, q \in \mathbb{N}).$$

To this end, let $\{f_i\}$ and $\{g_j\}$ be orthonormal bases of $\mathcal{Q}_\varphi$ and $\mathcal{Q}_\psi$, respectively, and let $\xi \in \tilde{\mathcal{E}}$ where

$$\xi = (\sum_{k,l} a_{kl} \varphi f_k \otimes g_l) \oplus (\sum_{k,l} b_{kl} f_k \otimes \psi g_l),$$

$\{a_{kl}\}, \{b_{kl}\} \subseteq \mathbb{C}$, and

$$\sum_{k,l} |a_{kl}|^2, \sum_{k,l} |b_{kl}|^2 < \infty.$$

Again we observe that for any inner function $\theta \in H^\infty(\mathbb{D})$ and $f = \sum_{m \geq 0} c_m z^m \in \mathcal{Q}_\theta$ we have

$$M_z^*(\theta \bar{f}) \in \mathcal{Q}_\theta,$$

where $\bar{f} = \sum_{m \geq 0} \bar{c}_m e^{-imt} \in L^2(\mathbb{T})$. This follows from the fact that θ is a bounded holomorphic function on $\mathbb{D}$ and $M_z^*(\theta \bar{f}) \perp z^m$ for all $m < 0$ (which gives that $M_z^*(\theta \bar{f}) \in H^2(\mathbb{D})$), and then $M_z^*(\theta \bar{f}) \perp \theta z^m$ in $L^2(\mathbb{T})$ for all $m \geq 0$ (which gives that $M_z^*(\theta \bar{f}) \in \mathcal{Q}_\theta$). It should be noted that $M_z^*(\theta \bar{f}) = \theta \overline{zf} = C_\theta(f)$, where the conjugation map $C_\theta : \mathcal{Q}_\theta \to \mathcal{Q}_\theta$, $f \mapsto M_z^*(\theta \bar{f})$, is

called a C-symmetry and it is used extensively in the study of Toeplitz operators on model spaces (for more details see [5]).
Coming back to our context, this immediately yields that

$$M_z^*(\varphi\overline{f_k}) \otimes M_w^*(\psi\overline{g_l}) \in \mathcal{Q}_\varphi \otimes \mathcal{Q}_\psi \qquad (k, l \geq 0),$$

and hence $s_0 \otimes s_1, t_0 \otimes t_1 \in \mathcal{Q}_\varphi \otimes \mathcal{Q}_\psi$, where

$$s_0 \otimes s_1 := -\sum_{k,l} \overline{a}_{kl} M_z^*(\varphi\overline{f}_k) \otimes M_w^*(\psi\overline{g}_l) = -(M_z^* \otimes M_w^*)(\varphi \otimes \psi)(\sum_{k,l} \bar{a}_{kl}\bar{f}_k \otimes \bar{g}_l)$$

and

$$t_0 \otimes t_1 := \sum_{k,l} \overline{b}_{kl} M_z^*(\varphi\overline{f}_k) \otimes M_w^*(\psi\overline{g}_l) = (M_z^* \otimes M_w^*)(\varphi \otimes \psi)(\sum_{k,l} \bar{b}_{kl}\bar{f}_k \otimes \bar{g}_l).$$

Set

$$\eta_\xi = (\varphi t_0 \otimes t_1) \oplus (s_0 \otimes \psi s_1) \in \tilde{\mathcal{E}}.$$

Then $\eta_\xi \neq 0$ and for every $p, q \in \mathbb{N}$ we have

$$\begin{aligned}
\langle (z^p \otimes w^q)\xi, \eta_\xi \rangle &= \langle (z^p \otimes w^q)((\sum_{k,l} a_{kl}\varphi f_k \otimes g_l) \oplus (\sum_{k,l} b_{kl} f_k \otimes \psi g_l)), (\varphi t_0 \otimes t_1) \oplus (s_0 \otimes \psi s_1) \rangle \\
&= \langle (z^p \otimes w^q)(\sum_{k,l} a_{kl}\varphi f_k \otimes g_l), \varphi t_0 \otimes t_1 \rangle \\
&\qquad\qquad + \langle (z^p \otimes w^q)(\sum_{k,l} b_{kl} f_k \otimes \psi g_l), s_0 \otimes \psi s_1 \rangle \\
&= \langle (z^p \otimes w^q)(\sum_{k,l} a_{kl} f_k \otimes g_l), t_0 \otimes t_1 \rangle + \langle (z^p \otimes w^q)(\sum_{k,l} b_{kl} f_k \otimes g_l), s_0 \otimes s_1 \rangle \\
&= \langle (z^{p+1} \otimes w^{q+1})(\sum_{k,l} a_{kl}\ f_k \otimes g_l), (\varphi \otimes \psi)(\sum_{k,l=1}^{\infty} \bar{b}_{kl}\bar{f}_k \otimes \bar{g}_l) \rangle \\
&\qquad\qquad - \langle (z^{p+1} \otimes w^{q+1})(\sum_{k,l} b_{kl} f_k \otimes g_l), (\varphi \otimes \psi)(\sum_{k,l} \bar{a}_{kl}\bar{f}_k \otimes \bar{g}_l) \rangle \\
&= 0.
\end{aligned}$$

We have thus shown that $\{\xi\}$ is not a minimal generating subset of $\tilde{\mathcal{E}}$ with respect to $(P_{\tilde{\mathcal{E}}} M_z|_{\tilde{\mathcal{E}}}, P_{\tilde{\mathcal{E}}} M_w|_{\tilde{\mathcal{E}}})$ as desired. □

As a consequence of the above theorem we have the following corollary which provides an affirmative answer of the question raised by Douglas and Yang [4].

Corollary 2.2. Let $\mathcal{S}$ be a co-doubly commuting submodule of $H^2(\mathbb{D}^2)$. Then rank $(\mathcal{S}) = 1$ if and only if $\mathcal{S} = \Theta H^2(\mathbb{D}^2)$ for some one variable inner function $\Theta \in H^\infty(\mathbb{D})$.

Proof. If $\mathcal{S} = \Theta H^2(\mathbb{D}^2)$ for some one variable inner function $\Theta \in H^\infty(\mathbb{D})$, then $\mathcal{S} \cong H^2(\mathbb{D}^2)$ and hence rank $\mathcal{S} = 1$. To prove the the sufficient part let $\mathcal{S}$ be a rank one co-doubly commuting submodule of $H^2(\mathbb{D}^2)$. Then there exist quotient modules $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of $H^2(\mathbb{D})$ such that (see [9, 11])

$$\mathcal{S} = (\mathcal{Q}_1 \otimes \mathcal{Q}_2)^{\perp}.$$

Since rank $(\mathcal{S}) = 1$, it follows from Theorem 2.1 that $\mathcal{Q}_j = H^2(\mathbb{D})$, for some $j = 1, 2$. This shows that

$$\mathcal{S} = \mathcal{S}_{\varphi} \otimes H^2(\mathbb{D}), \quad \text{or} \qquad \mathcal{S} = H^2(\mathbb{D}) \otimes \mathcal{S}_{\psi},$$

for some inner functions $\varphi, \psi \in H^{\infty}(\mathbb{D})$. This concludes the proof of the corollary. $\square$

There is now the following interesting and natural question: Let $m \geq 2$ and let $\{\varphi_j\}_{j=1}^m \subseteq H^{\infty}(\mathbb{D})$ be inner functions. Is then

$$\text{rank } (\mathcal{Q}_{\varphi_1} \otimes \cdots \otimes \mathcal{Q}_{\varphi_m})^{\perp} = m?$$

Our present approach does not seem to work for $m > 2$ case.

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