CHARACTERIZATIONS OF TOEPLITZ OPERATORS ON THE ANNULUS

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Abstract. We establish algebraic characterizations of Toeplitz operators on the Hardy space on the annulus. As an application, we characterize compact perturbations of Toeplitz operators.

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1. Introduction

Let $r \in (0,1)$ be a fixed number, and let $\mathbb{A}$ denote the annulus centered at the origin with inner radius r and outer radius 1:

$$\mathbb{A} = \{z \in \mathbb{C} : r < |z| < 1\}.$$

Define the Hardy space on $\mathbb{A}$ as

$$H^2(\mathbb{A}) = \left\{ f : \mathbb{A} \to \mathbb{C} \text{ holomorphic} : \sup_{r<t<1} \int_0^{2\pi} |f(te^{i\theta})|^2 d\theta < \infty \right\}.$$

This space was studied in detail by Sarason in his seminal paper [14]. It has since been studied from various function-theoretic and operator-theoretic perspectives. See, for example, [1, 2, 3, 11, 13], and the recent work [7, 12], as well as the references therein.

The purpose of this paper is to present characterizations of Toeplitz operators on $H^2(\mathbb{A})$. We then use this characterization to describe compact perturbations of Toeplitz operators on $H^2(\mathbb{A})$, which we call asymptotic Toeplitz operators, following Barría and Halmos [4] and Feintuch [8]. We point out that Abrahamse in [1] extended several properties of classical Toeplitz operators on the Hardy space of the open unit disc to Hardy spaces on the annulus (in fact, on multiply connected domains). To carry out our discussion, we introduce a well-known representation of $H^2(\mathbb{A})$ based on the boundary values of its functions.

Note that the boundary of the annulus consists of two disjoint circles:

$$\partial\mathbb{A} = \mathbb{T} \sqcup r\mathbb{T},$$

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where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $r\mathbb{T} := \{rz : z \in \mathbb{T}\}$. Define $L^2(\partial\mathbb{A})$ as the Hilbert space of all square-integrable functions on $\partial\mathbb{A}$:

$$L^2(\partial\mathbb{A}) = \Big\{ f : \partial\mathbb{A} \longrightarrow \mathbb{C} \text{ measurable} : \|f\|_{\partial\mathbb{A}} := \Big(\int_{\partial\mathbb{A}} |f(z)|^2 d\sigma(z) \Big)^{\frac{1}{2}} < \infty \Big\},$$

where σ is Lebesgue measure on $\partial\mathbb{A}$, scaled so that $\sigma(\partial\mathbb{A}) = 2$ (see Section 2 for more details). Denote by $H^2(\partial\mathbb{A})$ the closure in $L^2(\partial\mathbb{A})$ of the set of rational functions on $\mathbb{C}$ with no poles in $\overline{\mathbb{A}}$. Because functions in $H^2(\mathbb{A})$ have non-tangential limits at almost every point of $\partial\mathbb{A}$, each function in $H^2(\mathbb{A})$ corresponds to a function in $H^2(\partial\mathbb{A})$. This correspondence is an isometric isomorphism and is onto. We therefore identify these spaces and use the notations interchangeably (see [14, Theorem 1]).

Finally, let $L^\infty(\partial\mathbb{A})$ denote the space of all essentially bounded functions on $\partial\mathbb{A}$. This space forms a Banach algebra under the supremum norm. For $\varphi \in L^\infty(\partial\mathbb{A})$, the *Toeplitz operator* T_φ on $H^2(\partial\mathbb{A})$ with symbol φ is defined by

$$T_\varphi h = P_{H^2(\partial\mathbb{A})}(\varphi h),$$

for all $h \in H^2(\partial\mathbb{A})$, where $P_{H^2(\partial\mathbb{A})}$ denotes the orthogonal projection from $L^2(\partial\mathbb{A})$ onto $H^2(\partial\mathbb{A})$. The simplest nontrivial example of a Toeplitz operator corresponds to the holomorphic symbol $\varphi(z) = z$, $z \in \mathbb{A}$, which we denote by M_z. In this case,

$$M_z h = T_z h = zh,$$

for all $h \in H^2(\partial\mathbb{A})$. We now present our algebraic characterizations of Toeplitz operators. For a Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ denote the space of bounded linear operators on $\mathcal{H}$.

Theorem 1.1. *Let $T \in \mathcal{B}(H^2(\mathbb{A}))$. The following are equivalent:*

(1) *T is a Toeplitz operator.*
(2) *$-M_z^{*2} T M_z^2 + (1 + r^2) M_z^* T M_z = r^2 T$.*
(3) *There exists unique $X, Y \in \mathcal{B}(H^2(\mathbb{A}))$ such that $X + Y = T$ and*

$$M_z^* X M_z = X \text{ and } M_z^* Y M_z = r^2 Y.$$

The above result is in the spirit of Brown and Halmos's algebraic characterization of Toeplitz operators on the Hardy space of the unit disc (see [6, Theorem 6], also see (2.3)).

As a by-product of the above, we characterize compact perturbations of Toeplitz operators (see Theorem 4.2): Let $S_1 = M_z$ and $S_2 = rM_z^{-1}$. An operator $T \in \mathcal{B}(H^2(\mathbb{A}))$ is a compact perturbation of a Toeplitz operator if and only if there exist $J_1, J_2 \in \mathcal{B}(H^2(\mathbb{A}))$ such that

$$S_i^{*m} T S_i^m \longrightarrow J_i \text{ as } m \to \infty,$$

in norm for all $i = 1, 2$, and

$$S_p^{*m} \big(T - J_1 - J_2\big) S_q^m \longrightarrow 0 \text{ as } m \to \infty,$$

in norm for all $p, q = 1, 2$.

The corresponding result for the Hardy space on the disc is due to Feintuch [8, Theorem 4.1]. The case of the Hardy space on the polydisc was established in [10]; see also the remark at the end of Section 3.

The remainder of the paper is organized as follows. Section 2 establishes the main result, namely, characterizations of Toeplitz operators. Section 3 characterizes compact operators in terms of shift operators. In Section 4, the final section of the paper, we apply these characterizations to describe compact perturbations of Toeplitz operators.

2. Toeplitz operators

We begin with some basic background on the Hardy space of the annulus. For more details, we refer the reader to Sarason's seminal work [14]. The brief overview below follows the recent article [7].

First, we recall that $L^2(\partial\mathbb{A})$ is the space of all square-integrable, scalar-valued functions on $\partial\mathbb{A}$ with respect to the measure σ, defined for every measurable set $E \subseteq \partial\mathbb{A}$ by

$$\sigma(E) = \frac{1}{2\pi}\Big(\big(m\{a \in [0,2\pi) : e^{ia} \in E\}\big) + \big(m\{b \in [0,2\pi) : re^{ib} \in E\}\big)\Big),$$

where m denotes the Lebesgue measure on $\mathbb{R}$. In view of $\partial\mathbb{A} = \mathbb{T} \sqcup r\mathbb{T}$, the inner product on $L^2(\partial\mathbb{A})$ is given by:

$$\langle f, g\rangle_{L^2(\partial\mathbb{A})} = \int_{\partial\mathbb{A}} f(z)\overline{g(z)}\, d\sigma(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\overline{g(e^{it})}dm(t) + \frac{1}{2\pi}\int_0^{2\pi} f(re^{it})\overline{g(re^{it})}dm(t),$$

for all $f, g \in L^2(\partial\mathbb{A})$. In particular,

$$\|f\|^2_{L^2(\partial\mathbb{A})} = \int_{\partial\mathbb{A}} |f(z)|^2 d\sigma(z) = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 dm(t) + \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2 dm(t),$$

for all $f \in L^2(\partial\mathbb{A})$. In view of this, for each $h \in H^2(\mathbb{A})$, we have

$$\|h\|^2_{H^2(\mathbb{A})} = \|h\|_1^2 + \|h\|_r^2,$$

where

$$\|h\|_1^2 := \frac{1}{2\pi}\int_0^{2\pi} |h(e^{it})|^2 dm(t), \text{ and } \|h\|_r^2 := \frac{1}{2\pi}\int_0^{2\pi} |h(re^{it})|^2 dm(t).$$

As noted in the introduction, we identify $H^2(\mathbb{A})$ with $H^2(\partial\mathbb{A})$, the closure in $L^2(\partial\mathbb{A})$ of the rational functions on $\mathbb{C}$ with no poles in $\overline{\mathbb{A}}$. This identification is an onto isometry. We will use the two notations interchangeably. We also use the following orthonormal basis for $H^2(\partial\mathbb{A})$, given by $\{e_n(z)\}_{n\in\mathbb{Z}}$ (cf. [9]):

$$e_n(z) = \frac{1}{\sqrt{1+r^{2n}}} z^n \quad (n \in \mathbb{Z}). \tag{2.1}$$

We now turn to our main goal and prove the first result. Before doing so, we introduce a few more pieces of notation. As noted earlier, M_z denotes the multiplication operator by the coordinate function z on $H^2(\mathbb{A})$. Recall that (see [14, page 47])

$$\text{spectrum of } M_z = \overline{\mathbb{A}},$$

so, in particular, M_z is invertible on $H^2(\mathbb{A})$:

$$M_z^{-1} \in \mathcal{B}(H^2(\mathbb{A})).$$

It is convenient and instructive to introduce notation for the shift operator on $H^2(\mathbb{A})$ and its scaled inverse. We denote them by S_1 and S_2, respectively:

$$S_1 = M_z \text{ and } S_2 = rM_z^{-1}. \tag{2.2}$$

Clearly,

$$S_1 S_2 = rI_{H^2(\partial\mathbb{A})}.$$

Alongside the Hardy space on the annulus, we also consider the Hardy space on the unit disc $\mathbb{D}$, denoted by $H^2(\mathbb{D})$. Using boundary values, we identify $H^2(\mathbb{D})$ with the closed

subspace of $L^2(\mathbb{T})$ consisting of functions whose negative Fourier coefficients vanish. For each symbol $\varphi \in L^\infty(\mathbb{T})$, the *Toeplitz operator* $T_\varphi^{H^2(\mathbb{D})}$ on $H^2(\mathbb{D})$ is defined by

$$T_\varphi^{H^2(\mathbb{D})} h = P_{H^2(\mathbb{D})}(\varphi h) \qquad (h \in H^2(\mathbb{D})),$$

where $P_{H^2(\mathbb{D})}$ denotes the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$. The Toeplitz operator on $H^2(\mathbb{D})$ with symbol $\varphi(z) = z$, $z \in \mathbb{D}$, is the well-known *shift* operator, which we denote by S. Therefore,

$$Sh = T_z^{H^2(\mathbb{D})} h = zh,$$

for all $h \in H^2(\mathbb{D})$. We recall the classical result of Brown and Halmos [6, Theorem 6]: $T \in \mathcal{B}(H^2(\mathbb{D}))$ is a Toeplitz operator if and only if

$$S^* T S = T. \tag{2.3}$$

We now connect the Hardy spaces on the disc and the annulus, together with the operators S, S_1, and S_2, through the following elementary lemma. This lemma is central to our characterization of Toeplitz operators on the annulus and may also be of independent interest.

Lemma 2.1. *Define $G_1, G_2 : H^2(\mathbb{D}) \to H^2(\mathbb{A})$ by*

$$(G_1 h)(z) := h(z),$$

and

$$(G_2 h)(z) := h\left(\frac{r}{z}\right),$$

for all $h \in H^2(\mathbb{D})$ and $z \in \mathbb{A}$. Then the following hold:

1. *G_1 and G_2 are left-invertible operators with $\|G_1\| = \|G_2\| = \sqrt{2}$.*
2. *$G_1 S = S_1 G_1$.*
3. *$G_2 S = S_2 G_2$.*

Proof. For any $h(z) = \sum_{n=0}^\infty c_n z^n \in H^2(\mathbb{D})$, we have

$$(G_1 h)(z) = \sum_{n=0}^\infty c_n z^n,$$

and

$$(G_2 h)(z) = \sum_{n=0}^\infty c_n r^n z^{-n}.$$

Using the orthogonality of $\{z^n\}_{n\in\mathbb{Z}}$ in $H^2(\mathbb{A})$ and the fact that (see (2.1))

$$\|r^n z^{-n}\|_{H^2(\mathbb{A})}^2 = r^{2n}(1 + r^{-2n}) = 1 + r^{2n},$$

for all $n \in \mathbb{Z}_+$, we obtain

$$\|G_1 h\|_{H^2(\mathbb{A})}^2 = \|G_2 h\|_{H^2(\mathbb{A})}^2 = \sum_{n=0}^\infty (1 + r^{2n})|c_n|^2 \leq 2 \sum_{n=0}^\infty |c_n|^2 = 2\|h\|_{H^2(\mathbb{D})}^2,$$

with equality when h is a nonzero constant function. Thus, $\|G_1\| = \|G_2\| = \sqrt{2}$. This estimate also shows that G_1 and G_2 are bounded below and therefore left-invertible. Moreover, for any $h \in H^2(\mathbb{D})$ and $z \in \mathbb{A}$, we have

$$(S_1 G_1 h)(z) = zh(z) = (Sh)(z) = (G_1 Sh)(z),$$

which yields $G_1S = S_1G_1$. Similarly,

$$(S_2G_2h)(z) = \frac{r}{z}h\left(\frac{r}{z}\right) = (Sh)\left(\frac{r}{z}\right) = (G_2Sh)(z),$$

so $G_2S = S_2G_2$. □

We also need to analyze $L^\infty(\partial\mathbb{A})$, the space of essentially bounded measurable functions of $\partial\mathbb{A}$ with respect to the measure σ. As noted in the introduction, $\partial\mathbb{A} = \mathbb{T} \sqcup r\mathbb{T}$. Moreover,

$$\sigma(E) = \frac{1}{2\pi}(m(E \cap \mathbb{T}) + m(E \cap r\mathbb{T})),$$

for all σ-measurable set $E \subseteq \partial\mathbb{A}$. Recall that E is σ-measurable if and only if $E \cap \mathbb{T}$ and $E \cap r\mathbb{T}$ are Lebesgue measurable. It follows that for a set $N \subseteq \partial\mathbb{A}$,

$$\sigma(N) = 0 \iff m(N \cap \mathbb{T}) = 0 \quad \text{and} \quad m(N \cap r\mathbb{T}) = 0.$$

In view of this, $L^\infty(\partial\mathbb{A})$ admits a natural decomposition. Consider the Banach space $L^\infty(\mathbb{T}, m) \oplus L^\infty(r\mathbb{T}, m)$, which we abbreviate as

$$L^\infty(\mathbb{T}) \oplus L^\infty(r\mathbb{T}),$$

equipped with the norm

$$\|(g, h)\|_\infty = \max\{\|g\|_{L^\infty(\mathbb{T})}, \|h\|_{L^\infty(r\mathbb{T})}\}.$$

It is routine to verify that the map

$$\Psi(\varphi) = (\varphi|_{\mathbb{T}}, \varphi|_{r\mathbb{T}}) \qquad (\varphi \in L^\infty(\partial\mathbb{A})),$$

defines an isometric isomorphism from $L^\infty(\partial\mathbb{A})$ onto $L^\infty(\mathbb{T}) \oplus L^\infty(r\mathbb{T})$. For instance, to see that Ψ is surjective, pick $(g, h) \in L^\infty(\mathbb{T}) \oplus L^\infty(r\mathbb{T})$ and define $\varphi : \partial\mathbb{A} \to \mathbb{C}$ by

$$\varphi = g\chi_{\mathbb{T}} + h\chi_{r\mathbb{T}}.$$

It follows that $\varphi \in L^\infty(\partial\mathbb{A})$ and $\Psi(\varphi) = (g, h)$.

In view of the above, we will often identify elements of $L^\infty(\partial\mathbb{A})$ with elements of $L^\infty(\mathbb{T}) \oplus L^\infty(r\mathbb{T})$ via the isometric isomorphism Ψ, without explicitly mentioning the identification.

The final ingredient in our characterization is a formula for the matrix coefficients of Toeplitz operators, expressed using the Hardy space inner product. Let $\varphi \in L^2(\partial\mathbb{A})$. For each $n \in \mathbb{Z}$, define the pair of n-th Fourier coefficients by

$$\widehat{\varphi_{\mathbb{T}}}(n) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{it})e^{-int}dm(t),$$

and

$$\widehat{\varphi_{r\mathbb{T}}}(n) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(re^{it})e^{-int}dm(t).$$

If $\varphi \in L^\infty(\partial\mathbb{A})$, then for every $j, k \in \mathbb{Z}$, we have (cf. [7, page 988])

$$\langle T_\varphi z^k, z^j\rangle = \widehat{\varphi}_{\mathbb{T}}(j-k) + r^{j+k}\widehat{\varphi}_{r\mathbb{T}}(j-k). \tag{2.4}$$

We recall that $S_1 = M_z$ and $S_2 = rM_z^{-1}$, so

$$S_1S_2 = rI_{H^2(\mathbb{A})}.$$

Therefore, for $Y \in \mathcal{B}(H^2(\mathbb{A}))$, the condition

$$S_2^*YS_2 = Y,$$

is equivalent to $S_1^*YS_1 = r^2Y$, which, since $S_1 = M_z$, is equivalent to

$$M_z^*YM_z = r^2Y. \tag{2.5}$$

We are now ready to prove the central result of this paper.

Theorem 2.2. *Let $T \in \mathcal{B}(H^2(\mathbb{A}))$. The following are equivalent:*

(1) *T is a Toeplitz operator.*
(2) *$-S_1^{*2}TS_1^2 + (1+r^2)S_1^*TS_1 = r^2T$.*
(3) *There exists unique $X, Y \in \mathcal{B}(H^2(\mathbb{A}))$ such that $X + Y = T$ and*

$$S_1^*XS_1 = X \text{ and } S_2^*YS_2 = Y.$$

Proof. Suppose first that $T = T_\varphi$ for some $\varphi \in L^\infty(\partial\mathbb{A})$. For each $f, g \in H^2(\mathbb{A})$, we have

$$\begin{aligned}\left\langle \left(-M_z^{*2}T_\varphi M_z^2 + (1+r^2)M_z^*T_\varphi M_z - r^2T_\varphi\right) f, g\right\rangle \\ = \int_{\partial\mathbb{A}} \varphi(z)\left[-|z|^4 + (1+r^2)|z|^2 - r^2\right] f(z)\overline{g(z)}\, d\sigma(z).\end{aligned}$$

Since $|z| = 1$ for all $z \in \mathbb{T}$ and $|z| = r$ for all $z \in r\mathbb{T}$, we have

$$-|z|^4 + (1+r^2)|z|^2 - r^2 = 0,$$

on both $\mathbb{T}$ and $r\mathbb{T}$. Therefore, we conclude that

$$-S_1^{*2}T_\varphi S_1^2 + (1+r^2)S_1^*T_\varphi S_1 = r^2T_\varphi.$$

This proves (1) $\implies$ (2). To prove (2) $\implies$ (3), assume $-M_z^{*2}TM_z^2 + (1+r^2)M_z^*TM_z = r^2T$. We note a general fact: if we define

$$X = \frac{1}{1-r^2}(S_1^*TS_1 - r^2T) \text{ and } Y = \frac{1}{1-r^2}(T - S_1^*TS_1), \tag{2.6}$$

then it is straightforward to verify that

$$X + Y = T. \tag{2.7}$$

In addition, note that

$$S_1^*XS_1 = M_z^*XM_z = \frac{1}{1-r^2}(M_z^{*2}TM_z^2 - r^2M_z^*TM_z).$$

Since $M_z^{*2}TM_z^2 = (1+r^2)M_z^*TM_z - r^2T$, it follows that

$$M_z^*XM_z = \frac{1}{1-r^2}((1+r^2)M_z^*TM_z - r^2T - r^2M_z^*TM_z) = X.$$

Similarly,

$$M_z^*YM_z = \frac{1}{1-r^2}(M_z^*TM_z - M_z^{*2}TM_z^2) = \frac{1}{1-r^2}(-(1+r^2)M_z^*TM_z + r^2T + M_z^*TM_z).$$

Thus, $M_z^*YM_z = r^2Y$, and hence, by (2.5), we have $S_2^*YS_2 = Y$. We postpone the proof that the representation of T in (2.7) is unique until the end of this section.
Now we prove that (3) $\implies$ (1). In this part of the proof, we repeatedly use the notation and conclusions of Lemma 2.1. We know that

$$S_1^*XS_1 = X \qquad \text{and} \qquad S_2^*YS_2 = Y.$$

Define $A_1, A_2 \in \mathcal{B}(H^2(\mathbb{D}))$ by

$$A_1 := G_1^* X G_1, \qquad A_2 := G_2^* Y G_2.$$

We claim that A_1 and A_2 are Toeplitz operators on $H^2(\mathbb{D})$. Indeed, since $S_1^* X S_1 = X$ and $S^* G_1^* = G_1^* S_1^*$ (see Lemma 2.1), it follows that

$$S^* A_1 S = S^*(G_1^* X G_1)S = (G_1^* S_1^*)X(S_1 G_1) = G_1^*(S_1^* X S_1)G_1 = G_1^* X G_1 = A_1,$$

proving that A_1 is a Toeplitz operator. Similarly,

$$S^* A_2 S = S^*(G_2^* Y G_2)S = (G_2^* S_2^*)Y(S_2 G_2) = G_2^*(S_2^* Y S_2)G_2 = G_2^* Y G_2 = A_2,$$

showing that A_2 is a Toeplitz operator. By Brown and Halmos (2.3), there exist unique symbols $\varphi_1, \widetilde{\varphi} \in L^\infty(\mathbb{T})$ such that

$$A_1 = T_{\varphi_1}^{H^2(\mathbb{D})} \qquad \text{and} \qquad A_2 = T_{\widetilde{\varphi}}^{H^2(\mathbb{D})}.$$

Define $\varphi_r \in L^\infty(r\mathbb{T})$ by

$$\varphi_r(re^{i\theta}) := \widetilde{\varphi}(e^{-i\theta}).$$

Via the identification provided by the onto isometry Ψ, outlined preceding the statement of this theorem, we know that $(\varphi_1, 0)$ and $(0, \varphi_r)$ belong to $L^\infty(\partial\mathbb{A})$. We claim that

$$X = T_{(\varphi_1,0)} \text{ and } Y = T_{(0,\varphi_r)},$$

on $H^2(\mathbb{A})$. To show this, it suffices to show that matrix coefficients of X and Y agree with those of the Toeplitz operators $T_{(\varphi_1,0)}$ and $T_{(0,\varphi_r)}$, respectively (see (2.4)). To this end, fix $k, n \in \mathbb{Z}$. Choose an integer $m \geq 1$ large enough that

$$k + m, n + m \geq 0.$$

Then

$$z^{k+m}, z^{n+m} \in H^2(\mathbb{D}),$$

and by definition of G_1,

$$G_1(z^{k+m}) = z^{k+m} = S_1^m z^k,$$

and

$$G_1(z^{n+m}) = z^{n+m} = S_1^m z^n.$$

Since $S_1^* X S_1 = X$, it follows by iteration that $S_1^{*m} X S_1^m = X$. Therefore,

$$\begin{aligned}
\langle Xz^k, z^n\rangle_{H^2(\mathbb{A})} &= \langle S_1^{*m} X S_1^m z^k, z^n\rangle_{H^2(\mathbb{A})} \\
&= \langle XG_1 z^{k+m}, G_1 z^{n+m}\rangle_{H^2(\mathbb{A})} \\
&= \langle G_1^* X G_1 z^{k+m}, z^{n+m}\rangle_{H^2(\mathbb{D})} \\
&= \langle T_{\varphi_1}^{H^2(\mathbb{D})} z^{k+m}, z^{n+m}\rangle_{H^2(\mathbb{D})} \\
&= \widehat{\varphi_1}\big((n+m) - (k+m)\big) \\
&= \widehat{\varphi_1}(n-k) \\
&= \langle T_{(\varphi_1,0)} z^k, z^n\rangle_{H^2(\mathbb{A})},
\end{aligned}$$

where the last equality uses (2.4). In the above, we also used the well-known fact that the matrix entries of the Toeplitz operator $T_\psi^{H^2(\mathbb{D})}$ on $H^2(\mathbb{D})$, for symbol $\psi \in L^\infty(\mathbb{T})$, with respect to the orthonormal basis $\{z^k\}_{k\in\mathbb{Z}_+}$, are given by

$$\widehat{\psi}(p-q) = \langle T_\psi^{H^2(\mathbb{D})} z^q, z^p\rangle_{H^2(\mathbb{D})},$$

for all $p, q \in \mathbb{Z}_+$. Since $\{z^t : t \in \mathbb{Z}\}$ is an orthogonal basis for $H^2(\mathbb{A})$, it follows that

$$X = T_{(\varphi_1,0)}.$$

To prove that $Y = T_{(0,\varphi_r)}$, we again choose an integer m large enough so that

$$m-k, m-n \geq 0.$$

Since $S_2 = rM_z^{-1}$, we have

$$S_2^m z^k = r^m z^{k-m} = r^k \Big(\frac{r}{z}\Big)^{m-k} = r^k G_2(z^{m-k}),$$

and similarly,

$$S_2^m z^n = r^n G_2(z^{m-n}).$$

Proceeding as before, $S_2^* Y S_2 = Y$ implies $S_2^{*m} Y S_2^m = Y$, and hence

$$\begin{aligned}
\langle Y z^k, z^n\rangle_{H^2(\mathbb{A})} &= \langle S_2^{*m} Y S_2^m z^k, z^n\rangle_{H^2(\mathbb{A})} \\
&= r^{n+k}\langle Y G_2 z^{m-k}, G_2 z^{m-n}\rangle_{H^2(\mathbb{A})} \\
&= r^{n+k}\langle G_2^* Y G_2 z^{m-k}, z^{m-n}\rangle_{H^2(\mathbb{D})} \\
&= r^{n+k}\langle T_\psi^{H^2(\mathbb{D})} z^{m-k}, z^{m-n}\rangle_{H^2(\mathbb{D})} \\
&= r^{n+k}\widehat{\psi}\big((m-n)-(m-k)\big) \\
&= r^{n+k}\widehat{\psi}(k-n) \\
&= r^{n+k}\widehat{\varphi_r}(n-k) \\
&= \langle T_{(0,\varphi_r)} z^k, z^n\rangle_{H^2(\mathbb{A})}.
\end{aligned}$$

Similarly, we conclude that

$$Y = T_{(0,\varphi_r)}.$$

We know that $X + Y = T$. Since

$$\varphi := (\varphi_1, \varphi_r) = (\varphi_1, 0) + (0, \varphi_r) \in L^\infty(\partial\mathbb{A}),$$

and

$$X + Y = T_{(\varphi_1,0)} + T_{(0,\varphi_r)} = T_\varphi,$$

we conclude that $T = T_\varphi$ is a Toeplitz operator, completing the proof that (3) implies (1). □

We now prove that the representation of T in part (3) of Theorem 2.2 is unique. More specifically, we want to prove that the representation $T = X + Y$ in (2.7) is unique, where X and Y are given by (see (2.6))

$$X = \frac{1}{1-r^2}(S_1^* T S_1 - r^2 T) \text{ and } Y = \frac{1}{1-r^2}(T - S_1^* T S_1).$$

Suppose

$$T = \widetilde{X} + \widetilde{Y},$$

where $\widetilde{X}, \widetilde{Y} \in \mathcal{B}(H^2(\mathbb{A}))$ satisfy

$$S_1^* \widetilde{X} S_1 = \widetilde{X} \text{ and } S_2^* \widetilde{Y} S_2 = \widetilde{Y}.$$

Now $T = \widetilde{X} + \widetilde{Y}$ implies $S_1^* T S_1 = \widetilde{X} + S_1^* \widetilde{Y} S_1$. Since $S_1^* \widetilde{Y} S_1 = r^2 \widetilde{Y}$, it follows that

$$S_1^* T S_1 - r^2 T = \widetilde{X} + S_1^* \widetilde{Y} S_1 - r^2 \widetilde{X} - r^2 \widetilde{Y} = (1 - r^2)\widetilde{X},$$

which yields $\widetilde{X} = \frac{1}{1-r^2}(S_1^* T S_1 - r^2 T)$. Since $\widetilde{Y} = T - \widetilde{X}$, we finally conclude that

$$\widetilde{X} = \frac{1}{1 - r^2}(S_1^* T S_1 - r^2 T) \text{ and } \widetilde{Y} = \frac{1}{1 - r^2}(T - S_1^* T S_1).$$

This completes the uniqueness part of the proof of (2) $\implies$ (3), which was previously left to be established.

3. Compact operators

Compact operators constitute one of the most important classes of operators in infinite-dimensional analysis. They are often regarded as "small" operators and provide an important link between finite- and infinite-dimensional theory. There are several characterizations of compactness; here, we focus on one involving the shift operator. This problem for $H^2(\mathbb{D})$ was studied extensively by Barría and Halmos [4] and Feintuch [8]. Here, we continue this line of work on $H^2(\mathbb{A})$.

We continue with the notation as in (2.2):

$$S_1 = M_z \text{ and } S_2 = r M_z^{-1}.$$

Since $e_n(z) = \frac{1}{\sqrt{1+r^{2n}}} z^n$ (see (2.1)), we have

$$M_z e_n = \sqrt{\frac{1 + r^{2n+2}}{1 + r^{2n}}} e_{n+1} \qquad (n \in \mathbb{Z}).$$

Also, recall from the beginning of the previous section that $\|h\|^2_{H^2(\mathbb{A})} = \|h\|_1^2 + \|h\|_r^2$ for all $h \in H^2(\mathbb{A})$. Therefore,

$$\|S_1 h\|^2 = \|M_z h\|^2 = \|h\|_1^2 + r^2 \|h\|_r^2 \leq \|h\|^2,$$

and

$$\|S_2 h\|^2 = \|r M_z^{-1} h\|^2 = \|r M_{z^{-1}} h\|^2 = r^2 \|h\|_1^2 + \|h\|_r^2 \leq \|h\|^2,$$

for all $h \in H^2(\mathbb{A})$ (note that $M_z^{-1} = M_{z^{-1}}$ on $H^2(\mathbb{A})$). In particular, S_1 and S_2 are contractions. On the other hand, since $S_1^m e_n = M_z^m e_n$, a straightforward computation gives

$$S_1^m e_n = \sqrt{\frac{1 + r^{2(n+m)}}{1 + r^{2n}}} e_{n+m} \text{ and } S_1^{*m} e_n = \sqrt{\frac{1 + r^{2n}}{1 + r^{2(n-m)}}} e_{n-m}, \tag{3.1}$$

for all $n \in \mathbb{Z}$ and $m \geq 1$. In particular, for each fixed n,

$$\|S_1^{*m} e_n\|^2 = \frac{1 + r^{2n}}{1 + r^{2(n-m)}} \to 0 \text{ as } m \to \infty.$$

Since S_1 is a contraction, the operators S_1^{*m} are uniformly bounded. Thus, by density of the linear span of $\{e_n : n \in \mathbb{Z}\}$, it follows that $S_1^{*m} \to 0$ in the strong operator topology. Similarly, for $m \geq 1$, since $S_2^m e_n = (rM_z^{-1})^m e_n$, we have

$$S_2^m e_n = r^m \sqrt{\frac{1+r^{2(n-m)}}{1+r^{2n}}} e_{n-m} \text{ and } S_2^{*m} e_n = r^m \sqrt{\frac{1+r^{2n}}{1+r^{2(n+m)}}} e_{n+m}, \tag{3.2}$$

for all $n \in \mathbb{Z}$ and $m \geq 1$. We conclude that

$$\|S_2^{*m} e_n\|^2 = r^{2m} \frac{1+r^{2n}}{1+r^{2(n+m)}} \to 0 \text{ as } m \to \infty,$$

for each $n \in \mathbb{Z}$, and hence, $S_2^{*m} \to 0$ in the strong operator topology.

We are now ready to present a characterization of compact operators on $H^2(\mathbb{A})$.

Theorem 3.1. *Let $T \in \mathcal{B}(H^2(\mathbb{A}))$. Then T is compact if and only if*

$$S_i^{*m} T S_j^m \to 0,$$

in norm as $m \to \infty$ for all $i, j = 1, 2$.

Proof. The necessity follows from a more general fact. Suppose that T is compact. Since $S_1^{*m} \to 0$ and $S_2^{*m} \to 0$ in the strong operator topology as $m \to \infty$, it follows from [5, 1.3 (d), page 3] that

$$\|S_i^{*m} T\| \longrightarrow 0 \qquad (i = 1, 2),$$

as $m \to \infty$. Since S_j is a contraction for $j = 1, 2$, we have

$$\|S_i^{*m} T S_j^m\| \leq \|S_i^{*m} T\| \, \|S_j^m\| \longrightarrow 0,$$

for every $i, j = 1, 2$ as $m \to \infty$. This proves the necessity of the condition in the theorem. Conversely, assume that $\|S_i^{*m} T S_j^m\| \longrightarrow 0$ as $m \to \infty$ for $i, j = 1, 2$. For each $m \geq 1$, define

$$A_m = S_1^m S_1^{*m} \text{ and } B_m = S_2^m S_2^{*m}.$$

Since S_1 and S_2 are contractions, $\|A_m\| \leq 1$ and $\|B_m\| \leq 1$ for every $m \geq 1$. Define

$$F_m = (I - A_m)(I - B_m) \qquad (m \geq 1).$$

We claim that F_m is a compact diagonal operator for each $m \geq 1$. Indeed, (3.1) and (3.2) give

$$A_m e_n = \frac{1+r^{2n}}{1+r^{2(n-m)}} e_n \text{ and } B_m e_n = r^{2m} \frac{1+r^{2n}}{1+r^{2(n+m)}} e_n,$$

for every $n \geq 1$. Therefore,

$$F_m e_n = \left(1 - \frac{1+r^{2n}}{1+r^{2(n-m)}}\right) \left(1 - r^{2m} \frac{1+r^{2n}}{1+r^{2(n+m)}}\right) e_n,$$

for each $m \in \mathbb{N}$ and $n \in \mathbb{Z}$. For each fixed $m \in \mathbb{N}$, since the scalar coefficient

$$\left(1 - \frac{1+r^{2n}}{1+r^{2(n-m)}}\right) \left(1 - r^{2m} \frac{1+r^{2n}}{1+r^{2(n+m)}}\right) \longrightarrow 0,$$

as $|n| \to \infty$, we conclude that F_m is a compact diagonal operator. Define

$$K_m = TF_m + F_m T - F_m T F_m \qquad (m \geq 1).$$

Since F_m is compact, K_m is compact. Moreover,

$$T - K_m = T - TF_m - F_mT + F_mTF_m = (I - F_m)T(I - F_m).$$

Since $F_m = (I - A_m)(I - B_m)$, we have

$$T - K_m = (I - F_m)T(I - F_m) = (A_m + B_m - A_mB_m)T(A_m + B_m - A_mB_m).$$

Using $A_m = S_1^m S_1^{*m}$ and $B_m = S_2^m S_2^{*m}$, we conclude that

$$\|T - K_m\| \longrightarrow 0 \text{ as } m \to \infty,$$

that is, T is a norm limit of compact operators and is therefore compact. This completes the proof. □

The following result was proved by Abrahamse [1, Corollay 2.12] for multiply connected domains. In the case of the annulus, it follows from Theorems 2.2 and 3.1.

Corollary 3.2. *The only compact Toeplitz operator on the annulus is the zero operator.*

Proof. Let $T \in \mathcal{B}(H^2(\mathbb{A}))$ be a compact Toeplitz operator. Then from Theorem 2.2 (also see (2.7) and (2.5)), there exist $X, Y \in \mathcal{B}(H^2(\mathbb{A}))$ such that

$$M_z^* X M_z = X \text{ and } M_z^* Y M_z = r^2 Y,$$

and

$$X + Y = T.$$

For each $m \geq 1$, we have

$$X + r^{2m}Y = M_z^{*m} X M_z^m + M_z^{*m} Y M_z^m = S_1^{*m} T S_1^m,$$

and hence $\|X + r^{2m}Y\| = \|S_1^{*m} T S_1^m\|$. Since T is compact, Theorem 3.1 implies that $\|X + r^{2m}Y\| \to 0$ as $m \to \infty$. Because $r \in (0, 1)$, it follows that $X = 0$. Thus $T = Y$. Now for each $m \geq 1$,

$$\|Y\| = \|(rM_z^{-1})^* Y r M_z^{-1}\| = \|S_2^* T S_2\| = \|S_2^{*m} T S_2^m\|.$$

Again, by Theorem 3.1 and the compactness of T, $\|Y\| = \|S_2^{*m} T S_2^m\| \to 0$. Hence $Y = 0$, and therefore $T = 0$. □

It may be more than a coincidence that the characterization of compact operators on the annulus in Theorem 3.1 resembles the characterization on the polydisc in [10, Theorem 3.2], even though the two domains are quite different. From this perspective, we make the following two comments.

First, we recall the construction of the operator F_m in Theorem 3.1:

$$F_m = (I - A_m)(I - B_m) \qquad (m \geq 1).$$

Similar operators also appear in the proof of the compactness characterization on the polydisc [10, Theorem 3.2]. The main difference is that, in the polydisc case, these operators are always finite-rank, whereas in the annulus case, the operators F_m need not be finite-rank.

Second, we note that assuming only

$$\|S_i^{*m} T S_i^m\| \longrightarrow 0 \text{ as } m \to \infty \qquad (i = 1, 2),$$

in Theorem 3.1 would not yield the desired characterization of compactness. Indeed, consider the unitary operator $J \in \mathcal{B}(H^2(\mathbb{A}))$ defined by

$$Je_n = e_{-n}, \quad n \in \mathbb{Z}.$$

Clearly, J is not compact. However, it follows from (3.1) and (3.2) that $\|S_1^{*m} J S_1^m\| \leq r^m$ and $\|S_2^{*m} J S_2^m\| \leq r^m$ for each $m \geq 1$. Hence,

$$\|S_i^{*m} T S_i^m\| \longrightarrow 0 \text{ as } m \to \infty \qquad (i = 1, 2),$$

However,

$$\|S_1^{*m} J S_2^m\| = 1,$$

for every $m \geq 1$.

4. Asymptotic Toeplitz operators

Compact perturbations of well-known operators are of broad interest. Here, we study compact perturbations of Toeplitz operators and identify their key properties in the following definition. This class has previously been studied in [4, 8, 10].

Definition 4.1. *An operator $T \in \mathcal{B}(H^2(\mathbb{A}))$ is said to be asymptotic Toeplitz if there exist $J_1, J_2 \in \mathcal{B}(H^2(\mathbb{A}))$ such that*

$$S_1^{*m} T S_1^m \longrightarrow J_1 \text{ and } S_2^{*m} T S_2^m \longrightarrow J_2 \text{ as } m \to \infty,$$

and

$$S_p^{*m}\big(T - J_1 - J_2\big) S_q^m \longrightarrow 0 \text{ as } m \to \infty,$$

in norm for all $p, q = 1, 2$.

We now prove that the asymptotic Toeplitz operators are precisely the compact perturbations of Toeplitz operators.

Theorem 4.2. *An operator $T \in \mathcal{B}(H^2(\mathbb{A}))$ is asymptotically Toeplitz if and only if T is a compact perturbation of a Toeplitz operator.*

Proof. Suppose that T is asymptotic Toeplitz. There exist $J_j \in \mathcal{B}(H^2(\mathbb{A}))$, $j = 1, 2$, such that $S_j^{*m} T S_j^m \longrightarrow J_j$ as $m \to \infty$ and

$$S_p^{*m}\big(T - J_1 - J_2\big) S_q^m \longrightarrow 0 \text{ as } m \to \infty,$$

for all $p, q = 1, 2$. Since $M_z^{*m} T M_z^m = S_1^{*m} T S_1^m \longrightarrow J_1$ as $m \to \infty$ in norm, we have

$$M_z^{*k} J_1 M_z^k = J_1 \qquad (k \geq 1).$$

In particular, taking $k = 1, 2$, we have

$$-M_z^{*2} J_1 M_z^2 + (1 + r^2) M_z^* J_1 M_z = -J_1 + (1 + r^2) J_1 = r^2 J_1.$$

Hence, by Theorem 2.2, J_1 is a Toeplitz operator on $H^2(\mathbb{A})$. Therefore, there exists $\varphi_1 \in L^\infty(\partial\mathbb{A})$ such that $J_1 = T_{\varphi_1}$. Again, by Theorem 2.2, there exist $X_1, Y_1 \in \mathcal{B}(H^2(\mathbb{A}))$ such that $J_1 = X_1 + Y_1$. Moreover, we know that $M_z^* X_1 M_z = X_1$ and $M_z^* Y_1 M_z = r^2 Y_1$ (see also (2.5)). Combined with the fact that $M_z^* J_1 M_z = J_1$, this implies that

$$Y_1 = 0.$$

It follows from the proof of Theorem 2.2, specifically (2) $\implies$ (3) direction, that $\varphi_1 = 0$ on a.e. on $r\mathbb{T}$. Indeed, in the (2) $\implies$ (3) direction, $Y_1 = 0$ implies $A_2 = 0$, and hence the corresponding symbol $\widetilde{\varphi} = 0$ in $L^\infty(\mathbb{T})$. Thus, $\phi_r = 0$ on $r\mathbb{T}$. In the present setting, this implies that $\varphi_1 = 0$ a.e. on $\mathbb{T}$ (here, we also use the fact that a Toeplitz operator on $H^2(\mathbb{D})$ is zero if and only if its symbol is zero almost everywhere). As in the preceding case, the norm convergence $S_2^{*m} T S_2^m \longrightarrow J_2$ as $m \to \infty$ implies that J_2 is also a Toeplitz operator.

Hence, there exist $\varphi_2 \in L^\infty(\partial\mathbb{A})$ such that $J_2 = T_{\varphi_2}$. As before, we conclude that $\varphi_2 = 0$ a.e. on $\mathbb{T}$. Set

$$\varphi := \varphi_1 + \varphi_2.$$

Since T is asymptotic Toeplitz,

$$T - J_1 - J_2 = T - T_{\varphi_1} - T_{\varphi_2} = T - T_\varphi,$$

is compact by Theorem 3.1. Hence, T is a compact perturbation of a Toeplitz operator. Conversely, suppose $T = T_\varphi + K$ for some $\varphi \in L^\infty(\partial\mathbb{A})$ and a compact operator $K \in \mathcal{B}(H^2(\mathbb{A}))$. Define $\varphi_1, \varphi_2 \in L^\infty(\partial\mathbb{A})$ by

$$\varphi_1 = \varphi\chi_{\mathbb{T}} \text{ and } \varphi_2 = \varphi\chi_{r\mathbb{T}},$$

and define

$$J_1 := T_{\varphi_1} \text{ and } J_2 := T_{\varphi_2}.$$

For $f, g \in H^2(\mathbb{A})$, since

$$\begin{aligned}\langle M_z^{*m} T_\varphi M_z^m f, g\rangle &= \langle T_\varphi M_z^m f, M_z^m g\rangle \\ &= \int_{\partial\mathbb{A}} \varphi(z)|z|^{2m} f(z)\overline{g(z)}d\sigma(z) \\ &= \int_{\mathbb{T}} \varphi_1(z) f(z)\overline{g(z)}d\sigma(z) + r^{2m}\int_{r\mathbb{T}} \varphi_2(z) f(z)\overline{g(z)}d\sigma(z),\end{aligned}$$

we conclude that

$$M_z^{*m} T_\varphi M_z^m = J_1 + r^{2m} J_2.$$

Since $0 < r < 1$, it follows that

$$M_z^{*m} T_\varphi M_z^m \longrightarrow J_1 \text{ as } m \to \infty,$$

in operator norm. Again, for $f, g \in H^2(\mathbb{A})$, since

$$\begin{aligned}\langle S_2^{*m} T_\varphi S_2^m f, g\rangle &= \int_{\partial\mathbb{A}} \varphi(z) r^{2m}|z|^{-2m} f(z)\overline{g(z)}\, d\sigma(z) \\ &= r^{2m}\int_{\mathbb{T}} \varphi_1(z) f(z)\overline{g(z)}\, d\sigma(z) + \int_{\mathbb{T}_r} \varphi_2(z) f(z)\overline{g(z)}\, d\sigma(z),\end{aligned}$$

we have

$$S_2^{*m} T_\varphi S_2^m = r^{2m} J_1 + J_2,$$

and hence

$$S_2^{*m} T_\varphi S_2^m \longrightarrow J_2 \text{ as } m \to \infty,$$

in operator norm. The remaining part follows from Theorem 3.1. □

We conclude by noting that our results suggest that the operators

$$S_1 := M_z \qquad \text{and} \qquad S_2 := rM_z^{-1},$$

play a crucial role in understanding Toeplitz operators on $H^2(\mathbb{A})$. Of course, these operators are related: both are defined in terms of the shift operator M_z, they are connected to the shift on the Hardy space of the disc through Lemma 2.1, and they satisfy

$$S_1 S_2 = rI_{H^2(\mathbb{A})}.$$

Our results highlight their significance and suggest that these operators will continue to be useful in the study of operators on the Hardy space of the annulus.

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References

[1] M. B. Abrahamse, Toeplitz operators in multiply connected regions, *Amer. J. Math.* **96** (1974), 261-297.
[2] M. B. Abrahamse and R. G. Douglas, A class of subnormal operators related to multiply-connected domains. *Advances in Math.* **19** (1976), no. 1, 106–148.
[3] J. Agler, Rational dilation on an annulus, *Ann. of Math.* **121** (1985), no. 3, 537–563.
[4] J. Barría and P. R. Halmos, Asymptotic Toeplitz operators, *Trans. Amer. Math. Soc.* **273** (1982), no. 2, 621–630.
[5] A. Böttcher and B. Silbermann, *Analysis of Toeplitz operators*, Springer, Berlin, 1990.
[6] A. Brown and P. R. Halmos, Algebraic properties of Toeplitz operators, *J. Reine Angew. Math.* **213** (1963/64), 89–102.
[7] S. Das and E. K. Narayanan, Zero products of Toeplitz operators on the Hardy and Bergman spaces over an annulus, *New York J. Math.* **31** (2025), 984–1001.
[8] A. Feintuch, On asymptotic Toeplitz and Hankel operators, in *The Gohberg anniversary collection, Vol. II (Calgary, AB, 1988)*, 241–254, Oper. Theory Adv. Appl., 41, Birkhäuser, Basel.
[9] X. Huang, *Toeplitz operators on the Hardy space of the annulus*, Master's thesis, San Francisco State University, 2020. https://scholarworks.calstate.edu/concern/theses/1831cq97c.
[10] A. Maji, J. Sarkar, and S. Sarkar, Toeplitz and asymptotic Toeplitz operators on $H^2(\mathbb{D}^n)$, *Bull. Sci. Math.* **146** (2018), 33-49.
[11] M. Carlsson, On the Beurling-Lax theorem for domains with one hole. *New York J. Math.* **17A** (2011), 193–212.
[12] S. McCullough and J. E. Pascoe, Geometric dilations and operator annuli. *J. Funct. Anal.* **285** (2023), no. 7, Paper No. 110035, 20 pp.
[13] S. McCullough and S. Sultanic, Agler-commutant lifting on an annulus. *Integral Equations Operator Theory* **72** (2012), no. 4, 449–482.
[14] D. E. Sarason, The H^p spaces of an annulus, *Mem. Amer. Math. Soc.* **56** (1965), 78 pp.

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