PRODUCTS OF TWO ORTHOGONAL PROJECTIONS

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Abstract. We study operators that are products of two orthogonal projections. Our results complement some of the classical results of Crimmins and von Neumann. Particular emphasis has been given to projections associated with inner functions defined on the polydisc.

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1. Introduction

Elements from an algebra that are finite products of special elements from the same algebra are of general significance. One of the more specific cases would be the ring of all square matrices or the ring of all bounded linear operators acting on a Hilbert space $\mathcal{H}$, which we denote by $\mathcal{B}(\mathcal{H})$. The present investigation focuses on operators that are products of pairs of projections on Hilbert spaces. Here, all projections are orthogonal projections, and all Hilbert spaces are separable and over $\mathbb{C}$. Technically, an operator $P \in \mathcal{B}(\mathcal{H})$ is a *projection* if

$$P = P^* = P^2.$$

And, our aim is to study operators $T \in \mathcal{B}(\mathcal{H})$ such that

$$T = P_1 P_2,$$

for some projections P_1 and P_2 in $\mathcal{B}(\mathcal{H})$. Operators that admit the above factorizations have been examined in multiple contexts. Aronszajn, Browder, Dixnier, Kakutani, and Weiner are among the many more eminent names who have contributed to illuminate this subject (see [13, 22] for a thorough historical description). Two additional notable contributions relevant to our work are von Neumann's formula for iterated products of projections and Crimmins' analysis of products of two projections. Let us first revisit two characterizations from the list

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provided by Crimmins [5, 22] : An operator $T \in \mathcal{B}(\mathcal{H})$ is the product of two projections if and only if

$$T^2 = TT^*T,$$

if and only if

$$T = P_{\overline{\text{ran}T}} P_{\overline{\text{ran}T^*}}.$$

Given a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, we denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{S}$ by $P_{\mathcal{S}}$. The preceding factorization of T is referred to as the *canonical factorization* and will be crucial in the subsequent discussion. We present somewhat independent proofs of the above equivalence properties. Additionally, we offer the following new characterization: $T \in \mathcal{B}(\mathcal{H})$ is a product of two projections if and only if

$$TT^* = TP_{\overline{\text{ran}T^*}}.$$

Before turning to von Neumann's perspective, we pause for Nagy-Foias and Langer's viewpoint on contractive operators acting on Hilbert spaces. This is relevant because, if $T \in \mathcal{B}(\mathcal{H})$ is a product of two projections, then T is necessarily a contraction (that is, $\|Th\| \leq \|h\|$ for all $h \in \mathcal{H}$). Consequently, the theory of contractions [17]— an influential and impactful theory—applies to operators that are products of two projections. This paper presents some new perspectives on the structure of contractions that are products of two projections. Given a contraction $T \in \mathcal{B}(\mathcal{H})$, the celebrated *canonical decompositions* of T is the orthogonal decomposition of closed subspaces

$$\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_{cnu},$$

where $\mathcal{H}_u$ and $\mathcal{H}_{cnu}$ reduce T, and $T|_{\mathcal{H}_u}$ is unitary and $T|_{\mathcal{H}_{cnu}}$ is cnu (that is, $T|_{\mathcal{H}_{cnu}}$ on $\mathcal{H}_{cnu}$ does not have unitary summand). Moreover, the unitary summand $\mathcal{H}_u$ is given by

$$\mathcal{H}_u = \{h \in \mathcal{H} : \|T^m h\| = \|T^{*m} h\| = \|h\|, m \in \mathbb{N}\}.$$

This decomposition is due to Nagy-Foias and Langer [10, 18]. The spaces $\mathcal{H}_u$ and $\mathcal{H}_{cnu}$ are designated as the *unitary part* and the *cnu* part of T, respectively.

Now we assume that $T = P_1 P_2$ for some projections P_1 and P_2 in $\mathcal{B}(\mathcal{H})$. Theorems 3.1 and 3.2 give a concrete description of the canonical decomposition of such operators: The unitary and cnu parts of T are given by

$$\mathcal{H}_u = \ker(I - P_1P_2) = \text{ran}P_1 \cap \text{ran}P_2,$$

and

$$\mathcal{H}_{cnu} = \ker T \bigvee \ker T^*,$$

respectively, where $\bigvee$ denotes the span closure of subspaces. The above results also complement the classical von Neumann's alternating orthogonal projection formula. In fact, if P_1 and P_2 are projections, then the von Neumann's alternating orthogonal projection formula tells us that

$$\text{SOT-}\lim_{m\to\infty} T^m = P_{\text{ran}P_1 \cap \text{ran}P_2}.$$

From this perspective and in view of our results outlined above, we further assert that (see Theorem 4.1 for more details)

$$\text{SOT-}\lim_{m\to\infty} T^m = P_{\mathcal{H}_u},$$

where $\mathcal{H}_u$ is the unitary part of the canonical decomposition of the contraction $T = P_1P_2$. In the context of canonocial decomposition, we additionally include the following asymptotic properties of the cnu part of T (see Theorem 4.2):

$$\text{SOT} - \lim_{m\to\infty} (T|_{\mathcal{H}_{cnu}})^{*m} = \text{SOT} - \lim_{m\to\infty} (T|_{\mathcal{H}_{cnu}})^{m} = 0.$$

In view of the popular notations [17], we write the above asymptotic properties simply as:

$$T^*|_{\mathcal{H}_{cnu}}, T|_{\mathcal{H}_{cnu}} \in C_{\cdot 0}.$$

Now we look into a more concrete infinite dimensional Hilbert space, $H^2(\mathbb{D}^n)$, the Hardy space of square summable analytic functions defined on the polydisc $\mathbb{D}^n$. The commutative Banach algebra of all bounded analytic functions on $\mathbb{D}^n$ is denoted by $H^\infty(\mathbb{D}^n)$. Therefore, we have

$$H^\infty(\mathbb{D}^n) = \{\varphi \in \text{Hol}(\mathbb{D}^n) : \|\varphi\|_\infty := \sup_{z\in\mathbb{D}^n} |\varphi(z)| < \infty\}.$$

For each $\varphi \in H^\infty(\mathbb{D}^n)$, the analytic Toeplitz operator T_φ on $H^2(\mathbb{D}^n)$ is defined by

$$T_\varphi f = \varphi f,$$

for all $f \in H^2(\mathbb{D}^n)$. One knows that $T_\varphi \in \mathcal{B}(H^2(\mathbb{D}^n))$ for all $\varphi \in H^\infty(\mathbb{D}^n)$. The function $\varphi \in H^\infty(\mathbb{D}^n)$ is said to be *inner* if $|\varphi| = 1$ a.e. on $\mathbb{T}^n$ in the sense of radial limits. It is known that φ is inner if and only if T_φ is an isometry on $H^2(\mathbb{D}^n)$. In this case, $T_\varphi T_\varphi^*$ is an orthogonal projection onto $\varphi H^2(\mathbb{D}^n)$. In other words, we have

$$T_\varphi T_\varphi^* = P_{\varphi H^2(\mathbb{D}^n)}.$$

A projection $P \in \mathcal{B}(H^2(\mathbb{D}^n))$ is said to be an *inner projection* [6] if there exists an inner function $\varphi \in H^\infty(\mathbb{D}^n)$ such that $P = T_\varphi T_\varphi^*$. Equivalently, we have

$$P = P_{\varphi H^2(\mathbb{D}^n)}. \tag{1.1}$$

Here our goal is to classify operators that are products of two inner projections. Of course, our aim is to obtain an analytic answer to this problem. To achieve this, for each nonzero operator $T \in \mathcal{B}(H^2(\mathbb{D}^n))$, we associate an inner function that acts as the least common multiple within a suitable class of inner functions. Our approach is as follows: Given a bounded linear operator T on $H^2(\mathbb{D}^n)$, define the set (which appears to be a nonemtpy set)

$$\mathcal{I}_T = \{\varphi \in H^\infty(\mathbb{D}^n) : \varphi \text{ is inner, and } \text{ran}T \subseteq \varphi H^2(\mathbb{D}^n)\}.$$

Denote by $\varphi_T \in H^\infty(\mathbb{D}^n)$ the unique inner function, which is the lcm (that is, the least common multiple) of $\mathcal{I}_T$. Therefore (see Lemma 5.2 for more details)

$$\varphi_T := \text{lcm}\mathcal{I}_T.$$

Under this notation, we also have

$$\varphi_{T^*} = \text{lcm}\mathcal{I}_{T^*},$$

In the spirit of canonical factorizations, Theorem 5.3 yields an analytic characterization of products of two inner projections: Let $T \in \mathcal{B}(H^2(\mathbb{D}^n))$ be a nonzero operator. Then T is a product of two inner projections if and only if

$$T = P_{\varphi_T H^2(\mathbb{D}^n)} P_{\varphi_{T^*} H^2(\mathbb{D}^n)}.$$

Moreover, we observe in Corollary 3.3 that if $T \in \mathcal{B}(H^2(\mathbb{D}^n))$ is a cnu contraction, then T cannot be expressed as a product of two inner projections.

We also obtain similar results in the context of model spaces. A *model space* $\mathcal{Q}_\varphi$ is defined as the closed subspace obtained by quotienting $H^2(\mathbb{D}^n)$ by an inner function $\varphi \in H^\infty(\mathbb{D}^n)$ in the sense that

$$\mathcal{Q}_\varphi = H^2(\mathbb{D}^n)/\varphi H^2(\mathbb{D}^n).$$

We refer to $P_{\mathcal{Q}_\varphi}$ the projection onto the model space $\mathcal{Q}_\varphi$ as a *model projection.* We prove the following (see Theorem 6.3): Let $T \in \mathcal{B}(H^2(\mathbb{D}^n))$ be a nonzero operator. Then T is a product of two model projections if and only if

$$T = P_{\mathcal{Q}_{\psi_T}} P_{\mathcal{Q}_{\psi_{T^*}}},$$

where

$$\psi_T := \text{gcd}\mathcal{J}_T,$$

and

$$\mathcal{J}_T = \{\varphi \in H^\infty(\mathbb{D}^n) : \varphi \text{ is inner and } \varphi H^2(\mathbb{D}^n) \subseteq \ker T^*\}.$$

We note that, unlike $\mathcal{I}_T$, the set $\mathcal{J}_T$ has the potential to be empty. Therefore, part of the above result also guarantees that the set $\mathcal{J}_T$ remains nonempty. In the final section of this paper, we illustrate some of our results through concrete examples as above.

The remaining part of the paper is arranged as follows: Section 2 primarily focuses on proving some of the well-known results, including those by Crimmins and Sebestyén. However, the more general framework we've provided here also emphasizes the possibilities for the larger problem. In Section 3, we explain the classical concept of canonical decompositions of contractions and relate them to our theory of the product of two projections. This provides a precise description of the orthogonal decompositions applicable to the operators under consideration. Section 4 establishes an additional connection between our theory and a classical result, providing new insights into von Neumann's alternating orthogonal projection formula. In Section 5, we examine our underlying problem from a specific standpoint. We apply our theory to projections based on inner functions defined on the Hardy space over the polydisc. We continue to work on concrete situations in Section 6, where the projections are considered to be model projections. The final section, Section 7, illustrates some of our results on products of two projections using Blaschke products.

2. Crimmins' perspective

In this section, we prove some results on operators that can be represented as products of two projections. As stated in the very beginning, this is a problem that has been triggered by many. We take Crimmins' point of view and dedicate a part of the section to reproving his results. However, we address this within a marginally wider framework, anticipating more realistic uses in the structure of operators that can be expressed as products of operators.

Let us start with a lemma that states that operators with projection factors on the left side are obligated to have a canonical choice for the left projection factors.

Lemma 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T = PX$ for some projection $P \in \mathcal{B}(\mathcal{H})$ and linear operator $X \in \mathcal{B}(\mathcal{H})$, then*

$$T = P_{\overline{ranT}} X.$$

Proof. Since $T = PX$, by Douglas' lemma, it follows that $\overline{\text{ran}T} \subseteq \text{ran}P$. Then

$$PP_{\overline{\text{ran}T}} = P_{\overline{\text{ran}T}} = P_{\overline{\text{ran}T}}P.$$

Now, $T = P_{\overline{\mathrm{ran}T}}T$, and hence $T = P_{\overline{\mathrm{ran}T}}PX$. This implies $T = P_{\overline{\mathrm{ran}T}}X$, which completes the proof of the lemma. ∎

As a result, operators admitting the left and right side projection factors can be represented as follows:

Proposition 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$. If $T = P_1XP_2$ for some projections $P_1, P_2 \in \mathcal{B}(\mathcal{H})$ and linear operator $X \in \mathcal{B}(\mathcal{H})$, then*

$$T = P_{\overline{ranT}}XP_{\overline{ranT^*}}.$$

Proof. By Lemma 2.1, we know that $T = P_{\overline{\mathrm{ran}T}}XP_2$, and hence

$$T^* = P_2X^*P_{\overline{\mathrm{ran}T}}.$$

The proof can now be obtained by applying Lemma 2.1 one more time to T^* and the above factorization of T^*. ∎

In particular, if $X = I$, then we have the following classification of operators as products of two projections:

Corollary 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is a product of two projection if and only if*

$$T = P_{\overline{ranT}}P_{\overline{ranT^*}}.$$

Crimmins is credited with this result [5, page 1595]. Here, we draw the conclusion from a rather broad perspective.

We are now presenting a result of Sebestyén [16, Corollary 2]. The present proof, compared to [16], is more elementary (the sufficient part) and is also presented for completeness. The result is clearly in line with Douglas' range inclusion theorem.

Proposition 2.4. *Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$. Then there exists a projection $P \in \mathcal{B}(\mathcal{H})$ such that*

$$T_1 = T_2P,$$

if and only if

$$T_1T_1^* = T_2T_1^*.$$

Proof. If $T_1 = T_2P$ for some projection $P \in \mathcal{B}(\mathcal{H})$, then

$$T_1T_1^* = T_2PPT_2^* = T_2PT_2^* = T_2T_1^*.$$

Conversely, assume that $T_1T_1^* = T_2T_1^*$, then $T_1f = T_2f$ for all $f \in \overline{\mathrm{ran}T_1^*}$. Pick $f \in \mathcal{H}$ and write

$$f = P_{\ker T_1}f \oplus P_{\overline{\mathrm{ran}T_1^*}}f.$$

Then $T_1f = T_1P_{\overline{\mathrm{ran}T_1^*}}f$. Since $P_{\overline{\mathrm{ran}T_1^*}}f \in \overline{\mathrm{ran}T_1^*}$, we have

$$T_1f = T_2P_{\overline{\mathrm{ran}T_1^*}}f,$$

which completes the proof by choosing the projection $P = P_{\overline{\mathrm{ran}T_1^*}}$. ∎

In addition to its more elementary nature, the proof above also adds to our understanding, revealing that the projection P is given by

$$P = P_{\overline{\mathrm{ran}T_1^*}},$$

whenever $T_1 = T_2P$.

The subsequent corollary is yet another classification of operators that are products of two projections. The result is in line with Crimmins and Sebestyén, yet it offers some new perspective.

Corollary 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is a product of two projections if and only if*

$$TT^* = P_{\overline{ranT}} T^*.$$

Proof. If T is a product of two projections, then Corollary 2.3 implies $T = P_{\overline{\operatorname{ran} T}} P_{\overline{\operatorname{ran} T^*}}$. We now compute

$$TT^* = P_{\overline{\operatorname{ran} T}} P_{\overline{\operatorname{ran} T^*}} P_{\overline{\operatorname{ran} T}} = P_{\overline{\operatorname{ran} T}} (P_{\overline{\operatorname{ran} T^*}} P_{\overline{\operatorname{ran} T}}) = P_{\overline{\operatorname{ran} T}} T^*.$$

For the converse direction, assume that $TT^* = P_{\overline{\operatorname{ran} T^*}} T^*$. The conclusion now follows directly from the sufficient part of Proposition 2.4. ■

Next, we assemble all the results obtained so far, along with a new one.

Theorem 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$. The following are equivalent:*

(1) *T is a product of two projections.*
(2) *$T = P_{\overline{ranT}} P_{\overline{ranT^*}}$.*
(3) *$TT^* = TP_{\overline{ranT}}$.*
(4) *$TT^*T = T^2$.*

Proof. The equivalences (1) ⇔ (2) and (1) ⇔ (3) are simply Corollary 2.3 and Corollary 2.5, respectively. Next, assume (1), that is, T is a product of two projections, say, $T = PQ$. Then

$$TT^*T = PQ(QP)PQ = PQPQ = T^2.$$

This proves (4). To prove (4) ⇒ (3), we assume that $TT^*T = T^2$, equivalently, $T^*TT^* = T^{*2}$. Then

$$0 = T^*(TT^* - T^*) = T^*(P_{\overline{\operatorname{ran} T}} + (I - P_{\overline{\operatorname{ran} T}}))(TT^* - T^*) = T^* P_{\overline{\operatorname{ran} T}} (TT^* - T^*),$$

implies

$$(TT^* - TP_{\overline{\operatorname{ran} T}})T = 0.$$

This says $TT^* = TP_{\overline{\operatorname{ran} T}}$ on $\overline{\operatorname{ran} T}$. Since the identity holds trivially on $(\overline{\operatorname{ran} T})^\perp = \ker T^*$, we conclude that $TT^* = TP_{\overline{\operatorname{ran} T}}$. This completes the proof of the theorem. ■

Crimmins once again receives credit for the equivalence of (1) and (4) [5, page 1595]. The present proof is once again different. In this context, we also refer the reader to [5, Theorem 3.1]. The same paper contains results (cf. [5, Corollary 3.8]) related to parametrizations and the uniqueness of pairs of projections whose products yield a given operator.

3. Nagy-Foias and Langer's perspective

Note that if a bounded linear operator acting on a Hilbert space is a product of two projections, then it is necessarily a contraction. Consequently, the theory of contractions, widely recognized as being extremely rich, applies in this situation. This is a vast domain, and a thorough application of the theory of contractions to this particular class of operators and vice versa may not be entirely transparent. This requires a more in-depth investigation, which ought to be carried out. However, in this section, we will address a fundamental characteristic of the product of projections via the structure of contractions. To do that, we revisit the classic Nagy-Foias and Langer orthogonal decompositions of contractions [10, 18].

Let $T \in \mathcal{B}(\mathcal{H})$ be a contraction. Then there is a unique orthogonal decomposition $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_{cnu}$ such that both $\mathcal{H}_u$ and $\mathcal{H}_{cnu}$ reduce T, and $T|_{\mathcal{H}_u}$ is unitary whereas $T|_{\mathcal{H}_{cnu}}$ is completely non-unitary (cnu in short). This amounts to saying that $T|_{\mathcal{H}_{cnu}}$ does not have unitary summand (see Section 1). As a result, we get the diagonal decomposition:

$$T = \begin{bmatrix} T|_{\mathcal{H}_u} & 0 \\ 0 & T|_{\mathcal{H}_{cnu}} \end{bmatrix},$$

on $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_{cnu}$. Moreover, we have the representation of the unitary part $\mathcal{H}_u$ as [17, Chapt. 1, Theorem 3.2]

$$\mathcal{H}_u = \{h \in \mathcal{H} : \|T^m h\| = \|T^{*m} h\| = \|h\|, m \in \mathbb{N}\}.$$

This decomposition is popularly known as the *Nagy-Foiaş and Langer* or the *canonical decomposition* of contractions and stands as one of the most basic structures of contractions. In the following, we connect this with our theory of the product of two projections. For simplicity of notation, given projections P_1 and P_2, define the closed subspace $\mathcal{S}_{P_1 P_2}$ as

$$\mathcal{S}_{P_1 P_2} = \text{ran} P_1 \cap \text{ran} P_2. \tag{3.1}$$

Theorem 3.1. *Let P_1 and P_2 be projections defined on some Hilbert space $\mathcal{H}$. Then*

$$\mathcal{H}_u = \ker(P_1 P_2 - I) = \mathcal{S}_{P_1 P_2},$$

where $\mathcal{H}_u$ is the unitary part of the contraction $P_1 P_2$.

Proof. First, we claim that

$$\mathcal{S}_{P_1 P_2} = \{f \in \mathcal{H} : \|P_1 P_2 f\| = \|f\|\}. \tag{3.2}$$

Let $f \in \mathcal{H}$, and assume that $\|P_1 P_2 f\| = \|f\|$. Then

$$\|f\| = \|P_1 P_2 f\| \leq \|P_2 f\| \leq \|f\|.$$

Subsequently, each of the preceding inequities attains equality. Therefore

$$\|P_2 f\| = \|P_1 P_2 f\| = \|f\|.$$

Since P_2 is a projection, $\|P_2 f\| = \|f\|$ implies that $f \in \text{ran} P_2$, or equivalently

$$P_2 f = f.$$

Again, $\|P_2 f\| = \|P_1 P_2 f\|$ implies that $P_2 f \in \text{ran} P_1$. Therefore, $P_2 f = P_1 P_2 f$ and hence

$$P_1 f = f.$$

This proves that $f \in \text{ran} P_1 \cap \text{ran} P_2$. On the other hand, if $f \in \text{ran} P_1 \cap \text{ran} P_2$, then

$$P_1 P_2 f = f,$$

and evidently $\|P_1 P_2 f\| = \|f\|$. This completes the proof of the identity (3.2). Now we turn to the proof of the main body of the theorem. Let $f \in \mathcal{H}_u$. In particular, $\|P_1 P_2 f\| = \|f\|$, and then, (3.2) implies

$$f \in \text{ran} P_1 \cap \text{ran} P_2.$$

Therefore, $P_1P_2f = f$, and we conclude that $f \in \ker(P_1P_2 - I)$. This proves that $\mathcal{H}_u \subseteq \ker(P_1P_2 - I)$ and $\mathcal{H}_u \subseteq \mathcal{S}_{P_1P_2}$. For the reverse inclusion, pick $g \in \ker(P_1P_2 - I)$. Then $\|P_1P_2g\| = \|g\|$ and (3.2) together imply that

$$g \in \text{ran}P_1 \cap \text{ran}P_2,$$

and hence $P_2P_1g = g$. It follows that

$$P_1P_2g = (P_1P_2)^*g = g,$$

and hence $g \in \mathcal{H}_u$. This proves that $\ker(P_1P_2 - I) \subseteq \mathcal{S}_{P_1P_2}$ and $\mathcal{S}_{P_1P_2} \subseteq \mathcal{H}_u$. Therefore, $\mathcal{H}_u = \ker(P_1P_2 - I) = \mathcal{S}_{P_1P_2}$, which completes the proof of the theorem. ■

In particular, the cnu part of P_1P_2 is given by

$$\mathcal{H}_{cnu} = (\ker(P_1P_2 - I))^\perp.$$

However, some more can be said, and that is the content of the following result. Recall again that $\bigvee$ denotes the span closure of subspaces.

Theorem 3.2. *If $T \in \mathcal{B}(\mathcal{H})$ is a product of two projections, then*

$$\mathcal{H}_{cnu} = \ker T \bigvee \ker T^*.$$

Proof. We start with a general observation of such T as follows:

$$(T^* - I)\overline{T\mathcal{H}} \subseteq \ker T. \tag{3.3}$$

To show this, it suffices for us to prove that $T(T^* - I)Tf = 0$ for all $f \in \mathcal{H}$. Fix such $f \in \mathcal{H}$. Since T is a product of two projections, Theorem 2.6 (the one by Crimmins) implies $TT^*T = T^2$, and consequently

$$T(T^* - I)Tf = TT^*Tf - T^2f = 0.$$

This proves the inclusion in (3.3). We will now move on to the core part of the proof for the theorem. Since $\mathcal{H}_{cnu} = (\ker(T - I))^\perp$, as a general fact, we have

$$\mathcal{H}_{cnu} = \overline{\text{ran}(T^* - I)}.$$

Fix a vector $f \in \mathcal{H}$. In view of the decomposition $\mathcal{H} = \ker T^* \oplus \overline{\text{ran}T}$, we write $f = f_1 \oplus f_2 \in \ker T^* \oplus \overline{\text{ran}T}$. We focus on the half part of the decomposition that $f_2 \in \overline{\text{ran}T}$. Then (3.3) implies that

$$T(T^* - I)f_2 = 0.$$

In view of this, we write

$$\begin{aligned}(T^* - I)f &= (T^* - I)(f_1 \oplus f_2)\\ &= -f_1 + (T^* - I)f_2\\ &\in \ker T^* + \ker T.\end{aligned}$$

which proves that $\mathcal{H}_{cnu} \subseteq \ker T \bigvee \ker T^*$. For the reverse inclusion, we pick $g_1 \in \ker T^*$ and $g_2 \in \ker T$. We will prove that $g_1 + g_2 \perp \mathcal{H}_u$, which, in view of $\mathcal{H}_u^\perp = \mathcal{H}_{cnu}$, will lead to the conclusion. To this end, pick $g \in \mathcal{H}_u$. By Theorem 3.1, we know that $\mathcal{H}_u = \ker(T - I)$, and consequently

$$Tg = g.$$

Since $T^*g_1 = 0$, it follows that

$$\langle g_1 + g_2, g\rangle = \langle g_1 + g_2, Tg\rangle = \langle T^*g_1, g\rangle + \langle T^*g_2, g\rangle = \langle g_2, Tg\rangle = \langle g_2, g\rangle,$$

that is

$$\langle g_1 + g_2, g\rangle = \langle g_2, g\rangle.$$

Again, $Tg = g$ implies that $T^*Tg = T^*g$. On the other hand, since $T|_{\mathcal{H}_u} : \mathcal{H}_u \to \mathcal{H}_u$ is a unitary operator, and $g \in \mathcal{H}_u$, it follows that $T^*Tg = g$. When we combine this with the identity $T^*Tg = T^*g$, we find that

$$T^*g = g.$$

Then $\langle g_1 + g_2, g\rangle = \langle g_2, g\rangle$ implies

$$\langle g_1 + g_2, g\rangle = \langle g_2, T^*g\rangle = \langle Tg_2, g\rangle = 0,$$

as $g_2 \in \ker T$. This shows $g_1 + g_2 \perp \mathcal{H}_u$ and completes the proof of the theorem. ∎

Given this and Theorem 3.1, the canonical decompositions for contractions that are products of two projections are relatively concrete.

We end this section with an application of Theorem 3.1 to a concrete situation, like inner projections defined on the polydisc (see the definition in (1.1)). Let $T \in \mathcal{B}(H^2(\mathbb{D}^n))$ be a nonzero operator. Suppose T is a product of two inner projections, that is, there exist two inner functions φ and ψ in $H^\infty(\mathbb{D}^n)$ such that

$$T = P_{\varphi H^2(\mathbb{D}^n)} P_{\psi H^2(\mathbb{D}^n)}.$$

Theorem 3.1 yields

$$\ker(T - I) = H^2(\mathbb{D}^n)_u = \varphi H^2(\mathbb{D}^n) \cap \psi H^2(\mathbb{D}^n). \tag{3.4}$$

In terms of operators not being a product of two inner projections, the following specific observation is definitive:

Corollary 3.3. *If $T \in \mathcal{B}(H^2(\mathbb{D}^n))$ is a cnu contraction, then T cannot be expressed as a product of two inner projections.*

Proof. If possible, assume that T is a product of two inner projections. Since T is cnu, we have that $H^2(\mathbb{D}^n)_u = \{0\}$. By (3.4), we conclude

$$\{0\} = \varphi H^2(\mathbb{D}^n) \cap \psi H^2(\mathbb{D}^n),$$

which is a clear contradiction as $\varphi\psi \in H^\infty(\mathbb{D}^n)$ is an inner function and

$$\varphi\psi \in \varphi H^2(\mathbb{D}^n) \cap \psi H^2(\mathbb{D}^n).$$

This completes the proof of the corollary. ∎

In a more general way, operators with unitary parts cannot be expressed as products of inner projections.

4. von Neumann's perspectives

When discussing products of two projections, one immediately draws a connection with the classical von Neumann's alternating orthogonal projections formula [19]. This formula relates to the limit of powers of products of two projections. We begin with recalling a notation. Given projections P_1 and P_2 defined on some Hilbert space $\mathcal{H}$, the closed subspace $\mathcal{S}_{P_1P_2}$ is the common range space defined by (see (3.1))

$$\mathcal{S}_{P_1P_2} = \text{ran}P_1 \cap \text{ran}P_2.$$

The iterated product of projections then satisfies the following intriguing property, which was proved by von Neumann in 1933 [19]:

$$\text{SOT-}\lim_{m\to\infty}(P_1P_2)^m = P_{\mathcal{S}_{P_1P_2}}, \tag{4.1}$$

where SOT refers to the strong operator topology. This result, known as von Neumann's alternating orthogonal projections formula, has historically been associated with notable mathematicians, including Aronszajn [1], Browder [3], Kakutani [9], Nakano [12], and Wiener [21], among others. We direct the reader to [13] for a comprehensive account of the development and a proof. There is more to say in this line, and it will be crucial to our understanding. In fact, we further have (see the proof of Theorem 1 in [13])

$$\mathcal{S}_{P_1P_2} = \ker(I - P_2P_1).$$

Now we bring in Theorem 3.1, which tells us, in addition, that

$$\mathcal{H}_u = \ker(I - P_2P_1) = \mathcal{S}_{P_1P_2}.$$

This proves the following, which appears to be new information to von Neumann's alternating projection formula:

Theorem 4.1. *Let $P_1, P_2 \in \mathcal{B}(\mathcal{H})$ be two projections. Then*

$$SOT\text{-}\lim_{m\to\infty}(P_1P_2)^m = P_{\mathcal{H}_u},$$

where $\mathcal{H}_u$ is the unitary part of the canonical decomposition of the contraction P_1P_2.

Next, we introduce another significant class of contractions that serves a profound role in the theory of operators. Let $T \in \mathcal{B}(\mathcal{H})$ be a contraction. If in addition

$$\text{SOT} - \lim_{m\to\infty} T^{*m} = 0,$$

then we say that T is a $C_{\cdot 0}$ contraction. If both T and T^* are in $C_{\cdot 0}$, then we simply write [17, page 72]

$$T \in C_{00}.$$

This class of operators is interesting as they can be modeled in a more tangible manner by adhering to the classical theory of Sz.-Nagy and Foias. Evidently, if $T \in \mathcal{B}(\mathcal{H})$ is a C_{00} contraction, then its unitary part is zero, that is, $\mathcal{H}_u = \{0\}$. In the following, we focus on contractions that are products of pairs of projections. In fact, in the following result, we have added yet another new information to von Neumann's iterated products of projections:

Theorem 4.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be a product of two projections. Then*

$$T|_{\mathcal{H}_{cnu}} \in C_{00}.$$

Proof. Assume without loss of generality that $T \in \mathcal{B}(\mathcal{H})$ is a nonzero operator. Suppose $T = P_1P_2$ for some projections P_1 and P_2 in $\mathcal{B}(\mathcal{H})$. By a part of Theorem 2.6 (which is due to Crimmins), we know that

$$TT^*T = T^2.$$

Consider the canonical decomposition of T as $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_{cnu}$. We know that $T_{cnu} := T|_{\mathcal{H}_{cnu}}$ is a cnu contraction. Since $\mathcal{H}_{cnu}$ reduces T, it follows that

$$T = \begin{bmatrix} T_u & 0 \\ 0 & T_{cnu} \end{bmatrix},$$

where $T_u := T|_{\mathcal{H}_u}$ is the unitary part of T. This immediately implies that

$$T_{cnu}(T_{cnu})^* T_{cnu} = T_{cnu}^2.$$

Applying Theorem 2.6 again to T_{cnu} on $\mathcal{H}_{cnu}$, we conclude that it is the product of two projections. In particular, T_{cnu} on $\mathcal{H}_{cnu}$ is a contraction. We write the corresponding canonical decomposition of T_{cnu} as

$$\mathcal{H}_{cnu} = \tilde{\mathcal{H}}_u \oplus \tilde{\mathcal{H}}_{cnu},$$

where $\tilde{\mathcal{H}}_u$ is the unitary part of T_{cnu}. Theorem 4.1 now tells us that

$$\text{SOT-}\lim_{m\to\infty} T_{cnu}^m = P_{\tilde{\mathcal{H}}_u}.$$

However, T_{cnu} is a cnu contraction, and hence, its unitary part is trivial, that is,

$$\tilde{\mathcal{H}}_u = \{0\},$$

which says that $T|_{cnu} \in C_{\cdot 0}$. Finally, the fact that T is a product of two projections implies that T^* is also a product of two projections, thereby proving that $T|_{cnu}$ is in $C_{\cdot 0}$. This completes the proof of the theorem. ■

Finally, recall from Theorem 3.2, given $T \in \mathcal{B}(\mathcal{H})$, which is a product of two projections, we have

$$\mathcal{H}_{cnu} = \ker T \bigvee \ker T^*.$$

Therefore, we have somewhat clear picture of the cnu part $T|_{\mathcal{H}_{cnu}}$ of T. This observation shows significant potential for exploring the structure of this class of operators. However, we save this topic and direction for future work.

5. Inner projections

This section presents a specific scenario of the product of projections where one might expect analytic solutions. The objective is to relate the concept of inner functions to projections, which was first explored under the name of inner projections in [6]. We define an *invariant subspace* of $H^2(\mathbb{D}^n)$ as a nonzero closed subspace $\mathcal{S} \subseteq H^2(\mathbb{D}^n)$ satisfying $z_i\mathcal{S} \subseteq \mathcal{S}$, or, equivalently

$$T_{z_i}\mathcal{S} \subseteq \mathcal{S},$$

for all $i = 1, \ldots, n$. Let $\mathcal{S} \subseteq H^2(\mathbb{D}^n)$ be an invariant subspace. For each $i = 1, \ldots, n$, we define a bounded linear operator $R_i \in \mathcal{B}(\mathcal{S})$ by

$$R_i = T_{z_i}|_{\mathcal{S}}.$$

It is easy to see that R_i is an isometry, and consequently, $(R_1, \ldots, R_n)$ is an n-tuple of commuting isometries on $\mathcal{S}$. We say that $\mathcal{S}$ is *doubly commuting* or a *doubly commuting invariant subspace* if

$$R_i^* R_j = R_j R_i^*,$$

for all $i \neq j$. When discussing doubly commuting invariant subspaces, we always assume that $n > 1$. Reminiscent of the Beurling characterization of invariant subspaces of $H^2(\mathbb{D})$, the following connects doubly commuting invariant subspaces with inner functions [11]: An invariant subspace $\mathcal{S} \subseteq H^2(\mathbb{D}^n)$ is doubly commuting if and only if there exists an inner function $\varphi \in H^\infty(\mathbb{D}^n)$ such that

$$\mathcal{S} = \varphi H^2(\mathbb{D}^n).$$

This is also equivalent to the factorization of the projection $P_{\mathcal{S}}$:

$$P_{\mathcal{S}} = T_\varphi T_\varphi^*.$$

In view of the above, if $\mathcal{S} = \varphi H^2(\mathbb{D}^n)$ for some inner function $\varphi \in H^\infty(\mathbb{D}^n)$, then, for each $i \neq j$, we compute

$$P_{\mathcal{S}} T_{z_i}^* T_{z_j} P_{\mathcal{S}} = T_\varphi T_\varphi^* (T_{z_i}^* T_{z_j}) T_\varphi T_\varphi^* = T_\varphi T_{z_i}^* (T_\varphi^* T_\varphi) T_{z_j} T_\varphi^*.$$

We know that φ is inner, and hence T_φ is an isometry. This and the doubly commuting condition $T_{z_i}^* T_{z_j} = T_{z_j} T_{z_i}^*$ implies

$$\begin{aligned} P_{\mathcal{S}} T_{z_i}^* T_{z_j} P_{\mathcal{S}} &= T_\varphi T_{z_i}^* T_{z_j} T_\varphi^* \\ &= T_\varphi T_{z_j} T_{z_i}^* T_\varphi^* \\ &= T_{z_j} (T_\varphi T_\varphi^*) T_{z_i}^*, \end{aligned}$$

and hence

$$P_{\mathcal{S}} T_{z_i}^* T_{z_j} P_{\mathcal{S}} = T_{z_j} P_{\varphi H^2(\mathbb{D}^n)} T_{z_i}^*, \tag{5.1}$$

for all $i \neq j$. This well-known identity will be used extensively in what follows.

Let $T \in \mathcal{B}(H^2(\mathbb{D}^n))$. We already have learned from Corollary 2.3 that T is a product of two projections if and only if $T = P_{\overline{\text{ran}T}} P_{\overline{\text{ran}T^*}}$. In this section, we additionally demand that the factors be inner projections, and in exchange we look for an analytic answer. Stated differently, our focus will be on identifying analytic interpretations of the projection factors. For this, we introduce the notion of lcm of bounded linear operators on $H^2(\mathbb{D}^n)$. Given a nonzero operator $T \in \mathcal{B}(H^2(\mathbb{D}^n))$, define the set

$$\mathcal{I}_T = \{\varphi \in H^\infty(\mathbb{D}^n) : \varphi \text{ is inner, and } \text{ran}T \subseteq \varphi H^2(\mathbb{D}^n)\}.$$

Since the constant function $1 \in H^\infty(\mathbb{D}^n)$ is also in $\mathcal{I}_T$ (as, of course, $\text{ran}T \subseteq H^2(\mathbb{D}^n)$), it readily follows that

$$\mathcal{I}_T \neq \emptyset.$$

We need to recall the notion of the lcm of inner functions. In what follows, our index set will always be nonempty, and the collection of functions will never be a singleton zero function.

Definition 5.1. Given a set of inner functions $\{\varphi_\alpha\}_{\alpha\in\Lambda} \subseteq H^\infty(\mathbb{D}^n)$, an inner function $\varphi \in H^\infty(\mathbb{D}^n)$ is said to be the *least common multiple* (or *lcm* in short) of $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ if

(1) φ_α divides φ for all $\alpha \in \Lambda$, and
(2) if φ_α divides an inner function $\psi \in H^\infty(\mathbb{D})$ for all $\alpha \in \Lambda$, then φ also divides ψ.

Clearly, the lcm function φ is unique (if exists), and we simply express it as

$$\varphi = \text{lcm}\{\varphi_\alpha\}_{\alpha\in\Lambda}.$$

Returning to the range of inner projections, we now want to make sure that lcm exists at all times. The result is a straightforward application of Beurling's theorem for the case when $n = 1$, but it necessitates additional work when $n > 1$. We also notice that the lcm of $\{\varphi_\alpha\}_{\alpha\in\Lambda} \subseteq H^\infty(\mathbb{D}^n)$, as stated above, requires that it be an inner function.

Lemma 5.2. *lcm*$\mathcal{I}_T$ *exists for all nonzero* $T \in \mathcal{B}(H^2(\mathbb{D}^n))$.

Proof. Suppose $n = 1$. The proof in this case is well-known and is available in [2, page 21, Proposition 2.3]. Nevertheless, we sketch a proof just to be thorough. Set

$$\mathcal{S}_T = \bigcap_{\varphi \in \mathcal{I}_T} \varphi H^2(\mathbb{D}).$$

Since

$$\{0\} \neq \text{ran} T \subseteq \varphi H^2(\mathbb{D}),$$

for all $\varphi \in \mathcal{I}_T$, it follows that $\mathcal{S}_T \neq \{0\}$. Since $\varphi H^2(\mathbb{D})$ is a closed T_z-invariant subspace of $H^2(\mathbb{D})$ for all $\varphi \in \mathcal{I}_T$, it follows that $\mathcal{S}_T$ is also a closed T_z-invariant subspace of $H^2(\mathbb{D})$. According to the Beurling theorem, there exists an inner function $\varphi \in H^\infty(\mathbb{D})$ such that $\mathcal{S} = \varphi H^2(\mathbb{D})$. It is now easy to see that $\varphi = \text{lcm}\mathcal{I}_T$.
Assume that $n > 1$. In this case, a bit more effort is required. Like in the previous situation, we again define

$$\mathcal{S}_T = \bigcap_{\varphi \in \mathcal{I}_T} \varphi H^2(\mathbb{D}^n).$$

Again, as in the $n = 1$ case, we have that $\mathcal{S}_T$ is a nontrivial closed subspace of $H^2(\mathbb{D}^n)$, and

$$T_{z_i}\mathcal{S}_T \subseteq \mathcal{S}_T,$$

for all $i = 1, \dots, n$. At this stage, though, it is not apparent whether $\mathcal{S}_T$ is doubly commuting; we so claim that it is. To that end, first, we define partial order $\leq$ on $\mathcal{I}_T$ by

$$\varphi \leq \psi \text{ whenever } \varphi H^2(\mathbb{D}^n) \supseteq \psi H^2(\mathbb{D}^n),$$

for all $\varphi, \psi \in \mathcal{I}_T$. It is now easy to see that

$$P_{\mathcal{S}_T} = SOT - \lim_{\varphi \in \mathcal{I}_T} P_{\varphi H^2(\mathbb{D}^n)}.$$

Fix $f \in \mathcal{S}_T$. Note that, by the definition of $\mathcal{S}_T$, we have $f \in \varphi H^2(\mathbb{D}^n)$ for all $\varphi \in \mathcal{I}_T$. Then, for each $i \neq j$, we have

$$\begin{aligned}
R_i^* R_j f &= P_{\mathcal{S}_T} T_{z_i}^* P_{\mathcal{S}_T} T_{z_j} P_{\mathcal{S}_T} f \\
&= P_{\mathcal{S}_T} T_{z_i}^* T_{z_j} f \\
&= \lim_{\varphi \in \mathcal{I}_T} P_{\varphi H^2(\mathbb{D}^n)} T_{z_i}^* T_{z_j} f \\
&= \lim_{\varphi \in \mathcal{I}_T} \Big(P_{\varphi H^2(\mathbb{D}^n)} T_{z_i}^* T_{z_j} P_{\varphi H^2(\mathbb{D}^n)} \Big) f.
\end{aligned}$$

Recall from (5.1) that

$$P_{\varphi H^2(\mathbb{D}^n)} T_{z_i}^* T_{z_j} P_{\varphi H^2(\mathbb{D}^n)} = T_{z_j} P_{\varphi H^2(\mathbb{D}^n)} T_{z_i}^*,$$

for all $\varphi \in \mathcal{I}_T$. Therefore, we obtain

$$\begin{aligned}
R_i^* R_j f &= \lim_{\varphi \in \mathcal{I}_T} \Big(T_{z_j} P_{\varphi H^2(\mathbb{D}^n)} T_{z_i}^* \Big) f \\
&= T_{z_j} \Big(\lim_{\varphi \in \mathcal{I}_T} P_{\varphi H^2(\mathbb{D}^n)} \Big) T_{z_i}^* f \\
&= T_{z_j} P_{\mathcal{S}_T} T_{z_i}^* f.
\end{aligned}$$

On the other hand, we have

$$R_j R_i^* f = T_{z_j} P_{\mathcal{S}_T} T_{z_i}^* f,$$

and hence

$$R_i^* R_j = R_j R_i^*,$$

which implies that $\mathcal{S}_T$ is a doubly commuting invariant subspace. Then, there exists an inner function $\varphi_T \in H^\infty(\mathbb{D}^n)$ such that

$$\mathcal{S}_T = \varphi_T H^2(\mathbb{D}^n),$$

and consequently

$$\varphi_T H^2(\mathbb{D}^n) = \bigcap_{\varphi \in \mathcal{I}_T} \varphi H^2(\mathbb{D}^n). \tag{5.2}$$

Now, $\varphi_T H^2(\mathbb{D}^n) \subseteq \varphi H^2(\mathbb{D}^n)$ for all $\varphi \in \mathcal{I}_T$ implies that φ divides φ_T for all $\varphi \in \mathcal{I}_T$. Moreover, if ψ divides all $\varphi \in \mathcal{I}_T$, then $\psi H^2(\mathbb{D}^n) \subseteq \varphi H^2(\mathbb{D}^n)$ for all $\varphi \in \mathcal{I}_T$. Therefore

$$\psi H^2(\mathbb{D}^n) \subseteq \bigcap_{\varphi \in \mathcal{I}_T} \varphi H^2(\mathbb{D}^n) = \varphi_T H^2(\mathbb{D}^n),$$

implies that φ_T divides ψ. This proves that the lcm of $\mathcal{I}_T$ exists and is given by φ_T, which completes the proof of the lemma. ■

Let $T \in \mathcal{B}(H^2(\mathbb{D}^n))$ be a nonzero operator. In view of Lemma 5.2, we define the unique inner function $\varphi_T \in H^\infty(\mathbb{D}^n)$ by

$$\varphi_T = \text{lcm}\mathcal{I}_T.$$

From here, we note that (as also observed in (5.2) above)

$$\varphi_T H^2(\mathbb{D}^n) = \bigcap_{\varphi \in \mathcal{I}_T} \varphi H^2(\mathbb{D}^n).$$

Given this notation, we also have the following:

$$\varphi_{T^*} = \text{lcm}\mathcal{I}_{T^*},$$

where $\mathcal{I}_{T^*}$ is given by (as per the definition of $\mathcal{I}_T$)

$$\mathcal{I}_{T^*} = \{\varphi \in H^\infty(\mathbb{D}^n) : \varphi \text{ is inner, and } \text{ran}T^* \subseteq \varphi H^2(\mathbb{D}^n)\}.$$

Now that we have the pairs of inner functions φ_T and φ_{T^*} as constructed above, we are ready to provide an analytic description of operators that arise as products of inner projections.

Theorem 5.3. *Let $T \in \mathcal{B}(H^2(\mathbb{D}^n))$ be a nonzero operator. Then T is a product of inner projection if and only if*

$$T = P_{\varphi_T H^2(\mathbb{D}^n)} P_{\varphi_{T^*} H^2(\mathbb{D}^n)}.$$

Proof. We only need to prove the necessary part. As a result, we assume that there are inner functions φ_1 and φ_2 in $H^\infty(\mathbb{D}^n)$ such that $T = P_{\varphi_1 H^2(\mathbb{D}^n)} P_{\varphi_2 H^2(\mathbb{D}^n)}$. We recall, based on the construction of the set $\mathcal{I}_{T^*}$, that

$$\text{ran}T^* \subseteq \overline{\text{ran}T^*} \subseteq \varphi_{T^*} H^2(\mathbb{D}^n),$$

so that

$$P_{\overline{\text{ran}T^*}} = P_{\varphi_{T^*} H^2(\mathbb{D}^n)} P_{\overline{\text{ran}T^*}}. \tag{5.3}$$

In particular, $P^*_{\overline{\text{ran}T^*}} = P_{\overline{\text{ran}T^*}}$, or even the simple property of projections or set inclusions, implies that

$$P_{\varphi_{T^*} H^2(\mathbb{D}^n)} P_{\overline{\text{ran}T^*}} = P_{\overline{\text{ran}T^*}} P_{\varphi_{T^*} H^2(\mathbb{D}^n)}.$$

Similar to (5.3), or simply because $\overline{\text{ran}T^*} \subseteq \varphi_{T^*}H^2(\mathbb{D}^n)$, we have

$$TP_{\varphi_{T^*}H^2(\mathbb{D}^n)} = P_{\overline{\text{ran}T^*}}P_{\varphi_{T^*}H^2(\mathbb{D}^n)},$$

and then $P_{\varphi_{T^*}H^2(\mathbb{D}^n)}P_{\overline{\text{ran}T^*}} = P_{\overline{\text{ran}T^*}}P_{\varphi_{T^*}H^2(\mathbb{D}^n)}$ implies

$$TP_{\varphi_{T^*}H^2(\mathbb{D}^n)} = TP_{\varphi_{T^*}H^2(\mathbb{D}^n)}P_{\overline{\text{ran}T^*}}.$$

But, then (5.3) yields

$$TP_{\varphi_{T^*}H^2(\mathbb{D}^n)} = TP_{\overline{\text{ran}T^*}}.$$

Since $T|_{\ker T} = 0$, we immediately conclude that

$$T = TP_{\varphi_{T^*}H^2(\mathbb{D}^n)}.$$

This identity is true for all nonzero $T \in \mathcal{B}(H^2(\mathbb{D}^n))$. Consequently, if we apply this to T^*, then we see that

$$T^* = T^*P_{\varphi_T H^2(\mathbb{D}^n)}.$$

Now, we first take the adjoint of this, and then apply the identity $T = TP_{\varphi_{T^*}H^2(\mathbb{D}^n)}$ to deduce that

$$T = P_{\varphi_T H^2(\mathbb{D}^n)}TP_{\varphi_{T^*}H^2(\mathbb{D}^n)}.$$

Because we began with the factorization $T = P_{\varphi_1 H^2(\mathbb{D}^n)}P_{\varphi_2 H^2(\mathbb{D}^n)}$, for some inner functions $\varphi_1, \varphi_2 \in H^\infty(\mathbb{D}^n)$, we finally came to see that

$$T = \ P_{\varphi_T H^2(\mathbb{D}^n)}(P_{\varphi_1 H^2(\mathbb{D}^n)}P_{\varphi_2 H^2(\mathbb{D}^n)})P_{\varphi_{T^*}H^2(\mathbb{D}^n)}.$$

In addition, we are aware that φ_1 divides φ_T and φ_2 divides φ_{T^*} because of the property of lcm. This is equivalent to saying that

$$P_{\varphi_T H^2(\mathbb{D}^n)}P_{\varphi_1 H^2(\mathbb{D}^n)} = P_{\varphi_T H^2(\mathbb{D}^n)},$$

and

$$P_{\varphi_2 H^2(\mathbb{D}^n)}P_{\varphi_{T^*}H^2(\mathbb{D}^n)} = P_{\varphi_{T^*}H^2(\mathbb{D}^n)}.$$

As a result, we conclude that $T = P_{\varphi_T H^2(\mathbb{D}^n)}P_{\varphi_{T^*}H^2(\mathbb{D}^n)}$, which wraps up the proof of the theorem. ■

We conclude this section with the canonical decompositions of operators that result from products of inner projections, allowing us to once more link orthogonal decompositions with analytic objects. Let $T \in \mathcal{B}(H^2(\mathbb{D}^n))$ be a product of inner projection. By (3.4), we know that $H^2(\mathbb{D}^n)_u = \varphi_T H^2(\mathbb{D}^n) \cap \varphi_{T^*}H^2(\mathbb{D}^n)$. Now, we know that

$$\psi_T := gcd\{\varphi_T, \varphi_{T^*}\},$$

is an inner function in $H^\infty(\mathbb{D}^n)$, and consequently, with respect to the Hilbert space decomposition $H^2(\mathbb{D}^n) = \psi_T H^2(\mathbb{D}^n) \oplus \mathcal{Q}_{\psi_T}$, we can write

$$T = \begin{bmatrix} I_{\psi_T H^2(\mathbb{D}^n)} & 0 \\ 0 & T|_{\mathcal{Q}_{\psi_T}} \end{bmatrix}.$$

This decomposition bears some resemblance to the structure of matrices that result from the finite products of projections (see Wu [22, Theorem 4.6] and Oikhberg [14]).

6. Model projections

Recall that the model space $\mathcal{Q}_\varphi$ corresponding to an inner function $\varphi \in H^\infty(\mathbb{D}^n)$ is the quotient space defined by

$$\mathcal{Q}_\varphi = H^2(\mathbb{D}^n)/\varphi H^2(\mathbb{D}^n).$$

We refer to $P_{\mathcal{Q}_\varphi}$ the projection onto the model space $\mathcal{Q}_\varphi$ as a *model projection*. This section seeks to understand operators that can be represented as products of two model projections. Similarly to inner projections, if a nonzero operator $T \in \mathcal{B}(H^2(\mathbb{D}^n))$ can be represented as a product of two model projections, we can conclude that both the ran$T$ and ranT^* are nonzero. For each nonzero $T \in \mathcal{B}(H^2(\mathbb{D}^n))$, we define

$$\mathcal{J}_T = \{\varphi \in H^\infty(\mathbb{D}^n) : \varphi \text{ is inner and } \varphi H^2(\mathbb{D}^n) \subseteq \ker T^*\}.$$

Clearly, the set $\mathcal{J}_T$ defined above could potentially be empty (for instance, whenever T is a co-isometry). However, if the set is not trivial, then, as shown in the proof of Lemma 5.2, one may conclude that the gcd of $\mathcal{J}_T$ exists.

Definition 6.1. Given a collection of inner functions $\{\varphi_\alpha\}_{\alpha\in\Lambda} \subseteq H^\infty(\mathbb{D}^n)$, an inner function $\varphi \in H^\infty(\mathbb{D}^n)$ is said to be the greatest common divisor (or gcd in short) of $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ if

(1) φ divides φ_α for all $\alpha \in \Lambda$, and
(2) if an inner function $\psi \in H^\infty(\mathbb{D})$ divides φ_α for all $\alpha \in \Lambda$, then ψ also divides φ.

Similar to the proof of Lemma 5.2, we also have the following existential result. However, in this case, it is necessary to assume that the set $\mathcal{J}_T$ is nonempty (note that $\mathcal{I}_T$ is automatically nonempty).

Lemma 6.2. *Let $T \in \mathcal{B}(H^2(\mathbb{D}^n))$. If $\mathcal{J}_T \neq \emptyset$, then*

$$\psi_T := gcd\mathcal{J}_T,$$

exists.

We are now prepared to present the result related to the products of model operators. The proof most often deviates from that of Theorem 5.3. Also note that ψ_{T^*} is the gcd of $\mathcal{J}_{T^*}$ (the adjoint version of $\mathcal{J}_T$).

Theorem 6.3. *Let $T \in \mathcal{B}(H^2(\mathbb{D}^n))$ be a nonzero operator. Then T is a product of two model projections if and only if*

$$T = P_{\mathcal{Q}_{\psi_T}} P_{\mathcal{Q}_{\psi_{T^*}}}.$$

Proof. Suppose there are inner functions $\varphi_1, \varphi_2 \in H^\infty(\mathbb{D}^n)$ such that $T = P_{\mathcal{Q}_{\varphi_1}} P_{\mathcal{Q}_{\psi_{\varphi_2}}}$. In this case, $\varphi_1 \in \mathcal{J}_T$ (note that $\mathcal{Q}_{\varphi_1} = (\varphi_1 H^2(\mathbb{D}^n))^\perp$) and $\varphi_2 \in \mathcal{J}_{T^*}$ together imply that $\mathcal{J}_T$ and $\mathcal{J}_{T^*}$ are nonempty sets, and hence, Lemma 6.2 implies that ψ_T and ψ_{T^*} are nonzero inner functions in $H^\infty(\mathbb{D}^n)$. Since ψ_T in $\mathcal{J}_T$, and an inner function, it follows that $\psi_T H^2(\mathbb{D}^n) \subseteq \ker T^*$, and hence

$$\overline{\text{ran}T} = (\ker T^*)^\perp \subseteq \mathcal{Q}_{\psi_T},$$

that is, ran$T \subseteq \mathcal{Q}_{\psi_T}$. Now, we know in general that $\text{ran}(TP_{\mathcal{Q}_{T^*}}) \subseteq \text{ran}T$, which immediately implies that $\text{ran}(TP_{\mathcal{Q}_{\psi_{T^*}}}) \subseteq \mathcal{Q}_{\psi_T}$, and hence we have the identity

$$P_{\mathcal{Q}_{\psi_T}} T P_{\mathcal{Q}_{\psi_{T^*}}} = T P_{\mathcal{Q}_{\psi_{T^*}}}.$$

Again, we know the general fact that $T = TP_{(\ker T)^\perp}$. The above identity implies

$$TP_{\mathcal{Q}_{\psi_{T^*}}} = TP_{(\ker T)^\perp} P_{\mathcal{Q}_{\psi_{T^*}}}$$

Next, as $\psi_{T^*} \in \mathcal{J}_{T^*}$, we have $\psi_{T^*} H^2(\mathbb{D}^n) \subseteq \ker T$, equivalently, $(\ker T)^\perp \subseteq \mathcal{Q}_{\psi_{T^*}}$. This implies $P_{\mathcal{Q}_{\psi_{T^*}}} P_{(\ker T)^\perp} = P_{(\ker T)^\perp}$, and hence

$$TP_{\mathcal{Q}_{\psi_{T^*}}} = TP_{(\ker T)^\perp} = T.$$

This combined with $P_{\mathcal{Q}_{\psi_T}} TP_{\mathcal{Q}_{\psi_{T^*}}} = TP_{\mathcal{Q}_{\psi_{T^*}}}$ yields

$$T = P_{\mathcal{Q}_{\psi_T}} TP_{\mathcal{Q}_{\psi_{T^*}}}.$$

Now, as we know that T is a product of projections $T = P_{\mathcal{Q}_{\varphi_1}} P_{\mathcal{Q}_{\psi_{\varphi_2}}}$, the above implies

$$T = P_{\mathcal{Q}_{\psi_T}} P_{\mathcal{Q}_{\varphi_1}} P_{\mathcal{Q}_{\psi_{\varphi_2}}} P_{\mathcal{Q}_{\psi_{T^*}}}.$$

Looking at the product of two projections $P_{\mathcal{Q}_{\psi_T}} P_{\mathcal{Q}_{\varphi_1}}$, we observe that both ψ_T and φ_1 are in $\mathcal{J}_T$, which implies by the definition of gcd that φ_1 divides ψ_T; equivalently, $\mathcal{Q}_{\psi_T} \subseteq \mathcal{Q}_{\varphi_1}$. This gives $P_{\mathcal{Q}_{\psi_T}} P_{\mathcal{Q}_{\varphi_1}} = P_{\mathcal{Q}_{\psi_T}}$, and also, similarly, $P_{\mathcal{Q}_{\psi_{\varphi_2}}} P_{\mathcal{Q}_{\psi_{T^*}}} = P_{\mathcal{Q}_{\psi_{T^*}}}$. This completes the proof of the fact that $T = P_{\mathcal{Q}_{\psi_T}} P_{\mathcal{Q}_{\psi_{T^*}}}$. ■

At this time, we will make a general observation regarding kernels of product of two projections: When $T \in \mathcal{B}(\mathcal{H})$ is expressed as a product of two projections, $T = P_1 P_2$, we obtain:

$$\ker(P_1 P_2) = [\mathrm{ran}(I - P_1) \cap \mathrm{ran} P_2] \oplus \mathrm{ran}(I - P_2). \tag{6.1}$$

Indeed, if $P_1 P_2 h = 0$ for some $h \in \mathcal{H}$, then we write $h = h_r \oplus h_n \in \mathrm{ran} P_2 \oplus \ker P_2$. Therefore

$$0 = P_1 P_2 h = P_1 h_r,$$

implies $(I - P_1) h_r = h_r \in \mathrm{ran} P_2$, and consequently

$$h = h_r \oplus h_n \in [\mathrm{ran}(I - P_1) \cap \mathrm{ran} P_2] \oplus \mathrm{ran}(I - P_2),$$

proving that $\ker T$ is contained in the right side subspace of (6.1). The reverse set inclusion is straightforward.

This, when combined with Proposition 2.4 in the context of inner projections, results in the following characterization of operators as products of two inner projections:

Theorem 6.4. *Let $T \in \mathcal{B}(H^2(\mathbb{D}^n))$. Then T is a product of two inner projections if and only if*

$$TT^* = P_{\varphi_T H^2(\mathbb{D}^n)} T^*,$$

and

$$\ker T = [\varphi_{T^*} H^2(\mathbb{D}^n) \cap \mathcal{Q}_{\varphi_T}] \oplus \mathcal{Q}_{\varphi_{T^*}}.$$

Proof. If T is a product of two inner projections, then Theorem 5.3 implies

$$T = P_{\varphi_T H^2(\mathbb{D}^n)} P_{\varphi_{T^*} H^2(\mathbb{D}^n)}.$$

The representation of $\ker T$ then follows from (6.1). For the other identity, we compute:

$$TT^* = P_{\varphi_T H^2(\mathbb{D}^n)} P_{\varphi_{T^*} H^2(\mathbb{D}^n)} P_{\varphi_T H^2(\mathbb{D}^n)} = P_{\varphi_T H^2(\mathbb{D}^n)} T^*,$$

We now turn to prove the sufficient part. By Proposition 2.4 and $TT^* = P_{\varphi_T H^2(\mathbb{D}^n)} T^*$, we know that

$$T = P_{\varphi_T H^2(\mathbb{D}^n)} P_{(\ker T)^\perp}.$$

Define

$$\tilde{T} = P_{\varphi_T H^2(\mathbb{D}^n)} P_{\varphi_{T^*} H^2(\mathbb{D}^n)}.$$

We claim that $T = \tilde{T}$. The given assumption about $\ker T$ and the identity (6.1) yield

$$\ker T = \ker \tilde{T}.$$

In particular, $\mathcal{Q}_{\varphi_{T^*}} \subseteq \ker T$, that is, $(\ker T)^\perp \subseteq \varphi_{T^*} H^2(\mathbb{D}^n)$. This implies $P_{(\ker T)^\perp} = P_{\varphi_{T^*} H^2(\mathbb{D}^n)} P_{(\ker T)^\perp}$, and hence

$$T = P_{\varphi_T H^2(\mathbb{D}^n)} P_{(\ker T)^\perp} = P_{\varphi_T H^2(\mathbb{D}^n)} P_{\varphi_{T^*} H^2(\mathbb{D}^n)} P_{(\ker T)^\perp} = P_{\varphi_T H^2(\mathbb{D}^n)} P_{\varphi_{T^*} H^2(\mathbb{D}^n)} = \tilde{T}.$$

This completes the proof of the theorem. ■

Finally, we again consider model projections. The proof in this case is similar to the previous one.

Corollary 6.5. *Let $T \in \mathcal{B}(H^2(\mathbb{D}^n))$. Then T is a product of two model projections if and only if*

$$\mathcal{J}_T, \mathcal{J}_{T^*} \neq \emptyset,$$

and

$$TT^* = P_{\mathcal{Q}_{\psi_T}} T^*,$$

and

$$\ker T = [\mathcal{Q}_{\psi_{T^*}} \cap \psi_T H^2(\mathbb{D}^n)] \oplus \psi_{T^*} H^2(\mathbb{D}^n).$$

In the following section, we will use the identity (6.1) to concrete examples of projections. We will see that this simple observation brings some quick answers to questions related to the product of two projections.

7. Blaschke products

This section aims to provide concrete examples that illustrate some of the results obtained thus far. Given $\alpha \in \mathbb{D}$, the function $b_\alpha \in Aut(\mathbb{D})$ defined by

$$b_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha} z} \qquad (z \in \mathbb{D}),$$

is popularly known as a *Blaschke factor*. A finite Blaschke product is an inner function $\varphi \in H^\infty(\mathbb{D})$ such that

$$\varphi = \prod_{j=1}^m b_{\alpha_j},$$

for some finite subset $\{\alpha_j\}_{j=1}^m \subset \mathbb{D}$. We set the zero set of φ as follows:

$$\mathcal{Z}(\varphi) = \{\alpha_j\}_{j=1}^m,$$

counting the multiplicity. Blaschke products are important tools in Hilbert function spaces. In our context, we recall that finite Blaschke products provide finite codimensional invariant subspaces of $H^2(\mathbb{D})$. More specifically, if $\mathcal{Q}_\varphi$ is a model space for some inner function $\varphi \in H^\infty(\mathbb{D})$, then

$$\dim \mathcal{Q}_\varphi < \infty,$$

if and only if φ is a finite Blaschke product. The lemma that follows is elementary but of general interest and may have a broader scope when considered in its full generality. The cardinality of a set A is denoted by the notation $|A|$.

Lemma 7.1. *Let φ_1 and φ_2 be two finite Blaschke products. Then*

$$\mathcal{Q}_{\varphi_1} \cap \varphi_2 H^2(\mathbb{D}) = \{0\},$$

if and only if

$$|\mathcal{Z}(\varphi_1)| \leq |\mathcal{Z}(\varphi_2)|.$$

Proof. Define $m_i := |\mathcal{Z}(\varphi_i)|$ for $i = 1, 2$. We know that $m_1, m_2 < \infty$ (as φ_1 and φ_2 are finite Blaschke products). Assume that $m_1 \leq m_2$, and pick $f \in \mathcal{Q}_{\varphi_1} \cap \varphi_2 H^2(\mathbb{D})$. We claim that $f = 0$. Let $\lambda_1, \ldots, \lambda_t$ be the zeros of φ_1 with multiplicities $s_1, \ldots, s_t$, respectively. In this case, the model space $\mathcal{Q}_{\varphi_1}$ is given by

$$\mathcal{Q}_{\varphi_1} = \Big\{ \frac{g}{(1-\bar{\lambda}_1 z)^{s_1} \cdots (1-\bar{\lambda}_t z)^{s_t}} : g \in \mathbb{C}[z], \deg g \leq m_1 - 1 \Big\}.$$

Since $f \in \mathcal{Q}_{\varphi_1}$, there exists a polynomial $F \in \mathbb{C}[z]$ such that $\deg F \leq m_1 - 1$ and

$$f = \frac{F}{(1-\bar{\lambda}_1 z)^{s_1} \cdots (1-\bar{\lambda}_t z)^{s_t}}.$$

This implies, in particular, that $|\mathcal{Z}(f)| \leq m_1 - 1$. On the other hand, $f \in \varphi_2 H^2(\mathbb{D})$ implies that, counted with multiplicities, f has at least m_2 zeros (as $|\mathcal{Z}(\varphi_2)| \leq |\mathcal{Z}(f)|$). Therefore, we have

$$m_2 \leq |\mathcal{Z}(f)| \leq m_1 - 1,$$

and consequently, $m_2 < m_1$; a contradiction. This forces that $f = 0$. Now, we turn to prove the converse direction. Let $\mathcal{Q}_{\varphi_1} \cap \varphi_2 H^2(\mathbb{D}) = \{0\}$, and suppose $m_1 > m_2$. Assume that $\alpha_1, \ldots, \alpha_s$ are distinct zeros of φ_2 with multiplicities $n_1, \ldots, n_s$, respectively. Set

$$f = \frac{\prod_{j=1}^{s} (z-\alpha_j)^{n_j}}{(1-\bar{\alpha}_1 z)^{n_1} \cdots (1-\bar{\alpha}_s z)^{n_s}}.$$

Since $m_1 > m_2$, it is evident that $f \in \mathcal{Q}_{\varphi_1} \cap \varphi_2 H^2(\mathbb{D})$, contradicting the fact that $\mathcal{Q}_{\varphi_1} \cap \varphi_2 H^2(\mathbb{D}) = \{0\}$. Therefore, $m_1 \leq m_2$, which completes the proof of the lemma. ∎

In the following, we relate canonical factorizations of the products of inner projections with corresponding range spaces.

Proposition 7.2. *Let $\varphi_1, \varphi_2 \in H^\infty(\mathbb{D})$ be finite Blaschke products. Assume that*

$$|\mathcal{Z}(\varphi_1)| \neq |\mathcal{Z}(\varphi_2)|.$$

Then there does not exist any $T \in \mathcal{B}(H^2(\mathbb{D}))$ that is the product of two inner projections with the property that

$$\overline{ranT} = \varphi_1 H^2(\mathbb{D}) \text{ and } \overline{ranT^*} = \varphi_2 H^2(\mathbb{D}).$$

Proof. Let $T \in \mathcal{B}(H^2(\mathbb{D}))$ be a product of two inner projections. Moreover, assume that $\overline{\text{ran}T} = \varphi_1 H^2(\mathbb{D})$ and $\overline{\text{ran}T^*} = \varphi_2 H^2(\mathbb{D})$. Since $\overline{\text{ran}T} = \varphi_1 H^2(\mathbb{D})$, by the definition of lcm (also see the construction of $\mathcal{I}_T$), it follows that

$$\varphi_2 = \alpha \varphi_T,$$

for some $\alpha \in \mathbb{T}$. Similarly, we also have

$$\varphi_1 = \beta \varphi_{T^*},$$

for some $\beta \in \mathbb{T}$. In particular, we have

$$\text{ran}T = \varphi_2 H^2(\mathbb{D}) = \varphi_T H^2(\mathbb{D}),$$

and

$$\text{ran}T^* = \varphi_1 H^2(\mathbb{D}) = \varphi_{T^*} H^2(\mathbb{D}).$$

The later identity implies $\ker T = \mathcal{Q}_{\varphi_{T^*}}$. By (6.1), we also know that

$$\ker T = [\mathcal{Q}_{\varphi_T} \cap \varphi_{T^*} H^2(\mathbb{D})] \oplus \mathcal{Q}_{\varphi_{T^*}},$$

and consequently

$$\mathcal{Q}_{\varphi_T} \cap \varphi_{T^*} H^2(\mathbb{D}) = \{0\}.$$

In other words, we have $\mathcal{Q}_{\varphi_2} \cap \varphi_1 H^2(\mathbb{D}) = \{0\}$, and hence, Lemma 7.1 implies that $|\mathcal{Z}(\varphi_2)| \leq |\mathcal{Z}(\varphi_1)|$. Similarly, as T^* is also a product of two inner projections, working as above, we find that $|\mathcal{Z}(\varphi_1)| \leq |\mathcal{Z}(\varphi_2)|$. This yields $|\mathcal{Z}(\varphi_2)| = |\mathcal{Z}(\varphi_1)|$ – a contradiction, which completes the proof of the proposition. ∎

For operators that are products of two inner projections, the corresponding kernel spaces have a specific relationship with the set of all inner functions.

Lemma 7.3. *Let $T \in \mathcal{B}(H^2(\mathbb{D}))$ be a product of two inner projections. Then there does not exist any nonconstant inner function $\varphi \in H^\infty(\mathbb{D})$ such that*

$$\ker T \subseteq \varphi H^2(\mathbb{D}).$$

Proof. Suppose T is a product of two inner projections. We have, in particular, that $T = P_{\varphi_T H^2(\mathbb{D}^n)} P_{\varphi_{T^*} H^2(\mathbb{D}^n)}$. We again use the identity (6.1) and observe that

$$\ker T = [\mathcal{Q}_{\varphi_T} \cap \varphi_{T^*} H^2(\mathbb{D})] \oplus \mathcal{Q}_{\varphi_{T^*}},$$

In particular, $\mathcal{Q}_{\varphi_T} \subseteq \ker T$, and hence, by assumption, $\mathcal{Q}_{\varphi_T} \subseteq \varphi H^2(\mathbb{D})$. This can happen only if φ is a constant function. Indeed, $\mathcal{Q}_{\varphi_T} \subseteq \varphi H^2(\mathbb{D})$ implies that $z^m \mathcal{Q}_{\varphi_T} \subseteq \varphi H^2(\mathbb{D})$ for all $m \in \mathbb{Z}_+$, and then

$$H^2(\mathbb{D}) = \vee_{m \in \mathbb{Z}_+} z^m \mathcal{Q}_{\varphi_T} \subseteq \varphi H^2(\mathbb{D}),$$

proves the fact that φ is a constant function. ∎

As a result, we have the following result that says when an operator on $H^2(\mathbb{D})$ can't be represented as a product of two inner projections.

Corollary 7.4. *Let $T \in \mathcal{B}(H^2(\mathbb{D}))$ be a nonzero operator. If $\ker T$ is T_z-invariant, then T is not a product of two inner projections.*

Proof. Suppose T is a product of two inner projections. If there exists an inner function $\varphi \in H^\infty(\mathbb{D})$ such that $\ker T = \varphi H^2(\mathbb{D})$, then Lemma 7.3 will force that φ to be a constant function, implying that $T = 0$. ∎

For examples of bounded linear operators on $H^2(\mathbb{D})$ that meet the criteria stated in the above result, we note that for every inner function $\theta \in H^\infty(\mathbb{D})$, there exists a Hankel operator H_φ, $\varphi \in L^\infty(\mathbb{T})$, such that [20, page 15]

$$\ker H_\varphi = \theta H^2(\mathbb{D}).$$

Clearly, there is now an abundance of examples of Hankel operators, as well as the operators referenced in the above corollary.

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