STAR-GENERATING VECTORS OF RUDIN'S QUOTIENT MODULES

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ABSTRACT. The purpose of this paper is to study a class of quotient modules of the Hardy module $H^2(\mathbb{D}^n)$. Along with the two variables quotient modules introduced by W. Rudin, we introduce and study a large class of quotient modules, namely Rudin's quotient modules of $H^2(\mathbb{D}^n)$. By exploiting the structure of minimal representations we obtain an explicit co-rank formula for Rudin's quotient modules.

NOTATION

$\mathbb{N}$	Set of all natural numbers including 0.
n	Natural number $n \geq 2$, unless specifically stated otherwise.
$\mathbb{N}^n$	$\{\boldsymbol{k} = (k_1, \ldots, k_n) : k_i \in \mathbb{N}, i = 1, \ldots, n\}$.
$\mathbb{C}^n$	Complex n-space.
$\boldsymbol{z}$	$(z_1, \ldots, z_n) \in \mathbb{C}^n$.
$\boldsymbol{z}^{\boldsymbol{k}}$	$z_1^{k_1} \cdots z_n^{k_n}$.
T	n-tuple of commuting operators $(T_1, \ldots, T_n)$.
$T^{\boldsymbol{k}}$	$T_1^{k_1} \cdots T_n^{k_n}$.
$\mathbb{C}[\boldsymbol{z}]$	$\mathbb{C}[z_1, \ldots, z_n]$, the polynomial ring over $\mathbb{C}$ in n-commuting variables.
$\mathbb{D}^n$	Open unit polydisc $\{\boldsymbol{z} : |z_i| < 1\}$.

Throughout this note all Hilbert spaces are over the complex field and separable. Also for a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, we denote by $P_{\mathcal{S}}$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{S}$.

1. INTRODUCTION

This paper is concerned with the question of generating vectors for a commuting tuple of bounded linear operators on a separable Hilbert space. Let $T := (T_1, \ldots, T_n)$ be an n-tuple ($n \geq 1$) of commuting bounded linear operators on a separable Hilbert space $\mathcal{H}$, and let S be a non-empty subset of $\mathcal{H}$. The T-generating hull of S is defined by

$$[S]_T := \bigvee \{p(T_1, \ldots, T_n)h : p \in \mathbb{C}[\boldsymbol{z}], h \in S\},$$

where $\mathbb{C}[\boldsymbol{z}] := \mathbb{C}[z_1, \ldots, z_n]$ is the ring of polynomials of n-commuting variables. It is easy to verify that $[S]_T$ is the smallest closed subspace of $\mathcal{H}$ such that $S \subseteq [S]_T$ and $T_i([S]_T) \subseteq [S]_T$

for all $i = 1, \ldots, n$. That is,

$$[S]_T = \bigcap \{ \mathcal{S} : \mathcal{S} \text{ closed}, S \subseteq \mathcal{S}, T_i \mathcal{S} \subseteq \mathcal{S}, \ i = 1, \ldots, n \}.$$

The *rank* of the tuple T is the unique number rank T defined by

$$\text{rank } T = \inf\{ \#S : S \subseteq \mathcal{H}, \ [S]_T = \mathcal{H} \} \in \mathbb{N} \cup \{\infty\}.$$

We say that S is a T-generating subset if $[S]_T = \mathcal{H}$. In this case, we also say that the elements in S are T-*generating vectors* of $\mathcal{H}$.

One of the most intriguing and important open problems in operator theory, closely related to the invariant subspace problem, is the existence of nontrivial generating set for a tuple of operators. Also one may ask when the rank of T is finite.

This problem is known to be hard to solve in general. Nevertheless, one has a better chance in the special case of the restriction (compression) of tuple of multiplication operators, $M_z = (M_{z_1}, \ldots, M_{z_n})$, to a joint M_z-invariant (co-invariant) subspace of an analytic reproducing kernel Hilbert space $\mathcal{H}$ over a domain in $\mathbb{C}^n$. The reason behind this hope is that underlying analytic structure of the kernel function could provide useful information regarding the restriction (compression) of M_z to a joint M_z-invariant (co-invariant) subspace of $\mathcal{H}$.

In this paper, we confine our attention to a class of quotient Hilbert modules, namely "Rudin's quotient modules", after W. Rudin ([11]), of the Hardy module $H^2(\mathbb{D}^n)$ over the unit polydisc $\mathbb{D}^n$. We focus here mainly on the study of $(M_{z_1}^*|_{\mathcal{Q}}, \ldots, M_{z_n}^*|_{\mathcal{Q}})$-generating sets of a Rudin's quotient module $\mathcal{Q}$ of $H^2(\mathbb{D}^n)$. To be more precise we must introduce some notations first.

Let $n \geq 1$ be a natural number and $\mathbb{D}^n$ be the open unit polydisc in $\mathbb{C}^n$. The *Hardy module* $H^2(\mathbb{D}^n)$ over $\mathbb{D}^n$ is the Hilbert space of all holomorphic functions f on $\mathbb{D}^n$ such that

$$\|f\|_{H^2(\mathbb{D}^n)} := \left(\sup_{0 \leq r < 1} \int_{\mathbb{T}^n} |f(r\boldsymbol{z})|^2 \, d\boldsymbol{z} \right)^{\frac{1}{2}} < \infty,$$

where $d\boldsymbol{z}$ is the normalized Lebesgue measure on the torus $\mathbb{T}^n$, the distinguished boundary of $\mathbb{D}^n$, and $r\boldsymbol{z} := (rz_1, \ldots, rz_n)$ (cf. [11]). For each $i = 1, \ldots, n$, define the multiplication operator by the coordinate function z_i as

$$(M_{z_i} f)(\boldsymbol{w}) = w_i f(\boldsymbol{w}). \qquad (\boldsymbol{w} \in \mathbb{D}^n)$$

We will often identify $H^2(\mathbb{D}^n)$ with the n-fold Hilbert space tensor product $H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})$. In this identification, the multiplication operator M_{z_i} can be realized as $I_{H^2(\mathbb{D})} \otimes \cdots \otimes \underbrace{M_z}_{\text{i-th place}} \otimes \cdots \otimes I_{H^2(\mathbb{D})}$ for all $i = 1, \ldots, n$.

One can easily verify that

$$M_{z_i} M_{z_j} = M_{z_j} M_{z_i}, \quad M_{z_i}^* M_{z_i} = I_{H^2(\mathbb{D}^n)}, \qquad (i, j = 1, \ldots, n)$$

that is, $(M_{z_1}, \ldots, M_{z_n})$ is an n-tuple of commuting isometries. Moreover, for $n \geq 2$,

$$M_{z_i}^* M_{z_j} = M_{z_j} M_{z_i}^*, \qquad (1 \leq i < j \leq n)$$

that is, $(M_{z_1}, \ldots, M_{z_n})$ is a *doubly commuting* tuple of isometries.

A closed subspace $\mathcal{S} \subseteq H^2(\mathbb{D}^n)$ is said to be a *submodule* of $H^2(\mathbb{D}^n)$ if $M_{z_i}(\mathcal{S}) \subseteq \mathcal{S}$ for all $i = 1, 2, \ldots, n$, and a closed subspace $\mathcal{Q} \subseteq H^2(\mathbb{D}^n)$ is said to be a *quotient module* if $\mathcal{Q}^\perp (= H^2(\mathbb{D}^n) \ominus \mathcal{Q} \cong H^2(\mathbb{D}^n)/\mathcal{Q})$ is a submodule of $H^2(\mathbb{D}^n)$. For notational simplicity we shall let

$$\text{rank } \mathcal{S} := \text{rank } M_{\boldsymbol{z}}|_{\mathcal{S}}, \quad \text{and} \quad \text{co-rank } \mathcal{Q} = \text{rank } M_{\boldsymbol{z}}^*|_{\mathcal{Q}},$$

where $M_{\boldsymbol{z}}|_{\mathcal{S}} = (M_{z_1}|_{\mathcal{S}}, \ldots, M_{z_n}|_{\mathcal{S}})$ and $M_{\boldsymbol{z}}^*|_{\mathcal{Q}} = (M_{z_1}^*|_{\mathcal{Q}}, \ldots, M_{z_n}^*|_{\mathcal{Q}})$. It is quite natural to call the vectors of a $M_{\boldsymbol{z}}^*|_{\mathcal{Q}}$-generating set of a quotient module $\mathcal{Q}$ as *star-generating* vectors of $\mathcal{Q}$. We say that a quotient module $\mathcal{Q}$ is *star-cyclic* if co-rank $\mathcal{Q} = 1$.

Let us examine now the problem of generating vectors for submodules and quotient modules of $H^2(\mathbb{D})$. Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{D})$. Then from Beurling's theorem it follows that $\mathcal{Q} = \mathcal{Q}_\varphi$ for some inner function $\varphi \in H^\infty(\mathbb{D})$ (that is, φ is a bounded holomorphic function on $\mathbb{D}$ and $|\varphi| = 1$ a.e on $\mathbb{T}$), where

$$\mathcal{Q}_\varphi := H^2(\mathbb{D}) \ominus \mathcal{S}_\varphi, \text{ and } \mathcal{S}_\varphi := \varphi H^2(\mathbb{D}).$$

Next we consider the following two cases:

Case I: For the submodule $\mathcal{S}_\varphi$, it follows that φ is a generating vector for $M_z|_{\mathcal{S}_\varphi}$. In particular, $\mathcal{S}_\varphi$ is singly generated.

Case II: For the quotient module $\mathcal{Q}_\varphi$, it turns out that $M_z^*\varphi$ is a generating vector for $M_z^*|_{\mathcal{Q}_\varphi}$ (see Proposition 2.1). In particular, $\mathcal{Q}_\varphi$ is singly star-generated.

The above conclusions fail if we replace $H^2(\mathbb{D})$ by $H^2(\mathbb{D}^2)$. The following counterexample, due to Rudin [11], is particularly concise and illustrate the heuristic ideas behind our general consideration: Let $\mathcal{S}$ be a submodule of $H^2(\mathbb{D}^2)$ consists of all functions in $H^2(\mathbb{D}^2)$ which have a zero of order greater than or equal to m at $(0, \alpha_m) = (0, 1 - m^{-3})$. Then rank $\mathcal{S} = \infty$. Without giving a proof we note that the above submodule can be represented in the following way (see [15]):

$$\mathcal{S} = \mathcal{S}_\Phi := \bigvee_{m=0}^{\infty} \varphi_m H^2(\mathbb{D}) \otimes z^m H^2(\mathbb{D}),$$

where $\Phi = \{\varphi_m\}_{m \geq 0}$ is the decreasing sequence of Blaschke products defined by

$$\varphi_0 = \prod_{i=1}^{\infty} b_{\alpha_i}^i, \qquad \varphi_m = \frac{\varphi_{m-1}}{\prod_{i=m}^{\infty} b_{\alpha_i}}. \qquad (m \geq 1)$$

Here and throughout this paper, for each $\alpha \in \mathbb{D}$, we denote by b_α the Blaschke factor

$$b_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha}z}. \qquad (z \in \mathbb{D})$$

Subsequently, Rudin's result was improved and analyzed for submodules and quotient modules of $H^2(\mathbb{D}^2)$ by several authors, such as Ahern and Clark [2], Agrawal, Clark and Douglas [1], K. J. Izuchi, K. H. Izuchi and Y. Izuchi [5], [6], Seto and Yang [15].

Inspired and motivated by the above fact, we introduce the notion of a Rudin's quotient module: Let $\Phi_i = \{\varphi_{i,k}\}_{k=-\infty}^{\infty}$ be a sequence of Blaschke products (see (2.1)) for all $i = 1, \ldots, n$. Moreover, we assume that for each $i = 1, \ldots, n$, $\{\varphi_{i,k}\}_{k=-\infty}^{\infty}$ has a least common multiple φ_i, that is, $\varphi_{i,k}|\varphi_i$ for all $-\infty < k < \infty$ and if ψ_i is a Blaschke product with the

same property and $\psi_i | \varphi_i$, then ψ_i is a constant multiple of φ_i. The *Rudin's quotient module* corresponding to $\mathbf{\Phi} := (\Phi_1, \ldots, \Phi_n)$ is the quotient module $\mathcal{Q}_{\mathbf{\Phi}}$ of $H^2(\mathbb{D}^n)$ defined by

$$
\mathcal{Q}_{\mathbf{\Phi}} = \bigvee_{k=-\infty}^{\infty} \left(H^2(\mathbb{D}) \ominus \varphi_{1,k} H^2(\mathbb{D}) \right) \otimes \cdots \otimes \left(H^2(\mathbb{D}) \ominus \varphi_{n,k} H^2(\mathbb{D}) \right)
$$

$$
= \bigvee_{k=-\infty}^{\infty} \mathcal{Q}_{\varphi_{1,k}} \otimes \cdots \otimes \mathcal{Q}_{\varphi_{n,k}}.
$$

As we have already mentioned, when $n = 2$, in various situations, such Rudin's quotient module were already considered by Rudin [11], K. J. Izuchi et al. [5, 6], Young and Seto [15] and Seto [14] (for $n \geq 2$, see Douglas et al. [8], Guo [9, 10], Sarkar [13, 12] and Chattopadhyay et al. [3]).

The purpose of this paper is to compute and analyze the co-rank of $\mathcal{Q}_{\mathbf{\Phi}}$. We also consider the case when some of the inner sequences are increasing sequence of Blaschke products and rest of them are decreasing sequence of Blaschke products. Along the way, we obtain some results concerning minimal representations and compute co-ranks of a class of quotient modules of $H^2(\mathbb{D}^n)$.

Some of our main results are generalizations of theorems due to K. J. Izuchi, K. H. Izuchi and Y. Izuchi [5]. One of our main results, Theorem 5.2, concerning co-rank of a Rudin's quotient module is a refined and generalized version of results by Izuchi et al. (Theorems 4.2 and 4.3 in [5]). In particular, we point out and correct an error in the proof of the main result in the paper by Izuchi et al. (see Remark 5.3).

The remainder of the paper is organized as follows. In the preliminary Section 2, we set up notations, definitions and results needed further. In Section 3, we introduce and investigate minimal representations of a class of finite dimensional quotient modules of $H^2(\mathbb{D}^n)$. Section 4 is devoted to the minimal representation of a zero-based quotient module of $H^2(\mathbb{D}^n)$. In Section 5, the theory of Sections 3 and 4 is applied to obtain the main results of this paper. In the last section, we give some (counter-) examples of Rudin's quotient module of $H^2(\mathbb{D}^2)$.

2. PRELIMINARIES AND PREPARATORY RESULTS

In this section we gather some facts concerning quotient modules of $H^2(\mathbb{D})$. We begin with the result that any quotient module of $H^2(\mathbb{D})$ is singly generated.

PROPOSITION 2.1. For an inner function φ, let $\mathcal{Q}_\varphi = H^2(\mathbb{D}) \ominus \varphi H^2(\mathbb{D})$ be a quotient module of $H^2(\mathbb{D})$. Then $M_z^* \varphi$ is a star-cyclic vector of $\mathcal{Q}_\varphi$.

Proof. Since $M_\varphi^* M_z^* = M_z^* M_\varphi^*$, for all $m \geq 0$ we have

$$
\langle M_z^* \varphi, \varphi z^m \rangle = \langle M_\varphi^* M_z^* \varphi, z^m \rangle = \langle M_z^* M_\varphi^* M_\varphi 1, z^m \rangle = \langle M_z^* 1, z^m \rangle = 0.
$$

From this it follows that $M_z^* \varphi \in \mathcal{Q}_\varphi$. Then there exists an inner function θ such that

$$
\mathcal{Q}_\theta = \bigvee_{m \geq 0} M_z^{*m}(M_z^* \varphi) \subseteq \mathcal{Q}_\varphi.
$$

Now if the above inclusion is proper, then $\varphi = \theta\psi$ for some non-constant inner function ψ. On the other hand, $\theta \perp \mathcal{Q}_\theta = \bigvee_{m \geq 0} M_z^{*m}(M_z^*\varphi)$ implies that

$$0 = \langle M_z^{*m}\varphi, \theta \rangle = \langle M_z^{*m}\theta\psi, \theta \rangle = \langle \psi, z^m \rangle,$$

for all $m \geq 1$, which is a contradiction as ψ is a non-constant inner function. Therefore $\mathcal{Q}_\theta = \mathcal{Q}_\varphi$ and the proof follows. $\square$

The following well-known result furnish a rich supply of star-cyclic vectors for a quotient module of $H^2(\mathbb{D})$.

PROPOSITION 2.2. Let φ and ψ be two non-constant inner functions and q be the greatest common inner factor of φ and ψ. Let $\theta = \varphi/q$ and f be a star-cyclic vector of $\mathcal{Q}_\varphi$. Then $M_\psi^* f$ is also a star-cyclic vector of $\mathcal{Q}_\theta$.

Proof. Since $f \in \mathcal{Q}_\varphi$, we have

$$\langle M_\psi^* f, \theta z^m \rangle = \langle f, \varphi(\psi/q)z^m \rangle = 0. \qquad (m \geq 0)$$

Thus it follows that $M_\psi^* f \in \mathcal{Q}_\theta$ and $\bigvee_{m=0}^\infty M_z^{*m} M_\psi^* f \subseteq \mathcal{Q}_\theta$. We need to show $\mathcal{Q}_\theta \subseteq \bigvee_{m=0}^\infty M_z^{*m} M_\psi^* f$. Now for $g \in \mathcal{Q}_\theta$ and $g \perp \bigvee_{m=0}^\infty M_z^{*m} M_\psi^* f$, we have

$$\psi g \perp \bigvee_{m=0}^\infty M_z^{*m} f = \mathcal{Q}_\varphi,$$

and therefore $\psi g \in \psi H^2(\mathbb{D}) \cap \theta H^2(\mathbb{D}) = \psi\theta H^2(\mathbb{D})$. Consequently $\psi g = \psi\theta h$ for some $h \in H^2(\mathbb{D})$. Thus $g = \theta h$ and together with $g \in \mathcal{Q}_\theta$ imply that $g = 0$. The proof is complete. $\square$

A pair of non-constant inner functions φ and ψ is said to be *relatively prime* if φ and ψ do not have any common non-constant inner factor. An immediate consequence of the above proposition is as follows:

COROLLARY 2.3. Let φ and ψ be two relatively prime inner functions and f be a star-cyclic vector of $\mathcal{Q}_\varphi$. Then $M_\psi^* f$ is also a star-cyclic vector of $\mathcal{Q}_\varphi$.

We now specialize to the case where φ is a *Blaschke product*, that is,

$$(2.1) \qquad \varphi = \prod_{m=1}^\infty b_{\alpha_m}^{l_m},$$

where $\{l_m\}_{m=1}^\infty$ is a sequence of natural numbers and $\{\alpha_m\}_{m=1}^\infty \subseteq \mathbb{D}$ is a sequence of distinct scalars satisfying

$$\sum_{m=1}^\infty (1 - l_m|\alpha_m|) < \infty.$$

Let $\varphi \in H^\infty(\mathbb{D})$ be a Blaschke product, and let $\mathcal{I}_\varphi$ denote the set of all relatively prime inner factors of φ such that

$$(2.2) \qquad\qquad \varphi = \prod_{\xi \in \mathcal{I}_\varphi} \xi.$$

Note that $\mathcal{I}_\varphi$ is a countable set and contains non-constant inner functions. Moreover, for each $\xi \in \mathcal{I}_\varphi$, there exists a unique prime inner function $P(\xi)$ and an integer $m \in \mathbb{N} \setminus \{0\}$ such that

$$\xi = P(\xi)^m.$$

In particular if φ is of the form (2.1), then

$$\mathcal{I}_\varphi = \{b_{\alpha_m}^{l_m} : m \geq 1\},$$

and $P(b_{\alpha_m}^{l_m}) = b_{\alpha_m}$ for all $m \geq 1$.

The following result relates the aspect of relatively prime factors of a given inner function φ to the corresponding quotient module $\mathcal{Q}_\varphi$.

LEMMA 2.4. *Let φ be a Blaschke product. Then*

$$\mathcal{Q}_\varphi = \bigvee_{\xi \in \mathcal{I}_\varphi} \mathcal{Q}_\xi.$$

Proof. Since $\varphi = \prod_{\xi \in \mathcal{I}_\varphi} \xi$, it follows that $\varphi H^2(\mathbb{D}) \subseteq \xi H^2(\mathbb{D})$ for all $\xi \in \mathcal{I}_\varphi$. This implies that $\mathcal{Q}_\xi \subseteq \mathcal{Q}_\varphi$ for all $\xi \in \mathcal{I}_\varphi$, and therefore $\bigvee_{\xi \in \mathcal{I}_\varphi} \mathcal{Q}_\xi \subseteq \mathcal{Q}_\varphi$.

We now proceed to prove the other inclusion. Let $f \in \left(\bigvee_{\xi \in \mathcal{I}_\varphi} \mathcal{Q}_\xi \right)^\perp$. Then $f \in \mathcal{Q}_\xi^\perp = \xi H^2(\mathbb{D})$ for all $\xi \in \mathcal{I}_\varphi$, that is, $f \in \bigcap_{\xi \in \mathcal{I}_\varphi} \xi H^2(\mathbb{D})$. Since $\mathcal{I}_\varphi$ contains only relatively prime Blaschke products, it follows that

$$f \in \bigcap_{\xi \in \mathcal{I}_\varphi} \xi H^2(\mathbb{D}) = \Big(\prod_{\xi \in \mathcal{I}_\varphi} \xi \Big) H^2(\mathbb{D}) = \varphi H^2(\mathbb{D}) = \mathcal{Q}_\varphi^\perp.$$

This completes the proof. □

As a corollary, we obtain the following useful fact for tensor product of quotient modules:

COROLLARY 2.5. *Let $\{\varphi_j\}_{j=1}^n$ be a collection of Blaschke products. Then*

$$\mathcal{Q}_{\varphi_1} \otimes \cdots \otimes \mathcal{Q}_{\varphi_n} = \bigvee_{(\xi_1, \dots, \xi_n) \in \mathcal{I}_{\varphi_1} \times \cdots \times \mathcal{I}_{\varphi_n}} \mathcal{Q}_{\xi_1} \otimes \cdots \otimes \mathcal{Q}_{\xi_n}.$$

The following lemmas are both simple and useful.

LEMMA 2.6. *Let $\{\xi_i\}_{i=1}^n$ and $\{\eta_i\}_{i=1}^n$ be inner functions such that $\xi_j | \eta_j$ for some $1 \leq j \leq n$. Then*

$$M_{\eta_1}^* \otimes \cdots \otimes M_{\eta_n}^* (\mathcal{Q}_{\xi_1} \otimes \cdots \otimes \mathcal{Q}_{\xi_n}) = \{0\}.$$

Proof. Let $1 \leq j \leq n$ be such that $\xi_j | \eta_j$. Since $\mathcal{S}_{\xi_j} \supseteq \mathcal{S}_{\eta_j}$, we have $\mathcal{Q}_{\xi_j} \subseteq \mathcal{Q}_{\eta_j}$. Then the proof follows from the fact that $M_{\eta_j}^*(\mathcal{Q}_{\eta_j}) = \{0\}$. □

LEMMA 2.7. *Let $T = (T_1, \ldots, T_n)$ be a commuting tuple of operators on a Hilbert space $\mathcal{H}$, and let $\mathcal{Q}$ be a joint T^*-invariant closed subspace of $\mathcal{H}$. Then*

$$\operatorname{rank} P_{\mathcal{Q}} T|_{\mathcal{Q}} \leq \operatorname{rank} T,$$

where $P_{\mathcal{Q}} T|_{\mathcal{Q}} := (P_{\mathcal{Q}} T_1|_{\mathcal{Q}}, \ldots, P_{\mathcal{Q}} T_n|_{\mathcal{Q}})$.

Proof. If $\operatorname{rank} T = \infty$, then there is nothing to prove. So, let $\{f_1, \ldots, f_m\} \subseteq \mathcal{H}$ be a T-generating set for some $m \in \mathbb{N}$. Since $P_{\mathcal{Q}} T_j P_{\mathcal{Q}} = P_{\mathcal{Q}} T_j$ for all $j = 1, \ldots, n$, we have

$$(P_{\mathcal{Q}} T P_{\mathcal{Q}})^{\boldsymbol{k}}(P_{\mathcal{Q}} f_l) = P_{\mathcal{Q}} T^{\boldsymbol{k}} P_{\mathcal{Q}} f_l = P_{\mathcal{Q}}(T^{\boldsymbol{k}} f_l),$$

for all $l = 1, \ldots, m$ and $\boldsymbol{k} \in \mathbb{N}^n$. On the other hand, since $\bigvee \{T^{\boldsymbol{k}} f_j : \boldsymbol{k} \in \mathbb{N}^n, j = 1, \ldots, m\} = \mathcal{H}$, we have $\bigvee \{(P_{\mathcal{Q}} T P_{\mathcal{Q}})^{\boldsymbol{k}}(P_{\mathcal{Q}} f_j) : \boldsymbol{k} \in \mathbb{N}^n, j = 1, \ldots, m\} = \mathcal{Q}$. This shows in particular that $\{P_{\mathcal{Q}} f_1, \ldots, P_{\mathcal{Q}} f_m\}$ is a $P_{\mathcal{Q}} T|_{\mathcal{Q}}$-generating subset of $\mathcal{Q}$. This completes the proof. $\qquad\square$

3. Co-ranks of finite dimensional quotient modules

In this section we determine co-ranks of some finite dimensional quotient modules of $H^2(\mathbb{D}^n)$. This will be particularly useful in the next section when we consider minimal representations of quotient modules.

Before proceeding further, we find more useful descriptions of finite dimensional quotient modules of $H^2(\mathbb{D}^n)$. Recall that a quotient module $\mathcal{Q}_\varphi$ is finite dimensional if and only if φ is a finite Blaschke product, which is unique up to the circle group $\mathbb{T}$, and $\operatorname{order}\varphi = \dim \mathcal{Q}_\varphi$. Here our main interest concern the case of $\varphi = b_\alpha^m$, where $\alpha \in \mathbb{D}$ and $m \in \mathbb{N}$. We first observe that for $\alpha \in \mathbb{D}$ and $m \geq 1$, $\{b_\alpha^j M_z^* b_\alpha\}_{j=0}^{m-1}$ is an orthogonal basis of the quotient module $\mathcal{Q}_{b_\alpha^m}$. A simple calculation reveals that

$$M_z^* b_\alpha = (1 - |\alpha|^2)\mathbb{S}(\cdot, \alpha),$$

where $\mathbb{S}(\cdot, \alpha)$ is the Szegö kernel on $\mathbb{D}$ defined by

$$\mathbb{S}(\cdot, \alpha)(z) = (1 - \bar{\alpha}z)^{-1}. \qquad (z \in \mathbb{D})$$

Since $M_{b_\alpha^j} \in \mathcal{B}(H^2(\mathbb{D}))$ is an isometry, we have

$$(3.1) \qquad \|b_\alpha^j M_z^* b_\alpha\| = \|M_z^* b_\alpha\| = (1 - |\alpha|^2)\|\mathbb{S}(\cdot, \alpha)\| = (1 - |\alpha|^2)^{\frac{1}{2}},$$

for all $j \in \mathbb{N}$. Obviously

$$\langle b_\alpha^{m-1} M_z^* b_\alpha, M_z^*(b_\alpha^{m-1} M_z^* b_\alpha) \rangle = \langle M_z^* b_\alpha, M_z^{*2} b_\alpha \rangle = \bar{\alpha}(1 - |\alpha|^2)^2 \langle \mathbb{S}(\cdot, \alpha), \mathbb{S}(\cdot, \alpha) \rangle = \bar{\alpha}(1 - |\alpha|^2),$$

which yields

$$(3.2) \qquad P_{\mathbb{C}(b_\alpha^{m-1} M_z^* b_\alpha)} M_z^* b_\alpha^{m-1} M_z^* b_\alpha = \bar{\alpha}(b_\alpha^{m-1} M_z^* b_\alpha),$$

where $m \geq 1$ and $P_{\mathbb{C}(b_\alpha^{m-1} M_z^* b_\alpha)}$ denotes the orthogonal projection of $H^2(\mathbb{D})$ onto the one dimensional subspace generated by the vector $b_\alpha^{m-1} M_z^* b_\alpha$.

We next introduce a new class of quotient modules which is based on submodules vanishing at a point of $\mathbb{D}^n$. Given $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{D}^n$ and a finite subset A of $(\mathbb{N} \setminus \{0\})^n$, let $\mathcal{Q}(\boldsymbol{\alpha}; A)$ be the quotient module defined by

$$(3.3) \qquad \mathcal{Q}(\boldsymbol{\alpha}; A) := \bigvee_{(l_1, \ldots, l_n) \in A} \mathcal{Q}_{b_{\alpha_1}^{l_1}} \otimes \cdots \otimes \mathcal{Q}_{b_{\alpha_n}^{l_n}}.$$

Now we want to find a minimum cardinality subset $\tilde{A}$ of A such that $\mathcal{Q}(\boldsymbol{\alpha}; A) = \mathcal{Q}(\boldsymbol{\alpha}; \tilde{A})$. In order to find $\tilde{A}$, first observe that for $(l_1, \ldots, l_n), (l_1', \ldots, l_n') \in A$ if $l_i \le l_i'$ for all $i = 1, \ldots, n$, then

$$(3.4) \qquad \mathcal{Q}_{b_{\alpha_1}^{l_1}} \otimes \cdots \otimes \mathcal{Q}_{b_{\alpha_n}^{l_n}} \subseteq \mathcal{Q}_{b_{\alpha_1}^{l_1'}} \otimes \cdots \otimes \mathcal{Q}_{b_{\alpha_n}^{l_n'}},$$

which implies that

$$(3.5) \qquad (\mathcal{Q}_{b_{\alpha_1}^{l_1}} \otimes \cdots \otimes \mathcal{Q}_{b_{\alpha_n}^{l_n}}) \vee (\mathcal{Q}_{b_{\alpha_1}^{l_1'}} \otimes \cdots \otimes \mathcal{Q}_{b_{\alpha_n}^{l_n'}}) = \mathcal{Q}_{b_{\alpha_1}^{l_1'}} \otimes \cdots \otimes \mathcal{Q}_{b_{\alpha_n}^{l_n'}}.$$

Thus $\mathcal{Q}(\boldsymbol{\alpha}; A) = \mathcal{Q}(\boldsymbol{\alpha}; A \setminus \{(l_1, \ldots, l_n)\})$, and hence we remove $(l_1, \ldots, l_n)$ from A. By continuing this process we eventually obtain a set $\tilde{A} \subseteq A$ of minimal cardinality such that

$$(3.6) \qquad \mathcal{Q}(\boldsymbol{\alpha}; A) = \mathcal{Q}(\boldsymbol{\alpha}; \tilde{A}) = \bigvee_{(l_1, \ldots, l_n) \in \tilde{A}} \mathcal{Q}_{b_{\alpha_1}^{l_1}} \otimes \cdots \otimes \mathcal{Q}_{b_{\alpha_n}^{l_n}}.$$

It should be noted that for any pair $(l_1, \ldots, l_n)$ and $(l_1', \ldots, l_n')$ of $\tilde{A}$, one has the following:

$$(3.7) \qquad \exists\, i, j \in \{1, \ldots, n\}, i \ne j, \text{ such that } l_i < l_i' \text{ and } l_j > l_j'.$$

The new representation $\mathcal{Q}(\boldsymbol{\alpha}; \tilde{A})$ is called the *minimal representation* of $\mathcal{Q}(\boldsymbol{\alpha}; A)$.

We remark here that any finite dimensional quotient module of $H^2(\mathbb{D}^n)$ is span closure of finite number of quotient modules of the form $\mathcal{Q}(\boldsymbol{\alpha}; \tilde{A})$ (cf. [2], also see Douglas, Paulsen, Sah and Yan [8], Guo [9, 10] and Chen and Guo [4]).

This minimal representation of $\mathcal{Q}(\boldsymbol{\alpha}; A)$ plays a fundamental role in calculating the co-rank of a Rudin's quotient module $\mathcal{Q}$. Here is one example.

PROPOSITION 3.1. Let $\mathcal{Q}(\boldsymbol{\alpha}; A)$ be a quotient module as in (3.3). Then

$$\text{co-rank } \mathcal{Q}(\boldsymbol{\alpha}; A) = \text{co-rank } \mathcal{Q}(\boldsymbol{\alpha}; \tilde{A}) = \#\tilde{A}.$$

Proof. Let $\#\tilde{A} = r$. Without loss of generality we assume that $\tilde{A} = \{(l_{1,k}, l_{2,k}, \ldots, l_{n,k}) \in \mathbb{N}^n : k = 1, \ldots, r\}$. Let $f_{j,k}$ be a star-generator of $\mathcal{Q}_{b_{\alpha_j}^{l_{j,k}}}$, where $k = 1, \ldots, r$, and $j = 1, \ldots, n$. Then

$$[f_{1,k} \otimes \cdots \otimes f_{n,k}]_{M_z^*} = \mathcal{Q}_{b_{\alpha_1}^{l_{1,k}}} \otimes \cdots \otimes \mathcal{Q}_{b_{\alpha_n}^{l_{n,k}}},$$

for all $k = 1, \ldots, r$, so that co-rank $\mathcal{Q}(\boldsymbol{\alpha}; \tilde{A}) \le r$. The reverse inequality will follow, by virtue of Lemma 2.7, if we can construct a closed subspace $\mathcal{E} \subseteq \mathcal{Q}(\boldsymbol{\alpha}; \tilde{A})$ such that $\mathcal{Q}(\boldsymbol{\alpha}; \tilde{A}) \ominus \mathcal{E}$ is a quotient module of $H^2(\mathbb{D}^n)$ and rank $\mathcal{E} = r$ for $(P_{\mathcal{E}} M_{z_1}^*|_{\mathcal{E}}, \ldots, P_{\mathcal{E}} M_{z_n}^*|_{\mathcal{E}})$. To this end, let

$$g_k := b_{\alpha_1}^{l_{1,k}-1} M_z^* b_{\alpha_1} \otimes \cdots \otimes b_{\alpha_n}^{l_{n,k}-1} M_z^* b_{\alpha_n} \in \mathcal{Q}_{b_{\alpha_1}^{l_{1,k}}} \otimes \cdots \otimes \mathcal{Q}_{b_{\alpha_n}^{l_{n,k}}},$$

for all $k = 1, \ldots, r$. By virtue of (3.7) we conclude that $\{g_k\}_{k=1}^r$ is an orthogonal set, and hence $\mathcal{E} := \bigoplus_{k=1}^r \mathbb{C}g_k$ is an r dimensional subspace of $\mathcal{Q}(\boldsymbol{\alpha}, \tilde{A})$ and $\mathcal{Q}(\boldsymbol{\alpha}; \tilde{A}) \ominus \mathcal{E}$ is a quotient module of $H^2(\mathbb{D}^n)$. Now from (3.1) it follows that

$$\|g_k\|^2 = \prod_{j=1}^n (1 - |\alpha_j|^2),$$

and by (3.7) we have $\langle g_{k'}, M_{z_i}^* g_k \rangle = 0$ for any $1 \leq k < k' \leq r$ and $1 \leq i \leq n$. Thus using (3.2) one can have

$$P_\mathcal{E} M_{z_i}^* g_k = P_{\mathbb{C}g_k} M_{z_i}^* g_k = \bar{\alpha}_i g_k,$$

for all $i = 1, \ldots, n$, and $k = 1, \ldots, m$. This implies that

$$P_\mathcal{E} M_{z_i}^*|_\mathcal{E} = \bar{\alpha}_i I_\mathcal{E}. \qquad (i = 1, \ldots, n)$$

Since $\dim \mathcal{E} = r$, we see that rank $\mathcal{E} = r$ for $(P_\mathcal{E} M_{z_1}^*|_\mathcal{E}, \ldots, P_\mathcal{E} M_{z_n}^*|_\mathcal{E})$. This completes the proof. $\qquad \square$

4. Minimal representations of quotient modules

Let φ be a Blaschke product and ξ be a non-constant factor of φ. The *order* of ξ in φ, denoted by $\mathrm{ord}(\varphi, \xi)$, is the unique integer m such that $\varphi = \xi^m \psi$ for an inner function ψ and $\xi \nmid \psi$. In particular if b_α is a prime factor of φ, then $\mathrm{ord}(\varphi, b_\alpha)$ denotes the zero order of φ at α.

For the rest of this paper, we fix $\boldsymbol{\Phi} = (\Phi_1, \ldots, \Phi_n)$, where $\Phi_i = \{\varphi_{i,k}\}_{k=-\infty}^\infty$ is a sequence of Blaschke products with a least common multiple φ_i, $i = 1, \ldots, n$. Our main concern here is to analyze and compute the co-rank of the following Rudin's quotient module

$$(4.1) \qquad \mathcal{Q}_{\boldsymbol{\Phi}} = \bigvee_{k=-\infty}^\infty \mathcal{Q}_{\varphi_{1,k}} \otimes \cdots \otimes \mathcal{Q}_{\varphi_{n,k}}.$$

By defining

$$(4.2) \qquad \Lambda_k := \mathcal{I}_{\varphi_{1,k}} \times \cdots \times \mathcal{I}_{\varphi_{n,k}} \quad (k \in \mathbb{Z}) \quad \text{and} \quad \Lambda := \bigcup_{k \in \mathbb{Z}} \Lambda_k,$$

Corollary 2.5 shows that

$$\mathcal{Q}_{\boldsymbol{\Phi}} = \bigvee_{(\xi_1, \ldots, \xi_n) \in \Lambda} \mathcal{Q}_{\xi_1} \otimes \cdots \otimes \mathcal{Q}_{\xi_n}.$$

Now let $(\xi_1, \ldots, \xi_n) \in \Lambda_k$ and $k \in \mathbb{Z}$. Then $\xi_i = P(\xi_i)^{l_{i,k}}$, where $P(\xi_i)$ is the prime inner function corresponding to ξ_i and

$$(4.3) \qquad l_{i,k} = \mathrm{ord}(\xi_i, P(\xi_i)) = \mathrm{ord}(\varphi_{i,k}, P(\xi_i)). \qquad (i = 1, \ldots, n)$$

Thus $(\xi_1, \ldots, \xi_n) \in \Lambda_k$ corresponds precisely to a tuple of prime inner functions $(P(\xi_1), \ldots, P(\xi_n))$ and a tuple of natural numbers $(l_{1,k}, \ldots, l_{n,k}) \in \mathbb{N}^n$. Moreover,

$$(4.4) \qquad \mathcal{Q}_{\boldsymbol{\Phi}} = \bigvee_{(\xi_1, \ldots, \xi_n) \in \Lambda} \mathcal{Q}_{P(\xi_1)^{l_{1,k}}} \otimes \cdots \otimes \mathcal{Q}_{P(\xi_n)^{l_{n,k}}}.$$

Also, note that for each $i = 1, \ldots, n$, $P(\xi_i) = b_{\alpha_i}$ for some $\alpha_i \in \mathbb{D}$. Based on this observation, we define the *zero set* of the tuple $(\xi_1, \ldots, \xi_n) \in \Lambda_m$ as follows:

$$Z(\xi_1, \ldots, \xi_n) = \{k \in \mathbb{Z} : P(\xi_i) | \varphi_{i,k} \text{ for all } i = 1, 2, \ldots, n\}.$$

Note that $Z(\xi_1, \ldots, \xi_n)$ is a countable and non-empty set (since $m \in Z(\xi_1, \ldots, \xi_n)$).

If we define the quotient module $\mathcal{Q}(\xi_1, \ldots, \xi_n)$ by

$$(4.5) \qquad \mathcal{Q}(\xi_1, \ldots, \xi_n) := \bigvee_{k \in Z(\xi_1, \ldots, \xi_n)} \mathcal{Q}_{P(\xi_1)^{l_{1,k}}} \otimes \cdots \otimes \mathcal{Q}_{P(\xi_n)^{l_{n,k}}},$$

then by (4.4) it follows that

$$\mathcal{Q}_{\boldsymbol{\Phi}} = \bigvee_{(\xi_1, \ldots, \xi_n) \in \Lambda} \mathcal{Q}(\xi_1, \ldots, \xi_n).$$

This sets the stage for the following result concerning a minimal representation of $\mathcal{Q}(\xi_1, \ldots, \xi_n)$:

PROPOSITION 4.1. *Let* $\mathcal{Q}(\xi_1, \ldots, \xi_n)$ *be as in* (4.5) *for some* $(\xi_1, \ldots, \xi_n) \in \Lambda$. *Then there exists a finite subset* $\tilde{Z}(\xi_1, \ldots, \xi_n)$ *of* $Z(\xi_1, \ldots, \xi_n)$ *with minimal cardinality such that*

$$(4.6) \qquad \mathcal{Q}(\xi_1, \ldots, \xi_n) = \bigvee_{k \in \tilde{Z}(\xi_1, \ldots, \xi_n)} \mathcal{Q}_{P(\xi_1)^{l_{1,k}}} \otimes \cdots \otimes \mathcal{Q}_{P(\xi_n)^{l_{n,k}}}.$$

Proof. First consider the set of tuples $\{(l_{1,k}, \ldots, l_{n,k}) \in \mathbb{N}^n : k \in Z(\xi_1, \ldots, \xi_n)\}$, where $l_{i,k}$ is defined as in (4.3) for $i = 1, \ldots, n$. Then construct $\tilde{Z}(\xi_1, \ldots, \xi_n) \subseteq Z(\xi_1, \ldots, \xi_n)$ by removing those $k \in Z(\xi_1, \ldots, \xi_n)$ for which there exists $k' \in Z(\xi_1, \ldots, \xi_n)$ such that $l_{i,k'} \geq l_{i,k}$ for all $i = 1, \ldots, n$. Then the equality (4.6), for $\tilde{Z}(\xi_1, \ldots, \xi_n)$ as constructed above, follows from (3.4) and (3.5). Finally, since the sequence $\{\varphi_{i,k}\}_{k=-\infty}^{\infty}$ has a least common multiple, we obviously have

$$\sup_{k \in \mathbb{Z}} l_{i,k} = \sup_{k \in \mathbb{Z}} \mathrm{ord}(\varphi_{i,k}, P(\xi_i)) < \infty, \qquad (i = 1, \ldots, n)$$

and hence it follows that the cardinality of $\{(l_{1,k}, \ldots, l_{n,k}) \in \mathbb{N}^n : k \in Z(\xi_1, \ldots, \xi_n)\}$ is finite. Therefore $\tilde{Z}(\xi_1, \ldots, \xi_n)$ is a finite set. This concludes the proof of the proposition. $\square$

We will call the representation in (4.6) the *minimal representation* of $\mathcal{Q}(\xi_1, \ldots, \xi_n)$.

The following result is useful in connection with the existence of minimal index set $\tilde{Z}(\xi_1, \ldots, \xi_n)$.

PROPOSITION 4.2. *Let* $\{\varphi_{i,k}\}_{k=-\infty}^{\infty}$ *be a sequence of Blaschke products with a least common multiple inner function* φ_i, *for all* $i = 1, \ldots, n$. *Then for each* $(\xi_1, \ldots, \xi_n) \in \Lambda$,

$$\text{co-rank } \mathcal{Q}(\xi_1, \ldots, \xi_n) = \#\tilde{Z}(\xi_1, \ldots, \xi_n),$$

where $\tilde{Z}(\xi_1, \ldots, \xi_n)$ *is the minimal index set for* $\mathcal{Q}(\xi_1, \ldots, \xi_n)$ *as in Proposition 4.1.*

Proof. Let us set, for $i = 1, \ldots, n$,

$$P(\xi_i) = b_{\alpha_i},$$

for some $\alpha_i \in \mathbb{D}$, and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$. Using the notation in (3.3), we have

$$\mathcal{Q}(\xi_1, \ldots, \xi_n) = \mathcal{Q}(\boldsymbol{\alpha}; A(\xi_1, \ldots, \xi_n)) = \mathcal{Q}(\boldsymbol{\alpha}; \tilde{A}(\xi_1, \ldots, \xi_n)),$$

where

$$A(\xi_1, \ldots, \xi_n) = \{(l_{1,k}, \ldots, l_{n,k}) \in \mathbb{N}^n : k \in Z(\xi_1, \ldots, \xi_n)\},$$

and

$$\tilde{A}(\xi_1, \ldots, \xi_n) = \{(l_{1,k}, \ldots, l_{n,k}) \in \mathbb{N}^n : k \in \tilde{Z}(\xi_1, \ldots, \xi_n)\},$$

and $\mathcal{Q}(\boldsymbol{\alpha}; \tilde{A}(\xi_1, \ldots, \xi_n))$ is the minimal representation of $\mathcal{Q}(\boldsymbol{\alpha}; A(\xi_1, \ldots, \xi_n))$. Then the desired equality follows from Proposition 3.1. This completes the proof. $\qquad\square$

Now we observe that for $(\xi_1, \ldots, \xi_n), (\xi_1', \ldots, \xi_n') \in \Lambda$, if $P(\xi_i) = P(\xi_i')$ for all $i = 1, \ldots, n$, then $\mathcal{Q}(\xi_1, \ldots, \xi_n) = \mathcal{Q}(\xi_1', \ldots, \xi_n')$. Consequently, $\sim$ is an equivalence relation on Λ, where $(\xi_1, \ldots, \xi_n) \sim (\xi_1', \ldots, \xi_n')$ if $P(\xi_i) = P(\xi_i')$ for all $i = 1, \ldots, n$. This readily implies that

$$\mathcal{Q}_{\boldsymbol{\Phi}} = \bigvee_{(\xi_1, \ldots, \xi_n) \in [\Lambda]} \mathcal{Q}(\xi_1, \ldots, \xi_n),$$

where $[\Lambda] := \Lambda/\sim$ is the set of all equivalence classes in Λ.

5. Co-rank of $\mathcal{Q}_{\boldsymbol{\Phi}}$

In this section we compute the co-rank of the quotient module of the form (4.1).

THEOREM 5.1. Let $\{\varphi_{i,k}\}_{k=-\infty}^{\infty}$ be a sequence of Blaschke products with a least common multiple inner function φ_i, $i = 1, \ldots, n$, and let

$$\mathcal{Q}_{\boldsymbol{\Phi}} = \bigvee_{k=-\infty}^{\infty} \mathcal{Q}_{\varphi_{1,k}} \otimes \cdots \otimes \mathcal{Q}_{\varphi_{n,k}}.$$

Then

$$\text{co-rank } \mathcal{Q}_{\boldsymbol{\Phi}} = \sup_{(\xi_1, \ldots, \xi_n) \in \Lambda} \text{co-rank } \mathcal{Q}(\xi_1, \ldots, \xi_n) = \sup_{(\xi_1, \ldots, \xi_n) \in \Lambda} \#\tilde{Z}(\xi_1, \ldots, \xi_n),$$

where Λ is as in (4.2) and for $(\xi_1, \ldots, \xi_n) \in \Lambda$, $\tilde{Z}(\xi_1, \ldots, \xi_n)$ is the minimal index set for the minimal representation of $\mathcal{Q}(\xi_1, \ldots, \xi_n)$ as in (4.6).

Proof. By Proposition 4.2, we have

$$\sup_{(\xi_1, \ldots, \xi_n) \in \Lambda} \text{co-rank } \mathcal{Q}(\xi_1, \ldots, \xi_n) = \sup_{(\xi_1, \ldots, \xi_n) \in \Lambda} \#\tilde{Z}(\xi_1, \ldots, \xi_n).$$

Now to see the first equality, let $(\xi_1, \ldots, \xi_n) \in \Lambda$. Set

$$a_i := \sup\{l_{i,m} : m \in \tilde{Z}(\xi_1, \ldots, \xi_n)\},$$

where $l_{i,m} = \text{order }(\varphi_{i,m}, P(\xi_i))$, $m \in \tilde{Z}(\xi_1, \ldots, \xi_n)$, and $1 \le i \le n$. Since $\{\varphi_{i,k}\}_{k=-\infty}^{\infty}$ has a least common multiple, then $a_i < \infty$, and

$$\varphi_i(\xi_i) := \frac{\varphi_i}{P(\xi_i)^{a_i}},$$

is a Blaschke product for all $i = 1, \ldots, n$. Since $\varphi_i(\xi_i)$ and $P(\xi_i)^t$ are relatively prime for any $t \in \mathbb{N} \setminus \{0\}$ and $i = 1, 2, \ldots, n$, by Corollary 2.3 we conclude that

$$(5.1) \quad M_{\varphi_1(\xi_1)}^* \otimes \cdots \otimes M_{\varphi_n(\xi_n)}^* \left(\mathcal{Q}_{P(\xi_1)^{l_{1,m}}} \otimes \cdots \otimes \mathcal{Q}_{P(\xi_n)^{l_{n,m}}} \right) = \mathcal{Q}_{P(\xi_1)^{l_{1,m}}} \otimes \cdots \otimes \mathcal{Q}_{P(\xi_n)^{l_{n,m}}},$$

for all $m \in Z(\xi_1, \ldots, \xi_n)$. On the other hand, let $(\xi_1', \ldots, \xi_n') \in \Lambda$ be such that $P(\xi_i)$ is not a factor of ξ_i' for some $1 \le i \le n$. This implies that $\xi_i' | \varphi_i(\xi_i)$ for some $1 \le i \le n$. Consequently, by Lemma 2.6

$$(5.2) \qquad M_{\varphi_1(\xi_1)}^* \otimes \cdots \otimes M_{\varphi_n(\xi_n)}^* \left(\mathcal{Q}_{\xi_1'} \otimes \cdots \otimes \mathcal{Q}_{\xi_n'} \right) = \{0\}.$$

Thus combining (5.2) and (5.1), we have

$$M_{\varphi_1(\xi_1)}^* \otimes \cdots \otimes M_{\varphi_n(\xi_n)}^* \left(\mathcal{Q}(\xi_1, \ldots, \xi_n) \right) = \mathcal{Q}(\xi_1, \ldots, \xi_n),$$

and

$$M_{\varphi_1(\xi_1)}^* \otimes \cdots \otimes M_{\varphi_n(\xi_n)}^* \left(\mathcal{Q}_{\boldsymbol{\Phi}} \right) = \mathcal{Q}(\xi_1, \ldots, \xi_n).$$

This yields co-rank $\mathcal{Q}(\xi_1, \ldots, \xi_n) \le$ co-rank $\mathcal{Q}_{\boldsymbol{\Phi}}$ for all $(\xi_1, \ldots, \xi_n) \in \Lambda$.
To prove the reverse inequality, we may assume that

$$m_0 := \sup_{(\xi_1, \ldots, \xi_n) \in \Lambda} \# \tilde{Z}(\xi_1, \ldots, \xi_n) < \infty.$$

It is thus enough to show that $\mathcal{Q}_{\boldsymbol{\Phi}}$ is (co-)generated by m_0 vectors. We proceed next with the detailed construction of a co-generating set of vectors of cardinality m_0.
Let $k \in \mathbb{Z}$ and $(\xi_1, \ldots, \xi_n) \in \Lambda_k$. Also for all $m \in Z(\xi_1, \ldots, \xi_n)$, let $f_m(\xi_i) \in \mathcal{Q}_{P(\xi_i)^{l_{i,m}}}$ be a unit star-cyclic vector of $\mathcal{Q}_{P(\xi_i)^{l_{i,m}}}$, $i = 1, \ldots, n$. Obviously $f_m(\xi_1) \otimes \cdots \otimes f_m(\xi_n)$ is a star-cyclic vector of $\mathcal{Q}_{P(\xi_1)^{l_{1,m}}} \otimes \cdots \otimes \mathcal{Q}_{P(\xi_n)^{l_{n,m}}}$ for all $m \in Z(\xi_1, \ldots, \xi_n)$. In this setting, we relabel the set of unit vectors $\{f_m(\xi_1) \otimes \cdots \otimes f_m(\xi_n) : m \in \tilde{Z}(\xi_1, \ldots, \xi_n)\}$ by defining a bijective function

$$g : \{1, \ldots, \# \tilde{Z}(\xi_1, \ldots, \xi_n)\} \to \tilde{Z}(\xi_1, \ldots, \xi_n),$$

and letting

$$F_r(\xi_1, \ldots, \xi_n) = \begin{cases} f_{g(r)}(\xi_1) \otimes \cdots \otimes f_{g(r)}(\xi_n) & \text{if } 1 \le r \le \# \tilde{Z}(\xi_1, \ldots, \xi_n); \\ 0 & \text{if } \# \tilde{Z}(\xi_1, \ldots, \xi_n) < r \le m_0. \end{cases}$$

Thus corresponding to each $(\xi_1, \ldots, \xi_n) \in [\Lambda]$, we have m_0 number of vectors of the above form. We now use these facts to define

$$G_r = \sum_{(\xi_1, \ldots, \xi_n) \in [\Lambda]} C(\xi_1, \ldots, \xi_n) F_r(\xi_1, \ldots, \xi_n), \quad 1 \le r \le m_0,$$

where the sum is over a countable set and the constants $C(\xi_1, \ldots, \xi_n)$ are so that the above sum converges. Then $G_r \in \mathcal{Q}_{\boldsymbol{\Phi}}$ for $1 \le r \le m_0$. Next consider the subspace

$$\Omega = \bigvee_{t_1, t_2, \ldots, t_n \in \mathbb{N}} M_{z_1}^{*t_1} \otimes \cdots \otimes M_{z_n}^{*t_n} \{G_1, \ldots, G_{m_0}\}.$$

Since $G_r \in \mathcal{Q}_{\boldsymbol{\Phi}}$ for $1 \le r \le m_0$, we obviously have $\Omega \subseteq \mathcal{Q}_{\boldsymbol{\Phi}}$. Now for $(\xi_1, \ldots, \xi_n) \in \Lambda$ and $1 \le r \le \tilde{Z}(\xi_1, \ldots, \xi_n)$, we have

$$M_{\varphi_1(\xi_1)}^* \otimes \cdots \otimes M_{\varphi_n(\xi_n)}^* \left(G_r \right) \in \Omega,$$

and using (5.2) and (5.1) we conclude that

$$(5.3) \qquad M^*_{\varphi_1(\xi_1)} \otimes \cdots \otimes M^*_{\varphi_n(\xi_n)} (G_r)$$
$$= C(\xi_1,\ldots,\xi_n) M^*_{\varphi_1(\xi_1)} \otimes \cdots \otimes M^*_{\varphi_n(\xi_n)} (F_r(\xi_1,\ldots,\xi_n))$$
$$= C(\xi_1,\ldots,\xi_n) M^*_{\varphi_1(\xi_1)} \left(f_{g(r)}(\xi_1) \right) \otimes \cdots \otimes M^*_{\varphi_n(\xi_n)} \left(f_{g(r)}(\xi_n) \right).$$

By virtue of Corollary 2.3,

$$M^*_{\varphi_1(\xi_1)} \left(f_{g(r)}(\xi_1) \right) \otimes \cdots \otimes M^*_{\varphi_n(\xi_n)} \left(f_{g(r)}(\xi_n) \right)$$

is a star-cyclic vector of

$$\mathcal{Q}_{P(\xi_1)^{l_{1,g(r)}}} \otimes \cdots \otimes \mathcal{Q}_{P(\xi_n)^{l_{n,g(r)}}}.$$

Hence we obtain $\mathcal{Q}(\xi_1,\ldots,\xi_n) \subseteq \Omega$ for all $(\xi_1,\ldots,\xi_n) \in [\Lambda]$, and consequently $\Omega = \mathcal{Q}_{\boldsymbol{\Phi}}$. As a result we have co-rank $\mathcal{Q}_{\boldsymbol{\Phi}} \leq m_0$, and this concludes the proof. $\square$

Let $A \subsetneq \{1,\ldots,n\}$ and $\Phi_i = \{\varphi_{i,k}\}_{k=-\infty}^\infty$ be a sequence of Blaschke products with no common non-constant inner function, $i = 1,\ldots,n$. The contents of the last section can be adopted to a general class of Rudin's quotient modules $\mathcal{Q}_{\boldsymbol{\Phi}}$, where Φ_i is increasing for $i \in A$ and decreasing for $i \in B := \{1,\ldots,n\} \setminus A$.
In this case for each $(\xi_1,\ldots,\xi_n) \in \Lambda$,

$$Z(\xi_1,\ldots,\xi_n) = \{k \in \mathbb{Z} : r_1 \leq k \leq r_2\},$$

where

$$(5.4) \qquad r_1 = \min\{k \in \mathbb{Z} : P(\xi_i)|\varphi_{i,k} \text{ for all } i \in A\}, \text{ and}$$
$$r_2 = \max\{k \in \mathbb{Z} : P(\xi_i)|\varphi_{i,k} \text{ for all } i \in B\}.$$

Note first that $|r_1|, |r_2| < \infty$. This follows from the fact that Φ_i, $i = 1,\ldots,n$, does not have any common inner factor. Consequently, $Z(\xi_1,\ldots,\xi_n)$ is a finite set. Note that in the proof of Theorem 5.1, the assumption that each of the sequence has least common multiple has been used to ensure that $\#\tilde{Z}(\xi_1,\ldots,\xi_n) < \infty$ and also used to construct inner functions so that (5.1), (5.2) and (5.3) holds. In the present consideration, we can still do this by defining

$$(5.5) \qquad \varphi_i(\xi_i) = \begin{cases} \dfrac{\varphi_{i,r_1}}{P(\xi_i)^{l_{i,r_1}}} & \text{if } i \in B, \\[2mm] \dfrac{\varphi_{i,r_2}}{P(\xi_i)^{l_{i,r_2}}} & \text{if } i \in A, \end{cases}$$

where $i = 1,\ldots,n$, and r_1 and r_2 are as in (5.4). We can see now the proof of the co-rank equality, as in Theorem 5.1, for this quotient module follows along the same line as the proof of Theorem 5.1. Therefore, we have the following theorem:

THEOREM 5.2. Let A be a proper non-empty subset of $\{1,\ldots,n\}$ and $B := \{1,\ldots,n\} \setminus A$, and let $\Phi_i = \{\varphi_{i,k}\}_{k=-\infty}^\infty$ be a sequence of Blaschke products with no common non-constant inner function, $i = 1,\ldots,n$. Also let Φ_i be increasing for all $i \in A$ and decreasing for all $i \in B$. Then

$$\text{co-rank } \mathcal{Q}_{\boldsymbol{\Phi}} = \sup_{(\xi_1,\ldots,\xi_n) \in \Lambda} \#\tilde{Z}(\xi_1,\ldots,\xi_n).$$

REMARK 5.3. The above theorem, restricted to $n = 2$ case, is related to Theorem 4.2 in [5]. However, the formulation of Theorem 4.2 in [5] turns out to be incorrect. This will be discussed at the end of the final section.

In the present context, for $(\xi_1, \ldots, \xi_n) \in \Lambda$, it is also possible to describe the set $\tilde{Z}(\xi_1, \ldots, \xi_n)$. Let $l_{i,k} = \text{order}(\varphi_{i,k}, P(\xi_i))$, as before (see (4.3)), for all $i = 1, \ldots, n$ and $k \in Z(\xi_1, \ldots, \xi_n)$. Note that $l_{i,k} \geq l_{i,k+1}$ for all $i \in B$, and $l_{i,k} \leq l_{i,k+1}$ for all $i \in A$. Now we proceed to construct $\tilde{Z}(\xi_1, \ldots, \xi_n)$ as follows. Set

$$(5.6) \qquad \zeta_{i,k} := \frac{\varphi_{i,k}}{\varphi_{i,k-1}}, \qquad (i \in A, k \in Z(\xi_1, \ldots, \xi_n))$$

and

$$(5.7) \qquad I(\xi_1, \ldots, \xi_n) := \{k \in Z(\xi_1, \ldots, \xi_n) : P(\xi_i) | \zeta_{i,k} \text{ for some } i \in A\}.$$

It is clear that $r_1 \in I(\xi_1, \ldots, \xi_n)$ and hence $\tilde{Z}(\xi_1, \ldots, \xi_n) = \{r_1\}$ when $\#I(\xi_1, \ldots, \xi_n) = 1$. Now suppose we have $\#I(\xi_1, \ldots, \xi_n) = m + 1 > 1$, for some $m \in \mathbb{N}$, and (without loss of any generality)

$$I(\xi_1, \ldots, \xi_n) = \{k_0 = r_1 < k_1 < k_2 < \cdots < k_m \leq r_2\}.$$

Define

$$(5.8) \qquad \eta_{i,k_j} := \frac{\varphi_{i,k_j}}{\varphi_{i,k_{j+1}}}. \qquad (0 \leq j \leq m - 1, i \in B)$$

Then $\tilde{Z}(\xi_1, \ldots, \xi_n) = \{k_m\} \cup \{k_j \in I(\xi_1, \ldots, \xi_n) : k_j \neq k_m, P(\xi_i) | \eta_{i,k_j} \text{ for some } i \in B\}$.

The above discussion, along with Theorem 5.2, may be summarized in the following.

THEOREM 5.4. Let $\Phi_i = \{\varphi_{i,k}\}_{k=-\infty}^{\infty}$, $i = 1, \ldots, n$, be as in the statement of Theorem 5.2. Then

$$\text{co-rank } \mathcal{Q}_{\mathbf{\Phi}} = \sup_{(\xi_1, \ldots, \xi_n) \in \Lambda} \#\tilde{Z}(\xi_1, \ldots, \xi_n).$$

Moreover, for all $(\xi_1, \ldots, \xi_n) \in \Lambda$,

$$\#\tilde{Z}(\xi_1, \ldots, \xi_n) = 1 + \#\{k_j \in I(\xi_1, \ldots, \xi_n) : k_j \neq k_m, P(\xi_i) | \eta_{i,k_j} \text{ for some } i \in B\},$$

where $I(\xi_1, \ldots, \xi_n)$ is as in (5.7), and η_{i,k_j} is as in (5.8).

6. Concluding Remarks

We now present a simple example which illustrate the main idea of this paper. Let $\{\{\alpha_{i,k}\}_{k=-\infty}^{\infty} : i = 1, \ldots, n\}$ be a collection of sequences of distinct points in $\mathbb{D}$ such that

$$\sum_{k=-\infty}^{\infty} (1 - |\alpha_{i,k}|) < \infty. \qquad (i = 1, \ldots, n)$$

Let A be a proper non-empty subset of $\{1, \ldots, n\}$ and $B := \{1, \ldots, n\} \setminus A$. Also consider the following sequences of Blaschke products

$$\varphi_{i,k} = \begin{cases} \prod_{j=k}^{\infty} b_{\alpha_{i,j}} & \text{if } i \in B; \\ \prod_{j=-\infty}^{k} b_{\alpha_{i,j}} & \text{if } i \in A. \end{cases}$$

Consequently, $\{\varphi_{i,k}\}_{k=-\infty}^{\infty}$ is an increasing sequence for each $i \in A$ and decreasing sequence for each $i \in B$. Let $(\xi_1, \ldots, \xi_n) \in \Lambda_m$ for some $m \in \mathbb{Z}$. Then $\xi_i = b_{\alpha_{i,k_i}}$, where $k_i \geq m$ for all $i \in B$ and $k_i \leq m$ for all $i \in A$. In this case, $r_1 = \max\{k_i : i \in A\}$ and $r_2 = \min\{k_i : i \in B\}$. From the fact that the set of points are distinct, we deduce that $I(\xi_1, \ldots, \xi_n) = \{r_1\}$. Hence $\tilde{Z}(\xi_1, \ldots, \xi_n) = \{r_1\}$, and consequently, co-rank $\mathcal{Q}_{\mathbf{\Phi}} = 1$.

To end this paper, we construct a counter example, as promised in Remark 5.3, to point out an error in the formulation of Theorem 4.2 in [5].

Let $\Phi = \{\varphi_m\}_{m=-\infty}^{\infty}$ be a decreasing sequence of Blaschke products, and let $\Psi = \{\psi_m\}_{m=-\infty}^{\infty}$ be an increasing sequence of Blaschke products such that each of the sequence does not have any non-constant common inner factor. First, for the sake of convenience we state Theorem 4.2 from [5].

THEOREM 6.1 (Theorem 4.2, [5]). Let Φ and Ψ be as above. Then

$$\text{co-rank } \mathcal{Q}_{(\Phi,\Psi)} = \sup_{j \geq 1} \#\{n : \zeta_n(\alpha_j) = \xi_n(\beta_j) = 0, -\infty < n < \infty\},$$

where

$$\zeta_m = \varphi_m/\varphi_{m+1} \text{ and } \xi_m = \psi_m/\psi_{m-1}, \quad (m \in \mathbb{Z})$$

and $(\alpha_j, \beta_j)_{j \geq 1}$ is the enumeration of the countable set $Z = \{(\alpha, \beta) \in \mathbb{D}^2 : \varphi_m(\alpha) = \psi_m(\beta) = 0 \text{ for some } m \in \mathbb{Z}\}$.

We need some more notations in the spirit of [5]. For $j \geq 1$ and $(\alpha_j, \beta_j) \in Z$, define

$$Z_j = \{n : \varphi_n(\alpha_j) = \psi_n(\beta_j) = 0, -\infty < n < \infty\},$$

and

$$\mathcal{N}_j = \sum_{n \in Z_j} \mathcal{Q}_{b_{\alpha_j}^{r_{j,n}}} \otimes \mathcal{Q}_{b_{\beta_j}^{s_{j,n}}},$$

where $r_{j,n} = \text{ord}(\varphi_n, b_{\alpha_j})$ and $s_{j,n} = \text{ord}(\psi_n, b_{\beta_j})$ for all $n \in Z_j$.

We note first that the proof of Theorem 6.1 (or Theorem 4.2 in [5]), as pointed out by the authors, is based on the following identity:

(6.1) $$\#\{n : \zeta_n(\alpha_j) = \xi_n(\beta_j) = 0, -\infty < n < \infty\} = m_j,$$

where $j \geq 1$, $(\alpha_j, \beta_j) \in Z$ and m_j is the minimum number required to represent the quotient module $\mathcal{N}_j$. However, the above equalities does not hold in general, and hence Theorem 6.1 is also incorrect. The next example demonstrates that the above equality and Theorem 6.1 are incorrect.

Let $\{a_m\}_{m=-\infty}^{\infty}$ and $\{c_m\}_{m=-\infty}^{\infty}$ be a pair of sequences of points in $\mathbb{D}$ such that

$$\sum_{m=-\infty}^{\infty} (1 - |a_m|), \ \sum_{m=-\infty}^{\infty} (1 - |c_m|) < \infty,$$

and all elements are distinct but $a_k = a = a_{k+3}$ and $c_k = c = c_{k+2}$ for some fixed $k \in \mathbb{Z}$. Consider the following sequences of Blaschke products $\Phi = \{\varphi_m\}_{m \in \mathbb{Z}}$ and $\Psi = \{\psi_m\}_{m \in \mathbb{Z}}$,

where

$$\varphi_m := \prod_{j=m}^{\infty} b_{a_j} \text{ and } \psi_m := \prod_{j=-\infty}^{m} b_{c_j}. \qquad (m \in \mathbb{Z})$$

Notice that

$$\zeta_m = \varphi_m/\varphi_{m+1} = b_{a_m} \text{ and } \xi_m = \psi_m/\psi_{m-1} = b_{c_m}. \quad (m \in \mathbb{Z})$$

Furthermore we notice that if $\alpha_j = a$ and $\beta_j = c$, then

$$\#\{m : \zeta_m(a) = \xi_m(c) = 0, m \in \mathbb{Z}\} = \#\{m : b_a|b_{a_m}, b_c|b_{c_m}, m \in \mathbb{Z}\} = \#\{k\} = 1,$$

whereas

$$Z_j = \{k, k+1, k+2, k+3\},$$

and

(6.2) $N_j = \mathcal{Q}_{b_a{}^2} \otimes \mathcal{Q}_{b_c} \vee \mathcal{Q}_{b_a} \otimes \mathcal{Q}_{b_c} \vee \mathcal{Q}_{b_a} \otimes \mathcal{Q}_{b_c{}^2} \vee \mathcal{Q}_{b_a} \otimes \mathcal{Q}_{b_c{}^2} = \mathcal{Q}_{b_a{}^2} \otimes \mathcal{Q}_{b_c} \vee \mathcal{Q}_{b_a} \otimes \mathcal{Q}_{b_c{}^2}.$

The above identity shows that m_j has to be 2, and hence (6.1) is not correct.

   Also note that for any $(\alpha_j, \beta_j) \in Z$,

$$\#\{m : \zeta_m(\alpha_j) = \xi_m(\beta_j) = 0, m \in \mathbb{Z}\} \leq 1.$$

Therefore, by Theorem 6.1, the co-rank of the quotient module $\mathcal{Q}_{(\Phi,\Psi)}$ is 1.
However, since the the co-rank of $\mathcal{N}_j = 2$ (follows from (6.2)), by Theorem 5.1 (or by Theorem 4.1 in [5]) the co-rank of $\mathcal{Q}_{(\Phi,\Psi)}$ is at least 2. This shows that the formulation of Theorem 4.2 in [5] is also incorrect.

   On the other hand, one can easily calculate, using the formula in Theorem 5.4, that

$$\#\tilde{Z}(b_a, b_c) = 2,$$

and for any other $(b_{a_i}, b_{c_j}) \in \Lambda$

$$\#\tilde{Z}(b_{a_i}, b_{c_j}) \leq 1.$$

Consequently, the co-rank of $\mathcal{Q}_{(\Phi,\Psi)}$ is precisely 2.

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