PAIRED AND TOEPLITZ + HANKEL OPERATORS

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Abstract. We present complete classifications of paired operators on the Hilbert space $L^2(\mathbb{T})$ and Toeplitz + Hankel operators on vector-valued Hardy spaces. We introduce the notion of inner-paired operators defined on the Hardy space that use the classical model spaces. We fully classify inner-paired operators.

Contents

1. Introduction

The contribution of this paper that follows centres on Toeplitz operators, Hankel operators, and paired operators on Hilbert function spaces. Our precise goal is to offer thorough classifications of paired operators on $L^2(\mathbb{T})$ and Toeplitz + Hankel operators on the Hardy space $H^2(\mathbb{T})$, where $\mathbb{T}$ denotes the unit circle in the complex plane. A priori, these two classes of operators have different facades and flavours. They also made their first appearances at different times. However, for the record, the classification problem of paired operators on $L^2(\mathbb{T})$ prompted us to investigate Toeplitz + Hankel operators on $H^2(\mathbb{T})$. Indeed, we will note in the identity (1.3) that paired operators generate, in an appropriate sense, Toeplitz + Hankel operators, and vice versa. With this viewpoint, we begin introducing the concept of paired operators.

The idea of paired operators can be traced back at least to Widom's 1960s work in the context of singular integral equations [28] (also see Shinbrot [23], and [11, 19, 21]). However, the notion of paired operators can be presented in a much wider framework. Let $\mathcal{H}$ be a Hilbert space (in this paper, all Hilbert spaces are separable and over $\mathbb{C}$), and let P and Q be orthogonal projections. Suppose

$$P + Q = I_{\mathcal{H}},$$

that is, P and Q are complementary projections on $\mathcal{H}$. Let A and B be bounded linear operators on $\mathcal{H}$ ($A, B \in \mathcal{B}(\mathcal{H})$ in short). The *paired operator* corresponding to A and B

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and the complementary projections P and Q is defined by

$$S_{A,B} = AP + BQ.$$

Clearly, paired operators are bounded linear operators on $\mathcal{H}$. The order of the projections in the above identity is significant, and in (1.2), we shall explore the reverse order of projections in the context of transposed paired operators.

Our interest in paired operators stems from the $L^2(\mathbb{T})$-setting described in recent papers by Câmara, Guimarães, Partington, Speck [7, 8, 9, 26] and others, which is closely related to the classical context of singular integral operators. We denote by $L^\infty(\mathbb{T})$ the von Neumann algebra of all essentially bounded measurable functions on $\mathbb{T}$, and by $L^2(\mathbb{T})$ the Hilbert space of all Lebesgue square integrable functions on $\mathbb{T}$. The analytic counterpart of $L^2(\mathbb{T})$ is the *Hardy space* denoted by $H^2(\mathbb{T})$. This is a Hilbert space made of $L^2(\mathbb{T})$-functions whose negative Fourier coefficients are zero. Alternatively

$$H^2(\mathbb{T}) = \overline{\mathbb{C}[z]}^{L^2(\mathbb{T})},$$

where $\mathbb{C}[z]$ denotes the ring of polynomials. Throughout, P_+ will denote the orthogonal projection (also known as Szegö projection) of $L^2(\mathbb{T})$ onto the closed subspace $H^2(\mathbb{T})$. Also, we write

$$P_- = I_{L^2(\mathbb{T})} - P_+.$$

Therefore, P_+ and P_- are complementary projections on $L^2(\mathbb{T})$. The *Laurent operator* M_φ with the symbol $\varphi \in L^\infty(\mathbb{T})$ is the bounded linear operator on $L^2(\mathbb{T})$ definable by [17, Chapter III]

$$M_\varphi f = \varphi f \qquad (f \in L^2(\mathbb{T})).$$

The notion of paired operators that interests us can now be explained (cf. [9]).

Definition 1.1. Given $\varphi, \psi \in L^\infty(\mathbb{T})$, the paired operator $S_{\varphi,\psi} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is defined by

$$S_{\varphi,\psi} = M_\varphi P_+ + M_\psi P_-.$$

Theorem 4.1 provides an algebraic classification of paired operators, serving as one of the key results of this paper:

Theorem 1.2. *Let $X \in \mathcal{B}(L^2(\mathbb{T}))$. Then X is a paired operator if and only if*

$$X = M_z^* X M_z P_+ + M_z X M_z^* P_-.$$

Moreover, if there exist $\varphi, \psi \in L^\infty(\mathbb{T})$ such that

$$X = S_{\varphi,\psi},$$

then the representing functions φ and ψ of X are unique.

Now it is fitting to talk about the other two key ideas of this paper, the Toeplitz and the Hankel operators. Note that the Toeplitz and the Hankel operators are, in an appropriate sense, compressions and restrictions of the Laurent operators on $H^2(\mathbb{T})$ and $H^2(\mathbb{T})^\perp$. More specifically, with respect to the orthogonal decomposition $L^2(\mathbb{T}) = H^2(\mathbb{T}) \oplus H^2(\mathbb{T})^\perp$, we write the matrix representation of M_φ as

$$M_\varphi = \begin{bmatrix} T_\varphi & * \\ H_\varphi & * \end{bmatrix}.$$

Then, the *Toeplitz operator* with the symbol φ is identified as

$$T_\varphi = P_+ M_\varphi|_{H^2(\mathbb{T})},$$

and the *Hankel operator* with the symbol φ is identified as

$$H_\varphi = P_- M_\varphi|_{H^2(\mathbb{T})}.$$

One can easily deduce from the above that $T_\varphi : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ and $H_\varphi : H^2(\mathbb{T}) \to H^2(\mathbb{T})^\perp$ are bounded linear operators.

The most basic Toeplitz operator that holds significant importance is T_z, which is associated with the symbol $\varphi = z \in L^\infty(\mathbb{T})$. Therefore

$$T_z = P_+ M_z|_{H^2(\mathbb{T})}.$$

During the mid-1960s, Brown and Halmos [6] established a very useful algebraic characterization of Toeplitz operators, which can be summarized as follows: $A \in \mathcal{B}(H^2(\mathbb{T}))$ is a Toeplitz operator if and only if

$$T_z^* A T_z = A. \tag{1.1}$$

We view this result as an impetus for Theorem 1.2. In the same spirit, we derive algebraic classifications of Toeplitz + Hankel operators. First, we formally introduce these operators.

Although Toeplitz and Hankel operators are distinguishable topics, this paper will examine them together by focusing on bounded linear operators T that admit the form

$$T = \text{Toeplitz operator} + \text{Hankel operator}.$$

In short, we will write these operators as Toeplitz + Hankel. Before delving any further into the topic, let us first identify our entry point for the Toeplitz + Hankel operators through paired operators. We need the "dual" concept of paired operators: Given $\varphi, \psi \in L^\infty(\mathbb{T})$, the *transposed paired operator* $\Sigma_{\varphi,\psi}$ is defined by [8]

$$\Sigma_{\varphi,\psi} = P_+ M_\varphi + P_- M_\psi. \tag{1.2}$$

Paired and transposed paired operators are indeed dual to each other in the following sense:

$$S^*_{\varphi,\psi} = \Sigma_{\bar{\varphi},\bar{\psi}}.$$

Now restricting the transposed paired operator $\Sigma_{\varphi,\psi}$ to $H^2(\mathbb{T})$, we get

$$\Sigma_{\varphi,\psi}|_{H^2(\mathbb{T})} = P_+ M_\varphi\big|_{H^2(\mathbb{T})} + P_- M_\psi\big|_{H^2(\mathbb{T})},$$

that is

$$\Sigma_{\varphi,\psi}|_{H^2(\mathbb{T})} = T_\varphi + H_\psi \qquad (\varphi, \psi \in L^\infty(\mathbb{T})). \tag{1.3}$$

Therefore, $\Sigma_{\varphi,\psi}|_{H^2(\mathbb{T})}$ = Toeplitz + Hankel. These facts suggest a close relationship between paired operators and Toeplitz + Hankel operators. It is important to note that in this scenario, we treat the range spaces of T_φ and H_ψ as subspaces of $L^2(\mathbb{T})$.

Considering the above perspective and the classification of paired operators in Theorem 1.2, it is apparent that the classification of Toeplitz + Hankel operators is inescapable. Our next result gives algebraic classifications of Toeplitz + Hankel operators on vector-valued Hardy spaces. For this, an alternative representation of Hankel operators is essential for us. This time, the representation of Hankel operators in consideration will be in $\mathcal{B}(H^2(\mathbb{T}))$ (for simplicity, we will just cover the scalar case in point here). However, in both cases,

H_φ will stand for the Hankel operator's representation that corresponds to the symbol $\varphi \in L^\infty(\mathbb{T})$. In the $\mathcal{B}(H^2(\mathbb{T}))$ setting, H_φ, $\varphi \in L^\infty(\mathbb{T})$, is defined by

$$H_\varphi = P_+ M_\varphi J|_{H^2(\mathbb{T})},$$

where $J : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is the *flip operator* (or, linear involution) defined as

$$(Jf)(z) = f(\bar{z}) \qquad (f \in L^2(\mathbb{T}), z \in \mathbb{T}).$$

We now recall the important algebraic classification of Hankel operators on $H^2(\mathbb{T})$ [16, 18]: Let $H \in \mathcal{B}(H^2(\mathbb{T}))$. Then H is a Hankel operator if and only if

$$T_z^* H = H T_z. \tag{1.4}$$

Let $\mathcal{E}$ be a Hilbert space. Denote by $H^2_{\mathcal{E}}(\mathbb{T})$ the $\mathcal{E}$-valued Hardy space. Toeplitz and Hankel operators on $H^2_{\mathcal{E}}(\mathbb{T})$ can be defined similarly, as in the scalar case. Furthermore, these operators maintain the same algebraic classifications as in (1.1) and (1.4). The reader is directed to Section 2 for more information on these facts. In Theorem 2.2, we present classifications of Toeplitz + Hankel operators on $H^2_{\mathcal{E}}(\mathbb{T})$:

Theorem 1.3. *Let $A \in \mathcal{B}(H^2_{\mathcal{E}}(\mathbb{T}))$. Then the following are equivalent:*

(1) *$A =$ Toeplitz + Hankel.*
(2) *$AT_z - T_z^* A$ is a Toeplitz operator.*
(3) *$T_z^* A T_z - A$ is a Hankel operator.*
(4) *$AT_z + T_z^{*2} A T_z = T_z^* A + T_z^* A T_z^2$.*

In Theorem 3.1, we further offer an alternative proof of the above in the scalar case that reveals more function-theoretic features of the symbols. One important thing this does is add the Beurling inner functions [4] to the kernels of Hankel operators:

Theorem 1.4. *Let $\varphi, \psi \in L^\infty(\mathbb{T})$, and suppose $A = T_\varphi + H_\psi$. Then we have the following:*

(1) *There exists $\psi_1 \in L^\infty(\mathbb{T})$ such that $T_z^* A T_z - A = H_{\psi_1}$.*
(2) *Either $\ker H_{\psi_1} \neq \{0\}$, or $\ker H_{\psi_1}^* \neq \{0\}$.*

If $\ker H_{\psi_1} \neq \{0\}$, then there exist inner function $\theta \in H^\infty(\mathbb{D})$ and $\varphi_2 \in L^\infty(\mathbb{T})$ such that

$$\ker H_{\psi_1} = \theta H^2(\mathbb{T}),$$

and

$$AT_\theta = T_{\varphi_2},$$

and $\varphi = \overline{\theta}\varphi_2$.
If $\ker H_{\psi_1} = \{0\}$, then there exist inner function $\underline{\theta} \in H^\infty(\mathbb{D})$ and $\underline{\varphi_2} \in L^\infty(\mathbb{T})$ such that

$$\ker H_{\tilde{\psi}_1} = \underline{\theta} H^2(\mathbb{T}),$$

and

$$A^* T_{\underline{\theta}} = T_{\underline{\varphi_2}},$$

and $\varphi = \underline{\theta}\overline{\underline{\varphi_2}}$.

In the above, $H^\infty(\mathbb{D})$ denotes the Banach algebra of all bounded analytic functions on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Recall that a function $\theta \in H^\infty(\mathbb{D})$ is said to be inner if

$$|\theta(z)| = 1, \tag{1.5}$$

for $z \in \mathbb{T}$ a.e. Here the value $\theta(z)$ for z in $\mathbb{T}$ a.e. refers to radial limits (see [15, Chapter 3] for further details). Moreover, $\tilde{\psi}_1$ is defined by

$$\tilde{\psi}_1(z) = \overline{\psi_1(\bar{z})} \qquad (z \in \mathbb{T}).$$

Now we will take a look into the scalar version (that is, $\mathcal{E} = \mathbb{C}$) of Theorem 1.3's equivalence of (1) and (4) through the lens of the inner product. In view of the orthonormal basis $\{z^n\}_{n \in \mathbb{Z}_+}$ of $H^2(\mathbb{T})$, it is easy to see that condition (4) is equivalent to the following identity:

$$\langle Az^m, z^{n-1} \rangle + \langle Az^m, z^{n+1} \rangle = \langle Az^{m-1}, z^n \rangle + \langle Az^{m+1}, z^n \rangle,$$

for all $m, n \geq 1$. This identity has been recently proved by Ehrhardt, Hagger, and Virtanen in [14, Theorem 20]. For finite matrices, this was earlier obtained by Bevilacqua, Bonannie, and Bozzo [5] (also see Strang and MacNamara [27, Section 8]): An $N \times N$ complex matrix $A = (a_{i,j})_{i,j=1}^N$ is Toeplitz + Hankel if and only if

$$a_{i-1,j} + a_{i+1,j} = a_{i,j-1} + a_{i,j+1}$$

for all $1 < i, j < N$.

We note that our proofs of classifications of scalar and vector-valued Toeplitz + Hankel operators restricted to the above particular situations deviate significantly from the respective proofs. We add, however, that our proof of Theorem 1.3 uses a vector-valued version of a lemma from [14] (see Lemma 2.1 for more details).

We mentioned right at the beginning of this section that the concept of paired operators applies to a broader scope. Subsequently, we examine the notion of paired operators from a model space perspective. Fix an inner function $\theta \in H^\infty(\mathbb{D})$. This yields complementary projections $P_{\theta H^2(\mathbb{T})}$ and $P_{\mathcal{K}_\theta}$, where

$$\mathcal{K}_\theta = H^2(\mathbb{T}) \ominus \theta H^2(\mathbb{T}) \cong H^2(\mathbb{T})/\theta H^2(\mathbb{T}),$$

is the *model space* corresponding to the inner function θ [15, 16]. Given a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, we denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{S}$ by $P_{\mathcal{S}}$. The *θ-paired operator* with symbols $\varphi, \psi \in H^\infty(\mathbb{D})$ is defined by

$$S^\theta_{\varphi,\psi} = T_\varphi P_{\theta H^2(\mathbb{T})} + T_\psi P_{\mathcal{K}_\theta}.$$

In Theorem 5.2, we present a complete classification of θ-paired operators:

Theorem 1.5. *$X \in \mathcal{B}(H^2(\mathbb{T}))$ is a θ-paired operator if and only if the following two conditions are satisfied:*

(1) *$\theta H^2(\mathbb{T})$ is invariant under X.*
(2) *There exists a nonzero $h_0 \in \mathcal{K}_\theta$ and a function $\nu \in H^\infty(\mathbb{D})$ such that*

$$X h_0 = \nu h_0,$$

and

$$XT_z - T_z X = (X - T_\nu) P_{\theta H^2(\mathbb{T})} T_z P_{\mathcal{K}_\theta}.$$

Moreover, if $X = S^\theta_{\varphi,\psi}$, then the representing symbols φ and ψ in $H^\infty(\mathbb{D})$ are unique.

Before concluding this introductory section, we would like to provide some overarching comments. While the Toeplitz + Hankel operators are undoubtedly one of the most natural operators that comes to mind, research on their properties has only lately gained momentum. Nevertheless, progress in understanding these operators has been somewhat slow, perhaps due to the potential intricacy even at the individual level of the Toeplitz and the Hankel operators (however, see [1, 5, 12, 13, 25]). It is even challenging to anticipate the right results or the extent to which they will be pleasant and accessible.

We finally remark that, from a perturbation theory standpoint, the classification or general study of the Toeplitz + Hankel operators is also noteworthy. This is because one can view them as Hankel perturbations of Toeplitz operators, and vice versa. The same applies to paired operators. It is also worth noting that there are a considerable number of finite-rank Hankel operators.

The remainder of the paper is composed as follows: The paper's real contribution starts with the Toeplitz + Hankel operators. Section 2 classifies this type of operator in the setting of vector-valued Hardy spaces. Section 3 focuses on the scalar case and addresses the same classification problem. In this particular situation, we present an alternative proof of the classification that brings more function-theoretic flavours to the symbols. Section 4 deals with the classification of paired operators on $L^2(\mathbb{T})$, while Section 5 introduces and then classifies θ-paired operators on $H^2(\mathbb{T})$.

2. Toeplitz + Hankel operators: the general case

This section focuses on the characterizations of Toeplitz + Hankel operators defined on vector-valued Hardy spaces. In Section 3, we will revisit this result for the scalar case with a new proof and provide further information about the symbols of such a class of operators.

We need to go over a few basics about vector-valued Hardy spaces. We refer the reader to [16, Chapter IX] for more details. Given a Hilbert space $\mathcal{E}$, $L^2_{\mathcal{E}}(\mathbb{T})$ is defined as the Lebesgue-Bochner space of all $\mathcal{E}$-valued measurable functions $f : \mathbb{T} \to \mathcal{E}$ such that

$$\|f\| := \Big(\frac{1}{2\pi}\int_0^{2\pi} \|f(e^{i\theta})\|^2 d\theta\Big)^{\frac{1}{2}} < \infty.$$

Similar to the scalar case, the $\mathcal{E}$-valued Hardy space $H^2_{\mathcal{E}}(\mathbb{T})$ is the closed subspace of $L^2_{\mathcal{E}}(\mathbb{T})$ that is made up of functions of $L^2_{\mathcal{E}}(\mathbb{T})$ whose negative Fourier coefficients are equal to zero. Given the radial limits, $H^2_{\mathcal{E}}(\mathbb{T})$ is equivalent to the Hilbert space of analytic functions $f : \mathbb{D} \to \mathcal{E}$, such that

$$\|f\| := \sup_{0<r<1} \Big(\frac{1}{2\pi}\int_0^{2\pi} \|f(re^{i\theta})\|^2 d\theta\Big)^{\frac{1}{2}} < \infty.$$

Moreover, we define the Banach space $L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T})$ as the space of all measurable functions $\Phi : \mathbb{T} \to \mathcal{B}(\mathcal{E})$ such that

$$\|\Phi\| := \operatorname*{ess\,sup}_{\theta\in[0,2\pi]} \|f(e^{i\theta})\| < \infty.$$

For each $\Phi \in L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T})$, the multiplication operator $M_\Phi : L^2_{\mathcal{E}}(\mathbb{T}) \to L^2_{\mathcal{E}}(\mathbb{T})$ (the vector-valued Laurent operator) defined by

$$M_\Phi f = \Phi f \qquad (f \in L^2_{\mathcal{E}}(\mathbb{T})),$$

is a bounded linear operator, and $\|M_\Phi\| = \|\Phi\|$. The *Toeplitz operator* $T_\Phi \in \mathcal{B}(H^2_{\mathcal{E}}(\mathbb{T}))$ with symbol $\Phi \in L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T})$ is defined by

$$T_\Phi = P_{H^2_{\mathcal{E}}(\mathbb{T})} M_\Phi|_{H^2_{\mathcal{E}}(\mathbb{T})}.$$

Similarly, the *Hankel operator* $H_\Phi \in \mathcal{B}(H^2_{\mathcal{E}}(\mathbb{T}))$ with symbol $\Phi \in L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T})$ is defined by

$$H_\Phi = P_{H^2_{\mathcal{E}}(\mathbb{T})} M_\Phi J|_{H^2_{\mathcal{E}}(\mathbb{T})},$$

where

$$(Jf)(z) = f(\bar{z}) \qquad (f \in L^2_{\mathcal{E}}(\mathbb{T}), z \in \mathbb{T}),$$

is the conjugation operator on $L^2_{\mathcal{E}}(\mathbb{T})$.

Let $T, H \in \mathcal{B}(H^2_{\mathcal{E}}(\mathbb{T}))$. Similar to the scalar case, we have the following classifications [16, 22]: T is a Toeplitz operator if and only if

$$T_z^* T T_z = T,$$

and H is a Hankel operator if and only if

$$T_z^* H = H T_z.$$

We need a vector-valued variant of Lemma 12 from [14]. We merely rewrite the proof of [14, Lemma 12] in the present setting.

Lemma 2.1. *Let $\mathcal{E}$ be a Hilbert space, $M > 0$, and let $\Phi : \mathbb{T} \to \mathcal{B}(\mathcal{E})$ be a measurable function. Suppose*

$$\|\Phi(z)(1 - z^{2n})\|_\infty \leq M,$$

for all $n \geq 1$ and $z \in \mathbb{T}$ a.e. Then $\Phi \in L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T})$.

Proof. Consider the set $X = \{e^{2\pi i \alpha} : \alpha \in \mathbb{Q}^c\}$. Observe that the measure of $\mathbb{T} \setminus X$ is zero. For each $z \in X$, a well-known result of Kronecker and Jacobi [20, page 38] ensures that $\{z^{2n} : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}$. For $z \in X$, a.e. it follows that

$$\begin{aligned}
\|\Phi(z)\| &= \left\| \frac{1}{1 - z^{2n}} \Phi(z)(1 - z^{2n}) \right\| \\
&\leq \inf_{n \in \mathbb{Z}} \frac{M}{|1 - z^{2n}|} \\
&= \frac{M}{\sup_{n \in \mathbb{Z}} |1 - z^{2n}|} \\
&= \frac{M}{2},
\end{aligned}$$

as $\sup_{n \in \mathbb{Z}} |1 - z^{2n}| = 2$. Therefore, $\Phi \in L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T})$. □

Now we are in a position to state and prove the Brown-Halmos type algebraic characterization of Toeplitz + Hankel operators defined on vector-valued Hardy spaces.

Theorem 2.2. *Let $\mathcal{E}$ be a Hilbert space, and let $A \in \mathcal{B}(H^2_{\mathcal{E}}(\mathbb{T}))$. The following conditions are equivalent:*

1. *A is a Toeplitz + Hankel operator.*
2. *$AT_z - T_z^* A$ is a Toeplitz operator.*
3. *$T_z^* A T_z - A$ is a Hankel operator.*
4. *$AT_z + T_z^{*2} A T_z = T_z^* A + T_z^* A T_z^2$.*

Proof. First, we prove that (1) implies (4). Suppose there are $\Phi, \Psi \in L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T})$ such that

$$A = T_\Phi + H_\Psi.$$

We know that $T_z^* T_\Phi T_z = T_\Phi$ and $H_\Psi T_z = T_z^* H_\Psi$. We repeatedly exploit these identities and compute:

$$\begin{aligned} AT_z + T_z^{*2} AT_z &= T_\Phi T_z + H_\Psi T_z + T_z^{*2}(T_\Phi + H_\Psi)T_z \\ &= T_\Phi T_z + T_z^* H_\Psi + T_z^*(T_z^* T_\Phi T_z) + T_z^*(T_z^* H_\Psi)T_z \\ &= T_\Phi T_z + T_z^* H_\Psi + T_z^* T_\Phi + T_z^*(T_z^* H_\Psi)T_z \\ &= T_z^* T_\Phi T_z^2 + T_z^*(H_\Psi + T_\Phi) + T_z^* H_\Psi T_z^2 \\ &= T_z^*(T_\Phi + H_\Psi)T_z^2 + T_z^*(H_\Psi + T_\Phi). \end{aligned}$$

We conclude that $AT_z + T_z^{*2} AT_z = T_z^* A + T_z^* AT_z^2$. For the converse, we assume that A satisfies the identity in (4). Then

$$T_z^*(T_z^* A - AT_z)T_z = T_z^* A - AT_z,$$

which implies that $T_z^* A - AT_z$ is a Toeplitz operator. Therefore, there exists $\Phi_1 \in L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T})$ such that

$$T_z^* A - AT_z = T_{\Phi_1}.$$

Multiplying this by T_z and T_z^*, respectively, yields

$$T_z^* AT_z - AT_z^2 = T_{z\Phi_1},$$

and

$$T_z^{*2} A - T_z^* AT_z = T_{\bar{z}\Phi_1}.$$

By adding these two identities, we obtain

$$T_z^{*2} A - AT_z^2 = T_{z\Phi_1} + T_{\bar{z}\Phi_1} = T_{(z+\bar{z})\Phi_1},$$

and hence

$$T_z^{*2} A - AT_z^2 = T_{\frac{z^2 - \bar{z}^2}{z - \bar{z}}\Phi_1}.$$

With this calculation and view, we now proceed by induction. Fix a natural number $N > 1$. Assume that

$$T_z^{*n} A - AT_z^n = T_{\frac{z^n - \bar{z}^n}{z - \bar{z}}\Phi_1},$$

for all $1 \leq n \leq N$. Then

$$T_z^{*N} AT_z - AT_z^{N+1} = T_{\frac{z^N - \bar{z}^N}{z - \bar{z}} z\Phi_1},$$

and

$$T_z^{*N+1} A - T_z^* AT_z^N = T_{\frac{z^N - \bar{z}^N}{z - \bar{z}} \bar{z}\Phi_1}.$$

Adding the above two identities yields

$$T_z^{*N+1} A - AT_z^{N+1} + T_z^*(T_z^{*N-1} A - AT_z^{N-1})T_z = T_{\frac{z^N - \bar{z}^N}{z - \bar{z}}(z+\bar{z})\Phi_1}.$$

By our induction hypothesis

$$T_z^{*N+1} A - AT_z^{N+1} + T_{\frac{z^{N-1} - \bar{z}^{N-1}}{z - \bar{z}}\Phi_1} = T_{\frac{z^N - \bar{z}^N}{z - \bar{z}}(z+\bar{z})\Phi_1},$$

and hence

$$\begin{aligned} T_z^{*N+1}A - AT_z^{N+1} &= T_{[(z^N-\bar{z}^N)(z+\bar{z})-(z^{N-1}-\bar{z}^{N-1})]\frac{\Phi_1}{z-\bar{z}}} \\ &= T_{\frac{z^{N+1}-\bar{z}^{N+1}}{z-\bar{z}}\Phi_1}. \end{aligned}$$

Therefore, by induction, we conclude that

$$T_z^{*n}A - AT_z^n = T_{\frac{(z^n-\bar{z}^n)\Phi_1}{z-\bar{z}}},$$

for all $n \in \mathbb{Z}_+$. By the norm identity of vector-valued Toeplitz operators [22, Section 6.2, page 110], we have

$$\left\| T_{\frac{(z^n-\bar{z}^n)\Phi_1}{z-\bar{z}}} \right\| = \left\| \frac{z^n-\bar{z}^n}{z-\bar{z}}\Phi_1 \right\|_\infty,$$

and consequently

$$\left\| \frac{z^n-\bar{z}^n}{z-\bar{z}}\Phi_1 \right\|_\infty \le 2\|A\|.$$

Moreover, observe that

$$\begin{aligned} \left\| \frac{z^n-\bar{z}^n}{z-\bar{z}}\Phi_1 \right\|_\infty &= \operatorname*{ess\,sup}_{z\in\mathbb{T}} \left\| \frac{z^n-\bar{z}^n}{z-\bar{z}}\Phi_1(z) \right\| \\ &= \operatorname*{ess\,sup}_{z\in\mathbb{T}} \left\| \frac{1-z^{2n}}{z-\bar{z}}\Phi_1(z) \right\|. \end{aligned}$$

Lemma 2.1 yields

$$\Phi := \frac{1}{\bar{z}-z}\Phi_1 \in L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T}).$$

It is clear from the definition of Φ that $T_{\Phi_1} = T_{(\bar{z}-z)\Phi}$, which implies

$$T_{\Phi_1} = T_z^* T_\Phi - T_\Phi T_z.$$

On the other hand, by the definition of Φ_1, we know that $T_{\Phi_1} = T_z^* A - AT_z$, and hence

$$T_z^* T_\Phi - T_\Phi T_z = T_z^* A - AT_z,$$

or equivalently

$$T_z^*(A - T_\Phi) = (A - T_\Phi)T_z.$$

This is the same as saying that $A - T_\Phi$ is a Hankel operator. Therefore, there exists $\Psi \in L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T})$ such that $A = T_\Phi + H_\Psi$. This proves (1) and completes the proof of the equivalence of (1) and (4).

The equivalence between (2) and (4) follows immediately once we rewrite condition (4) as

$$T_z^*(AT_z - T_z^* A)T_z = AT_z - T_z^* A.$$

Indeed, the above identity is equivalent to the statement that $AT_z - T_z^* A$ is a Toeplitz operator. Similarly, condition (4), rewritten as

$$(T_z^* A T_z - A)T_z = T_z^*(T_z^* A T_z - A),$$

leads directly to the equivalence between (3) and (4). $\square$

Now we turn to the uniqueness of the symbols representing Toeplitz + Hankel operators: Let $\mathcal{E}$ be a Hilbert space and let $A \in \mathcal{B}(H^2_{\mathcal{E}}(\mathbb{T}))$. If A is a Toeplitz + Hankel operator, then there exist a unique symbol $\Phi \in L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T})$ and a symbol $\Psi \in L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T})$ unique up to translation by functions from $\overline{zH^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})}$ such that

$$A = T_\Phi + H_\Psi,$$

where $\overline{zH^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})}$ denotes the set of all $\Phi \in L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T})$ of the form

$$\Phi = \sum_{n=1}^{\infty} \Phi_n e^{-in\theta},$$

where $\Phi_n \in \mathcal{B}(\mathcal{E})$ for all $n \geq 1$.
Proof of the claim: Suppose there exist symbols $\Phi, \Phi_1, \Psi, \Psi_1 \in L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T})$ such that

$$A = T_\Phi + H_\Psi = T_{\Phi_1} + H_{\Psi_1}.$$

Since

$$AT_z - T_z^* A = T_\Phi T_z - T_z^* T_\Phi = T_{\Phi_1} T_z - T_z^* T_{\Phi_1},$$

and, in general

$$T_\chi T_z - T_z^* T_\chi = T_{(z-\bar{z})\chi} \qquad (\chi \in L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T})),$$

it follows that $T_{(z-\bar{z})(\Phi-\Phi_1)} = 0$. Then

$$\|T_{(z-\bar{z})(\Phi-\Phi_1)}\| = \|(z-\bar{z})(\Phi-\Phi_1)\|_\infty,$$

implies that

$$(z-\bar{z})(\Phi-\Phi_1) = 0,$$

and hence $\Phi = \Phi_1$ almost everywhere on $\mathbb{T}$. Now we have that $H_\Psi = H_{\Psi_1}$. Then [16, Lemma 3.2, Chapter IX] yields

$$\Psi - \Psi_1 \in \overline{zH^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})}.$$

This completes the proof of the claim.

3. Toeplitz + Hankel operators: the scalar case

This section provides yet another proof of the classifications of the Toeplitz + Hankel operators, but restricted to the scalar case. Although establishing a weaker version of the general classification is something that should worry one right away, there are at least two good reasons to consider the scalar case separately. First, this special case's proof is based more on classical function theory. Second, the function theory-based element of the proof establishes enhanced interactions among the symbols of the Toeplitz + Hankel operators.

The first component of the function-theoretic approach is the inner function-based representations of closed subspaces of the Hardy space. Recall that a function $\theta \in H^\infty(\mathbb{D})$ is said to be *inner* if (see (1.5)) the radial limit function of θ satisfies the following condition:

$$|\theta(z)| = 1,$$

for all $z \in \mathbb{T}$ a.e. Inner functions are of importance in the theory of the Hardy space because, among many other reasons, they parameterize invariant subspaces of the shift operators. To be more specific, a nonzero closed subspace $\mathcal{S} \subseteq H^2(\mathbb{T})$ is invariant under T_z if and only if there is an inner function $\theta \in H^\infty(\mathbb{D})$ that makes

$$\mathcal{S} = \theta H^2(\mathbb{T}).$$

This inner function is unique up to the multiplication of the circle group $\mathbb{T}$. This is one of Beurling's most famous and well-known results [4]. We note that θ inner makes the (analytic) Toeplitz operator T_θ isometric, implying right away that the subspace $\theta H^2(\mathbb{T})$ is closed.

The second tool is a distinctive feature of Hankel operators. Recall that a bounded linear operator H on $H^2(\mathbb{T})$ is Hankel if and only if

$$T_z^* H = H T_z.$$

In particular, if H_φ is a Hankel operator with the symbol $\varphi \in L^\infty(\mathbb{T})$, then the above intertwiner property in particular implies that

$$T_z \ker H_\varphi \subseteq \ker H_\varphi,$$

that is, $\ker H_\varphi$ is invariant under the shift T_z on $H^2(\mathbb{T})$. By the Beurling theorem, if $\ker H_\varphi \neq \{0\}$ then there exists an inner function $\theta \in H^\infty(\mathbb{D})$ such that

$$\ker H_\varphi = \theta H^2(\mathbb{T}). \tag{3.1}$$

The Beurling representation of the kernel of the Hankel operator will play a key role in what follows. Moreover, given the Fourier series representation

$$\varphi(z) = \sum_{n\in\mathbb{Z}} \varphi_n z^n \in L^\infty(\mathbb{T}),$$

we define $\tilde{\varphi} \in L^\infty(\mathbb{T})$ by

$$\tilde{\varphi}(z) = \sum_{n\in\mathbb{Z}} \overline{\varphi}_n z^n.$$

Since $J^* = J$, it follows that $H_\varphi^* = P_{H^2(\mathbb{T})} J M_{\bar{\varphi}}|_{H^2(\mathbb{T})}$. For each $f \in H^2(\mathbb{T})$ and $z \in \mathbb{T}$, we compute

$$\begin{aligned}(JM_{\bar{\varphi}} f)(z) &= (J(\bar{\varphi} f))(z)\\ &= \bar{\varphi}(\bar{z}) f(\bar{z})\\ &= (M_{\tilde{\varphi}} J f)(z).\end{aligned}$$

This yields $JM_{\bar{\varphi}} = M_{\tilde{\varphi}} J$, and consequently

$$H_\varphi^* = H_{\tilde{\varphi}}. \tag{3.2}$$

In the scalar case, Theorem 2.2 now modifies as follows:

Theorem 3.1. *Let $A \in \mathcal{B}(H^2(\mathbb{T}))$. Then A is a Toeplitz + Hankel operator if and only if*

$$AT_z + T_z^{*2} A T_z = T_z^* A + T_z^* A T_z^2.$$

Additionally, if A is a Toeplitz + Hankel operator, then we have the following:

(1) *There exists $\psi_1 \in L^\infty(\mathbb{T})$ such that $T_z^* A T_z - A = H_{\psi_1}$.*
(2) *Either* $\ker H_{\psi_1} \neq \{0\}$, *or* $\ker H_{\psi_1}^* \neq \{0\}$.

If $\ker H_{\psi_1} \neq \{0\}$, *then:*

(a) *there exist inner function* $\theta \in H^\infty(\mathbb{D})$ *and* $\varphi_2 \in L^\infty(\mathbb{T})$ *such that*

$$\ker H_{\psi_1} = \theta H^2(\mathbb{T}),$$

and

$$AT_\theta = T_{\varphi_2}.$$

(b) $A = T_\varphi + H_\psi$, *where* $\varphi = \overline{\theta}\varphi_2$ *and* $\psi \in L^\infty(\mathbb{T})$.

If $\ker H_{\psi_1} = \{0\}$, *then:*

(a) *there exist inner function* $\underline{\theta} \in H^\infty(\mathbb{D})$ *and* $\underline{\varphi_2} \in L^\infty(\mathbb{T})$ *such that*

$$\ker H_{\tilde{\psi}_1} = \underline{\theta} H^2(\mathbb{T}),$$

and

$$A^* T_{\underline{\theta}} = T_{\underline{\varphi_2}},$$

and $A = T_{\underline{\varphi}} + H_{\underline{\psi}}$, *where* $\underline{\varphi} = \underline{\theta}\overline{\underline{\varphi_2}}$ *and* $\underline{\psi} \in L^\infty(\mathbb{T})$.

Proof. We treat the proof of the necessary part as the same as the proof of (1) implies (4) of Theorem 2.2. For the sufficient part, we assume that

$$AT_z + T_z^{*2} AT_z = T_z^* A + T_z^* AT_z^2.$$

By appropriately rearranging the above identity, we get

$$T_z^*(AT_z - T_z^* A)T_z = AT_z - T_z^* A,$$

and

$$(T_z^* AT_z - A)T_z = T_z^*(T_z^* AT_z - A).$$

This says that $AT_z - T_z^* A$ is a Toeplitz operator and $T_z^* AT_z - A$ is a Hankel operator. Therefore, there exist $\varphi_1, \psi_1 \in L^\infty(\mathbb{T})$ such that

$$T_{\varphi_1} = AT_z - T_z^* A,$$

and

$$H_{\psi_1} = T_z^* AT_z - A.$$

At this point, we recall the general fact that Hankel operators are never invertible [18, page 159]. Therefore, in our situation, either $\ker H_{\psi_1} \neq \{0\}$ or $\ker H_{\psi_1}^* \neq \{0\}$. Let us assume that

$$\ker H_{\psi_1} \neq \{0\}.$$

By the discussion preceding the statement of this theorem, or more specifically by (3.1), there exists a nonconstant inner function $\theta \in H^\infty(\mathbb{D})$ such that

$$\ker H_{\psi_1} = \theta H^2(\mathbb{T}).$$

Therefore, for any $f \in H^2(\mathbb{T})$, we know that $H_{\psi_1}(\theta f) = 0$, which is equivalent to saying that

$$H_{\psi_1} T_\theta = 0.$$

Consequently

$$(T_z^* AT_z - A)T_\theta = H_{\psi_1} T_\theta = 0,$$

and hence

$$T_z^* AT_\theta T_z = AT_\theta.$$

This says that AT_θ is a Toeplitz operator. Then there exists $\varphi_2 \in L^\infty(\mathbb{T})$ such that

$$AT_\theta = T_{\varphi_2}.$$

By multiplying the identity $T_{\varphi_1} = AT_z - T_z^* A$ by T_θ from the right-hand side and then using the above equation, we obtain

$$\begin{aligned} T_{\varphi_1\theta} &= (AT_z - T_z^* A)T_\theta \\ &= AT_\theta T_z - T_z^* AT_\theta \\ &= T_{\varphi_2}T_z - T_z^* T_{\varphi_2}. \end{aligned}$$

Making use of the fact that θ is an inner function, we have the representation of T_{φ_1} as

$$T_{\varphi_1} = T_{\bar{\theta}\varphi_2}T_z - T_z^* T_{\bar{\theta}\varphi_2}.$$

Since $T_{\varphi_1} = AT_z - T_z^* A$, it follows that

$$AT_z - T_z^* A = T_{\bar{\theta}\varphi_2}T_z - T_z^* T_{\bar{\theta}\varphi_2},$$

and hence

$$(A - T_{\bar{\theta}\varphi_2})T_z = T_z^*(A - T_{\bar{\theta}\varphi_2}).$$

That is, $(A - T_{\bar{\theta}\varphi_2})$ is a Hankel operator. Therefore, there exists $\psi \in L^\infty(\mathbb{T})$ such that

$$H_\psi = (A - T_{\bar{\theta}\varphi_2}),$$

which implies $A = H_\psi + T_{\bar{\theta}\varphi_2}$. Thus $A = H_\psi + T_\varphi$, where

$$\varphi = \bar{\theta}\varphi_2 \in L^\infty(\mathbb{T}).$$

This settles the case $\ker H_{\psi_1} \neq \{0\}$. Now assume that $\ker H_{\psi_1} = \{0\}$. So

$$\ker H_{\psi_1}^* \neq \{0\}.$$

Then, in view of (3.2), we have

$$\ker H_{\tilde{\psi}_1} \neq \{0\}.$$

The identical outcome can now be concluded, following the same steps as in the proof in case $\ker H_{\psi_1} \neq \{0\}$. This completes the proof of the sufficient part. The remaining existential part of the proof is encompassed within the proof of the sufficient part. ☐

Clearly, the representing symbols φ and ψ of the Toeplitz + Hankel operator in the above theorem are more informative. This is due to function theoretic techniques, which are less effective in the vector-valued case.

4. Paired operators on $L^2(\mathbb{T})$

The goal of this section is to fully classify paired operators acting on $L^2(\mathbb{T})$. Recall that P_+ denotes the orthogonal projection of $L^2(\mathbb{T})$ onto the closed subspace $H^2(\mathbb{T}) \subseteq L^2(\mathbb{T})$, and $P_- = I_{L^2(\mathbb{T})} - P_+$. Therefore

$$P_- = P_{H^2(\mathbb{T})^\perp}.$$

Now we observe a general fact about projections P_+ and P_-. Since $M_z(H^2(\mathbb{T})) \subseteq H^2(\mathbb{T})$, it follows that

$$P_- M_z P_+ = 0. \tag{4.1}$$

We will use this identity in what follows. We also need to recollect a standard fact about certain bounded linear operators on $L^2(\mathbb{T})$ [15, Lemma 8.12]: Let $\varphi \in L^2(\mathbb{T})$. If there exists $M > 0$ such that

$$\|p\varphi\| \leq M\|p\|,$$

for all trigonometric polynomials p, then $\varphi \in L^\infty(\mathbb{T})$, and

$$\|M_\varphi\| = \|\varphi\|_\infty \leq M. \tag{4.2}$$

Now we are ready for the classification of paired operators on $L^2(\mathbb{T})$.

Theorem 4.1. *Let $X \in \mathcal{B}(L^2(\mathbb{T}))$. Then X is a paired operator if and only if*

$$X = M_z^* X M_z P_+ + M_z X M_z^* P_-.$$

Moreover, if there exist $\varphi, \psi \in L^\infty(\mathbb{T})$ such that

$$X = S_{\varphi,\psi},$$

then the symbols φ and ψ in the representation of X are unique.

Proof. Assume first that $X = S_{\varphi,\psi}$ for some $\varphi, \psi \in L^\infty(\mathbb{T})$. In view of (4.1), we have

$$\begin{aligned} M_z^* S_{\varphi,\psi} M_z P_+ &= M_{\bar{z}}(M_\varphi P_+ + M_\psi P_-) M_z P_+ \\ &= M_{\bar{z}} M_\varphi P_+ M_z P_+ \\ &= M_\varphi P_+. \end{aligned}$$

Also, by (4.1), we have $P_+ M_{\bar{z}} P_- = 0$, and hence

$$\begin{aligned} M_z S_{\varphi,\psi} M_z^* P_- &= M_z(M_\varphi P_+ + M_\psi P_-) M_{\bar{z}} P_- \\ &= M_z M_\psi P_- M_{\bar{z}} P_- \\ &= M_\psi P_-. \end{aligned}$$

It is now evident that

$$\begin{aligned} S_{\varphi,\psi} &= M_\varphi P_+ + M_\psi P_- \\ &= M_z^* S_{\varphi,\psi} M_z P_+ + M_z S_{\varphi,\psi} M_z^* P_-, \end{aligned}$$

which proves the necessary part of the theorem. For the reverse direction, assume that

$$X = M_z^* X M_z P_+ + M_z X M_z^* P_-.$$

It follows that $XP_+ = M_z^* X M_z P_+$, and hence

$$X M_z P_+ = M_z X P_+.$$

By using this repeatedly, we obtain, for each $n \geq 1$, that

$$\begin{aligned} X(z^n) &= X P_+ M_z^n(1) \\ &= X M_z P_+ M_z^{n-1}(1) \\ &= M_z X P_+ M_z^{n-1}(1), \end{aligned}$$

and hence $X(z^n) = M_z^n X(1)$. Set

$$\varphi := X(1).$$

Now, since $Xz^n = z^n \varphi$ for all $n \geq 0$, we have

$$Xp = p\varphi \qquad (p \in \mathbb{C}[z]).$$

Let us now note that for a given trigonometric polynomial $p \in L^2(\mathbb{T})$, there exists $m \in \mathbb{Z}_+$ such that

$$z^m p \in \mathbb{C}[z] \subseteq H^2(\mathbb{T}).$$

Moreover, observe, in general, that

$$\|z^m g\|_2 = \|g\|_2,$$

for all $g \in L^2(\mathbb{T})$. Therefore, for a trigonometric polynomial $p \in L^2(\mathbb{T})$, with the above setting, we compute

$$\begin{aligned}\|\varphi p\|_2 &= \|\varphi z^m p\|_2\\ &= \|X(z^m p)\|_2\\ &\leq \|X\|\|z^m p\|_2\\ &= \|X\|\|p\|_2.\end{aligned}$$

By the fact recorded in (4.2), we immediately conclude that $\varphi \in L^\infty(\mathbb{T})$. Fix $f \in H^2(\mathbb{T})$. There exists a sequence of polynomials $\{p_n\}_{n=1}^\infty \subseteq H^2(\mathbb{T})$ such that

$$p_n \longrightarrow f,$$

in $L^2(\mathbb{T})$. Therefore

$$\begin{aligned}\|Xf - \varphi f\|_2 &\leq \|Xf - Xp_n\|_2 + \|Xp_n - \varphi p_n\|_2 + \|\varphi p_n - \varphi f\|_2\\ &\leq \|X\|\|f - p_n\|_2 + \|\varphi p_n - \varphi p_n\|_2 + \|\varphi\|_\infty \|p_n - f\|_2\\ &= (\|X\| + \|\varphi\|_\infty)\|p_n - f\|_2\\ &\longrightarrow 0.\end{aligned}$$

This implies

$$Xf = \varphi f \qquad (f \in H^2(\mathbb{T})),$$

or, in other words

$$XP_+ = M_\varphi P_+. \tag{4.3}$$

Once again, we observe that $X = M_z^* X M_z P_+ + M_z X M_z^* P_-$ implies that

$$XP_- = M_z X M_z^* P_-,$$

from which it immediately follows that

$$XM_z^* P_- = M_z^* X P_-.$$

For each $n \geq 1$, we use the same procedure as before to calculate

$$\begin{aligned}X(\overline{z}^n) &= XM_z^{*n-1}P_-(\overline{z})\\ &= M_z^{*n-1}XP_-(\overline{z})\\ &= M_z^{*n-1}X\overline{z}\\ &= M_z^{*n}(zX\overline{z}).\end{aligned}$$

By setting

$$\psi := zX(\overline{z}),$$

we have $X(\overline{z}^n) = \psi \overline{z}^n$ for all $n \geq 1$. Clearly, for any polynomial $p \in (H^2(\mathbb{T}))^\perp$, we have

$$X(p) = \psi p.$$

Again, for a fixed polynomial $p \in L^2(\mathbb{T})$, there exists $m \in \mathbb{Z}_+$ such that

$$\overline{z}^m p \in (H^2(\mathbb{T}))^\perp.$$

Since $\|\bar{z}^m g\|_2 = \|g\|_2$ for all $g \in L^2(\mathbb{T})$, it follows that

$$\begin{aligned}\|\psi p\|_2 &= \|\psi(\bar{z}^m p)\|_2 \\ &= \|X(\bar{z}^m p)\|_2 \\ &\leq \|X\| \|\bar{z}^m p\|_2 \\ &= \|X\| \|p\|_2.\end{aligned}$$

Therefore, (4.2) guarantees that $\psi \in L^\infty(\mathbb{T})$. Furthermore, it is easy to see that for any $f \in (H^2(\mathbb{T}))^\perp = \overline{zH^2(\mathbb{T})}$, there exists a sequence of polynomials $\{p_n\}_{n=1}^\infty \subseteq (H^2(\mathbb{T}))^\perp$ such that

$$p_n \longrightarrow f,$$

in $L^2(\mathbb{T})$. From here onwards, we can follow exactly similar lines of arguments as in the proof of (4.3) to conclude that

$$Xf = \psi f \qquad (f \in (H^2(\mathbb{T}))^\perp),$$

that is, $XP_- = M_\psi P_-$. Combining this and (4.3), we have

$$\begin{aligned}X &= X(P_+ + P_-) \\ &= M_\varphi P_+ + M_\psi P_-,\end{aligned}$$

which proves the sufficient part of the theorem. For the uniqueness part, suppose $X \in \mathcal{B}(L^2(\mathbb{T}))$ is a paired operator, and suppose

$$X = S_{\varphi,\psi} = S_{\varphi_1,\psi_1},$$

for some $\varphi, \psi, \varphi_1, \psi_1 \in L^\infty(\mathbb{T})$. Then $X1 = S_{\varphi,\psi}(1) = S_{\varphi_1,\psi_1}(1)$ yields

$$M_\varphi 1 = M_{\varphi_1} 1,$$

that is, $\varphi = \varphi_1$. Similarly, $X(\bar{z}) = S_{\varphi,\psi}(\bar{z}) = S_{\varphi_1,\psi_1}(\bar{z})$ implies $\psi = \psi_1$. □

Recall that given $\varphi, \psi \in L^\infty(\mathbb{T})$, the *transposed paired operator* $\Sigma_{\varphi,\psi} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is defined by (see Section 1)

$$\Sigma_{\varphi,\psi} = P_+ M_\varphi + P_- M_\psi.$$

$S_{\varphi,\psi}$ and $\Sigma_{\varphi,\psi}$ are called *dual* to each other due to the relation (cf. [8, Proposition 2.1])

$$S^*_{\varphi,\psi} = \Sigma_{\bar{\varphi},\bar{\psi}}.$$

As an immediate consequence of Theorem 4.1, we have the following: Let $X \in \mathcal{B}(L^2(\mathbb{T}))$. Then X is a transposed paired operator if and only if

$$X = P_+ M_z^* X M_z + P_- M_z X M_z^*.$$

Moreover, if $X = \Sigma_{\varphi,\psi}$ for $\varphi, \psi \in L^\infty(\mathbb{T})$, then φ and ψ are unique representing functions.

5. Paired operators on $H^2(\mathbb{T})$

The purpose of this section is to examine the concept of paired operators on $H^2(\mathbb{T})$, or more specifically, paired operators corresponding to model spaces. Model spaces of $H^2(\mathbb{T})$ are essentially T_z^*-invariant closed subspaces of $H^2(\mathbb{T})$. We start our discussion by revisiting the Beurling theorem from Section 3: A nonzero closed subspace $\mathcal{S} \subseteq H^2(\mathbb{T})$ is T_z-invariant if and only if there is an inner function $\theta \in H^\infty(\mathbb{D})$ (unique up to the multiplication of the group $\mathbb{T}$) such that

$$\mathcal{S} = \theta H^2(\mathbb{T}).$$

The subsequent orthocomplement is then a closed subspace that is invariant under T_z^*. This space is also very significant for general research in Hilbert function spaces. We commonly refer to the space in question as the *model space* associated with the inner function $\theta \in H^\infty(\mathbb{D})$ and denote it by $\mathcal{K}_\theta$. Therefore

$$\mathcal{K}_\theta = H^2(\mathbb{T}) \ominus \theta H^2(\mathbb{T}) \cong H^2(\mathbb{T})/\theta H^2(\mathbb{T}).$$

Consequently, given an inner function $\theta \in H^\infty(\mathbb{D})$, we have the splitting of the space $H^2(\mathbb{T})$ as follows:

$$H^2(\mathbb{T}) = \theta H^2(\mathbb{T}) \oplus \mathcal{K}_\theta,$$

and subsequently, we can define a new notion (and many more, just as similar to this) of paired operator. Throughout the rest of this section, $\theta \in H^\infty(\mathbb{D})$ will be an arbitrary nonconstant inner function.

Definition 5.1. Let $\theta \in H^\infty(\mathbb{D})$ be an inner function. The θ-paired operator with (ordered) symbols $\varphi, \psi \in H^\infty(\mathbb{D})$ is defined by

$$S^\theta_{\varphi,\psi} = T_\varphi P_{\theta H^2(\mathbb{T})} + T_\psi P_{\mathcal{K}_\theta}.$$

Clearly, $S^\theta_{\varphi,\psi} : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ is a bounded linear operator. An operator $X \in \mathcal{B}(H^2(\mathbb{T}))$ is said to be θ-paired, if there exist φ and ψ in $H^\infty(\mathbb{D})$ such that

$$X = S^\theta_{\varphi,\psi}.$$

Our goal is to characterize θ-paired operators. Before we do that, let us go over a common and well-known fact about model spaces. Note that

$$f \in H^2(\mathbb{T}) \cap \theta \overline{zH^2(\mathbb{T})},$$

if and only if $f \in H^2(\mathbb{T})$ and

$$f = \theta \bar{z} \bar{g},$$

for some $g \in H^2(\mathbb{T})$, which is further equivalent to

$$M_\theta^* f = \bar{z}\bar{g} \in \overline{zH^2(\mathbb{T})},$$

for some $g \in H^2(\mathbb{T})$. Therefore, we conclude that

$$\mathcal{K}_\theta = H^2(\mathbb{T}) \cap \theta \overline{zH^2(\mathbb{T})}.$$

Finally, one piece of notation. For a pair of operators $A, B \in \mathcal{B}(\mathcal{H})$, the commutator $[A, B]$ is defined by

$$[A, B] = AB - BA.$$

Now we are ready for the characterization of θ-paired operators.

Theorem 5.2. *$X \in \mathcal{B}(H^2(\mathbb{T}))$ is a θ-paired operator if and only if the following two conditions are satisfied:*

1. *$\theta H^2(\mathbb{T})$ is invariant under X.*
2. *There exists a nonzero $h_0 \in \mathcal{K}_\theta$ and $\nu \in H^\infty(\mathbb{D})$ such that*
$$Xh_0 = \nu h_0,$$
and
$$[X, T_z] = (X - T_\nu) P_{\theta H^2(\mathbb{T})} T_z P_{\mathcal{K}_\theta}.$$

Moreover, if $X = S^\theta_{\varphi,\psi}$ is a θ-paired operator, then the representing symbols $\varphi, \psi \in H^\infty(\mathbb{D})$ are unique for fixed θ.

Proof. Suppose $X = S^\theta_{\varphi,\psi}$ is a θ-paired operator. For $h \in H^2(\mathbb{T})$, we have
$$S^\theta_{\varphi,\psi}(\theta h) = \varphi\theta h \in \theta H^2(\mathbb{T}),$$
implying the invariance condition in (1). To verify (2), fix $f \in H^2(\mathbb{T})$, and write
$$f = \theta h_1 + \theta\bar{z}\overline{h_2},$$
where $h_1, h_2 \in H^2(\mathbb{T})$ and $\theta\overline{zh_2} \in H^2(\mathbb{T}) \cap \theta\overline{zH^2(\mathbb{T})}$. Now, we compute
$$\begin{aligned}
S^\theta_{\varphi,\psi} T_z f &= (T_\varphi P_{\theta H^2(\mathbb{T})} + T_\psi P_{\mathcal{K}_\theta})(z\theta h_1 + z\theta\bar{z}\overline{h_2})\\
&= z\varphi\theta h_1 + T_\varphi P_{\theta H^2(\mathbb{T})}(z\theta\bar{z}\overline{h_2}) + T_\psi P_{\mathcal{K}_\theta}(z\theta\bar{z}\overline{h_2})\\
&= z\varphi\theta h_1 + T_\varphi P_{\theta H^2(\mathbb{T})}(\theta\overline{h_2}) + T_\psi P_{\mathcal{K}_\theta}(\theta\overline{h_2}),
\end{aligned}$$
and, on the other hand, we also have
$$\begin{aligned}
T_z S^\theta_{\varphi,\psi} f &= T_z(T_\varphi P_{\theta H^2(\mathbb{T})} + T_\psi P_{\mathcal{K}_\theta})(\theta h_1 + \theta\bar{z}\overline{h_2})\\
&= T_z(T_\varphi\theta h_1 + T_\psi\theta\bar{z}\overline{h_2})\\
&= z\varphi\theta h_1 + \psi\theta\overline{h_2}.
\end{aligned}$$
As a result, we have
$$\begin{aligned}
[X, T_z](f) &= S^\theta_{\varphi,\psi} T_z f - T_z S^\theta_{\varphi,\psi} f\\
&= \varphi P_{\theta H^2(\mathbb{T})}(\theta\overline{h_2}) - \psi(I - P_{\mathcal{K}_\theta})(\theta\overline{h_2})\\
&= \varphi P_{\theta H^2(\mathbb{T})}(\theta\overline{h_2}) - \psi P_{\theta H^2(\mathbb{T})}(\theta\overline{h_2})\\
&= X P_{\theta H^2(\mathbb{T})}(\theta\overline{h_2}) - \psi P_{\theta H^2(\mathbb{T})}(\theta\overline{h_2}).
\end{aligned}$$
But
$$P_{\theta H^2(\mathbb{T})} T_z P_{\mathcal{K}_\theta} f = P_{\theta H^2(\mathbb{T})}(\theta\overline{h_2}),$$
and hence
$$[X, T_z](f) = (X - T_\psi) P_{\theta H^2(\mathbb{T})} T_z P_{\mathcal{K}_\theta}(f).$$
Define $\nu = \psi \in H^\infty(\mathbb{D})$. Clearly, for all $h_0 \in \mathcal{K}_\theta$, we have $\nu h_0 = \psi h_0$. This proves the necessary part of the theorem. For the converse direction, assume that $X \in \mathcal{B}(H^2(\mathbb{T}))$ satisfies the conditions (1) and (2). It is readily seen from condition (2) that $[X, T_z] P_{\theta H^2(\mathbb{T})} = 0$, that is
$$X T_z P_{\theta H^2(\mathbb{T})} = T_z X P_{\theta H^2(\mathbb{T})}.$$

Since $P_{\theta H^2(\mathbb{T})} = T_\theta T_{\overline{\theta}}$ (cf. [15, Theorem 14.11]), it follows that

$$XT_zT_\theta T_{\overline{\theta}} = T_zXT_\theta T_{\overline{\theta}}.$$

Multiplying both sides of the equality by T_θ from the right-hand side, we get

$$XT_\theta T_z = T_zXT_\theta,$$

and therefore there exists $\chi \in H^\infty(\mathbb{D})$ such that

$$XT_\theta = T_\chi.$$

At the same time, the fact that X leaves $\theta H^2(\mathbb{T})$ invariant ensures existence of $\varphi \in H^2(\mathbb{T})$ such that

$$\chi = X\theta = \theta\varphi.$$

This asserts that $\varphi \in L^\infty(\mathbb{T}) \cap H^2(\mathbb{T})$, that is, $\varphi \in H^\infty(\mathbb{D})$. As a result, we have

$$\begin{aligned} XT_\theta T_{\overline{\theta}} &= T_\chi T_{\overline{\theta}} \\ &= T_{\varphi\theta} T_{\overline{\theta}} \\ &= T_\varphi T_\theta T_{\overline{\theta}}. \end{aligned}$$

In other words

$$XP_{\theta H^2(\mathbb{T})} = T_\varphi P_{\theta H^2(\mathbb{T})}. \tag{5.1}$$

Define $A \in \mathcal{B}(H^2(\mathbb{T}))$ by

$$A = XP_{\mathcal{K}_\theta} + T_\nu P_{\theta H^2(\mathbb{T})},$$

where ν is as defined in the condition (2). Also, the second part of condition (2) implies

$$(XT_z - T_zX)P_{\mathcal{K}_\theta} = (X - T_\nu)P_{\theta H^2(\mathbb{T})}T_zP_{\mathcal{K}_\theta}.$$

In particular, we have

$$T_zXP_{\mathcal{K}_\theta} = XT_zP_{\mathcal{K}_\theta} - (X - T_\nu)P_{\theta H^2(\mathbb{T})}T_zP_{\mathcal{K}_\theta}.$$

We compute

$$\begin{aligned} T_zA &= T_z(XP_{\mathcal{K}_\theta} + T_\nu P_{\theta H^2(\mathbb{T})}) \\ &= T_zXP_{\mathcal{K}_\theta} + T_zT_\nu P_{\theta H^2(\mathbb{T})} \\ &= XT_zP_{\mathcal{K}_\theta} - XP_{\theta H^2(\mathbb{T})}T_zP_{\mathcal{K}_\theta} + T_\nu P_{\theta H^2(\mathbb{T})}T_zP_{\mathcal{K}_\theta} + T_\nu T_zP_{\theta H^2(\mathbb{T})} \\ &= X(I - P_{\theta H^2(\mathbb{T})})T_zP_{\mathcal{K}_\theta} + T_\nu P_{\theta H^2(\mathbb{T})}T_zP_{\mathcal{K}_\theta} + T_\nu T_zP_{\theta H^2(\mathbb{T})} \\ &= XP_{\mathcal{K}_\theta}T_zP_{\mathcal{K}_\theta} + T_\nu P_{\theta H^2(\mathbb{T})}T_zP_{\mathcal{K}_\theta} + T_\nu T_zP_{\theta H^2(\mathbb{T})} \\ &= XP_{\mathcal{K}_\theta}T_zP_{\mathcal{K}_\theta} + T_\nu P_{\theta H^2(\mathbb{T})}T_zP_{\mathcal{K}_\theta} + T_\nu P_{\theta H^2(\mathbb{T})}T_zP_{\theta H^2(\mathbb{T})} \\ &= XP_{\mathcal{K}_\theta}T_zP_{\mathcal{K}_\theta} + T_\nu P_{\theta H^2(\mathbb{T})}T_zP_{\mathcal{K}_\theta} + (XP_{\mathcal{K}_\theta} + T_\nu P_{\theta H^2(\mathbb{T})})T_zP_{\theta H^2(\mathbb{T})} \\ &= AT_zP_{\mathcal{K}_\theta} + AT_zP_{\theta H^2(\mathbb{T})} \\ &= AT_z, \end{aligned}$$

and consequently, there exists $\psi \in H^\infty(\mathbb{D})$ such that $A = T_\psi$. At the same time, we know that there exists a nonzero $h_0 \in \mathcal{K}_\theta$ such that $X(h_0) = \nu h_0$. Therefore

$$\nu h_0 = X(h_0) = A(h_0) = T_\psi(h_0) = \psi h_0,$$

which means $\psi = \nu$, and hence

$$XP_{\mathcal{K}_\theta} = T_\nu P_{\mathcal{K}_\theta} = T_\psi P_{\mathcal{K}_\theta}.$$

Combining with (5.1), this implies

$$X = XP_{\theta H^2(\mathbb{T})} + XP_{\mathcal{K}_\theta} = T_\varphi P_{\theta H^2(\mathbb{T})} + T_\psi P_{\mathcal{K}_\theta}.$$

This settles the characterization part of the theorem. For the uniqueness part, assume that

$$X = S^\theta_{\varphi,\psi} = S^\theta_{\varphi_1,\psi_1},$$

$\varphi, \psi, \varphi_1, \psi_1 \in H^\infty(\mathbb{D})$. Then

$$X\theta = S^\theta_{\varphi,\psi}\theta = S^\theta_{\varphi_1,\psi_1}\theta,$$

implies that

$$\varphi\theta = \varphi_1\theta,$$

that is, $\varphi = \varphi_1$. Similarly, for a nonzero $h_0 \in \mathcal{K}_\theta$, we have

$$Xh_0 = S^\theta_{\varphi,\psi}h_0 = S^\theta_{\varphi_1,\psi_1}h_0,$$

which yields $\psi = \psi_1$. Our proof is therefore complete. □

Remember that the Toeplitz and the Hankel operators have a close association with the paired operators (see the identity in (1.3)). In particular, given a paired operator $S_{\varphi,\psi} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$, one has the Toeplitz operator

$$T_\varphi = P_{H^2(\mathbb{T})} S_{\varphi,\psi}|_{H^2(\mathbb{T})}.$$

Similar to this, θ-paired operators are associated with another important class of operators, namely truncated Toeplitz operators. These are the compressions of Toeplitz operators on model spaces. More specifically, given an inner function $\theta \in H^\infty(\mathbb{D})$, the *truncated Toeplitz operator* with symbol $\varphi \in L^\infty(\mathbb{T})$ is defined by

$$A^\theta_\varphi f = P_{\mathcal{K}_\theta}(\varphi f) \qquad (f \in \mathcal{K}_\theta).$$

Therefore, for a θ-paired operator $S^\theta_{\varphi,\psi} : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ with $\varphi, \psi \in H^\infty(\mathbb{D})$, we have

$$A^\theta_\psi = P_{\mathcal{K}_\theta} S^\theta_{\varphi,\psi}|_{\mathcal{K}_\theta}.$$

Since its inception by Sarason in [24], numerous researchers have used this framework to formulate theories of diverse nature. At times, results in this direction are comparable to the classical theory of Toeplitz operators. For more details and further development, we refer the reader to [2, 3, 10] and the references therein.

In closing, we mention that our primary reason for studying θ-paired operators is to see how they match up with some other ideas about paired operators (say, paired operators on $L^2(\mathbb{T})$). The classifications of model space-based θ-paired operators on $H^2(\mathbb{T})$ derived in this work vividly suggest that the general theory of paired operators will shift from case to case. We finally remark that the potential impact of θ-paired operators on the theory of truncated Toeplitz operators, or vice versa, is the subject of future investigation.

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