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On CNC commuting contractive tuples

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Abstract. The characteristic function has been an important tool for studying completely non-unitary contractions on Hilbert spaces. In this note, we consider completely non-coisometric contractive tuples of commuting operators on a Hilbert space $\mathcal{H}$. We show that the characteristic function, which is now an operator-valued analytic function on the open Euclidean unit ball in $\mathbb{C}^n$, is a complete unitary invariant for such a tuple. We prove that the characteristic function satisfies a natural transformation law under biholomorphic mappings of the unit ball. We also characterize all operator-valued analytic functions which arise as characteristic functions of pure commuting contractive tuples.

Keywords. Characteristic function; invariant subspaces; biholomorphic automorphisms; functional model; coincidence.

1. Introduction

The characteristic function for a single contraction on a Hilbert space was defined by Sz-Nagy and Foias in [23]. Since then it has drawn a lot of attention and several interesting results are known.

A tuple $T = (T_1, \ldots, T_n)$ of bounded operators on a Hilbert space $\mathcal{H}$ is called contractive if $\|T_1h_1 + \cdots + T_nh_n\|^2 \le \|h_1\|^2 + \cdots + \|h_n\|^2$ for all $h_1, \ldots, h_n$ in $\mathcal{H}$ or equivalently $\sum_{i=1}^n T_iT_i^* \le 1_{\mathcal{H}}$. The positive operator $(1_{\mathcal{H}} - \sum_{i=1}^n T_iT_i^*)^{1/2}$ and the closure of its range will be called the *defect operator* D_{T^*} and the *defect space* $\mathcal{D}_{T^*}$ of T.

We shall also denote by T the row operator from $\mathcal{H}^n$ to $\mathcal{H}$ which maps $(h_1, \ldots, h_n)$ to $T_1h_1 + \cdots + T_nh_n$. The adjoint $T^*: \mathcal{H} \to \mathcal{H}^n$ maps h to the column vector $(T_1^*h, \ldots, T_n^*h)$ and, in fact, T is a contractive tuple if and only if the operator T is a contraction. Thus for a contractive tuple T one can also consider the defect operator $D_T = (1_{\mathcal{H}^n} - T^*T)^{1/2} = ((\delta_{ij}1_{\mathcal{H}} - T_i^*T_j))^{1/2}$ in $\mathcal{B}(\mathcal{H}^n)$ and the associated defect space $\mathcal{D}_T = \overline{\operatorname{Ran}} D_T \subset \mathcal{H}^n$.

We use the notation $\mathbb{B}_n$ for the open Euclidean unit ball in $\mathbb{C}^n$. The prototypical example, which has been used by Arveson [2] and Müller and Vasilescu [17] in the construction of appropriate models, is the shift on H_n^2 defined as follows. Given a complex Hilbert space $\mathcal{E}$, let $\mathcal{O}(\mathbb{B}_n, \mathcal{E})$ be the class of all $\mathcal{E}$-valued analytic functions on $\mathbb{B}_n$. For any multi-index $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$, we write $|k| = k_1 + \cdots + k_n$. Then consider the Hilbert space

$$H_n^2(\mathcal{E}) = \left\{ f = \sum_{k \in \mathbb{N}^n} a_k z^k \in \mathcal{O}(\mathbb{B}, \mathcal{E}) \colon a_k \in \mathcal{E} \text{ with } \|f\|^2 = \sum_{k \in \mathbb{N}^n} \frac{\|a_k\|^2}{\gamma_k} < \infty \right\}, \quad (1.1)$$

where $\gamma_k = |k|!/k!$. One can show that $H_n^2(\mathcal{E})$ is the $\mathcal{E}$-valued functional Hilbert space given by the reproducing kernel $(1 - \langle z, w\rangle)^{-1} 1_{\mathcal{E}}$. When $\mathcal{E} = \mathbb{C}$, we use the abbreviation H_n^2. For $n = 1$, this space is the usual Hardy space on the unit disk. The space $H_n^2(\mathcal{E})$ is isometrically isomorphic to the Hilbertian tensor product $H_n^2 \otimes \mathcal{E}$ in a canonical way. Given complex Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$, the multiplier space $M(\mathcal{E}, \mathcal{E}_*)$ consists of all $\varphi \in \mathcal{O}(\mathbb{B}_n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ such that $\varphi H_n^2(\mathcal{E}) \subset H_n^2(\mathcal{E}_*)$. By the closed graph theorem, for each function $\varphi \in M(\mathcal{E}, \mathcal{E}_*)$,the induced multiplication operator $M_\varphi\colon H(\mathcal{E}) \to H(\mathcal{E}_*)$, $f \mapsto \varphi f$, is continuous. The *standard shift* on $H_n^2(\mathcal{E})$ is the tuple $M_z^{\mathcal{E}} = (M_{z_1}^{\mathcal{E}}, \ldots, M_{z_n}^{\mathcal{E}})$ consisting of the multiplication operators $M_{z_i}^{\mathcal{E}}\colon H_n^2(\mathcal{E}) \to H_n^2(\mathcal{E})$ with the coordinate functions z_i. When $\mathcal{E} = \mathbb{C}$, we shall write M_z for $M_z^{\mathcal{E}}$. Arveson studied both the space H_n^2 and the standard shift M_z in great detail in [2]. It can be seen without much difficulty that M_z is a commuting contractive tuple. In fact, $D_{M_z^*}$ is the one-dimensional projection onto the space of constant functions. The space H_n^2 was first used by Drury [12] who generalized von Neumann's inequality to operator tuples.

With a commuting contractive tuple T on a Hilbert space $\mathcal{H}$, one associates a completely positive map $P_T\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by $P_T(X) = \sum_{i=1}^n T_i X T_i^*$. We denote by $A_T \in \mathcal{B}(\mathcal{H})$ the strong limit of the decreasing sequence of positive operators $I \geq P_T(I) \geq P_T^2(I) \geq \cdots \geq 0$. The tuple T is called pure if $A_T = 0$. For $n = 1$, this corresponds to the $C_{\cdot 0}$ case in the Sz-Nagy and Foias classification [23] of contractions. The standard shift M_z on H_n^2 is pure.

An operator-valued bounded function on $\mathbb{B}_n$ is a triple $\{\mathcal{E}, \mathcal{E}_*, \varphi\}$, where $\mathcal{E}$ and $\mathcal{E}_*$ are Hilbert spaces and φ is a $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$-valued bounded function on $\mathbb{B}_n$. If $\|\varphi(z)\| \leq 1$, then the function is called contractive. Two operator-valued bounded functions $\{\mathcal{E}, \mathcal{E}_*, \varphi\}$ and $\{\mathcal{F}, \mathcal{F}_*, \psi\}$ are said to coincide if there exist unitary operators $\tau\colon \mathcal{E} \to \mathcal{F}$ and $\tau_*\colon \mathcal{E}_* \to \mathcal{F}_*$ such that the following diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\varphi(z)"] \arrow[d, "\tau"'] & \mathcal{E}_* \arrow[d, "\tau_*"] \\
\mathcal{F} \arrow[r, "\psi(z)"'] & \mathcal{F}_*
\end{tikzcd}$$

commutes for all z in $\mathbb{B}_n$.

The characteristic function for a commuting contractive tuple is defined as the operator-valued contractive function $\{\mathcal{D}_T, \mathcal{D}_{T^*}, \theta_T\}$, where

$$\theta_T(z) = -T + D_{T^*}(1_{\mathcal{H}} - ZT^*)^{-1} Z D_T, \quad z \in \mathbb{B}_n. \tag{1.2}$$

Here $Z = (z_1 I_{\mathcal{H}}, \ldots, z_n I_{\mathcal{H}})$ denotes the row multiplication induced by $z \in \mathbb{B}_n$. The characteristic function was defined in [8], and it was proved to be a complete unitary invariant in the case of pure tuples T. Theorem 4.4 in [8] states that if T and R are two pure commuting contractive tuples on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively, then T and R are unitarily equivalent (that is, $T_i = U R_i U^*$ for all $i = 1, \ldots, n$ and a suitable unitary operator $U\colon \mathcal{K} \to \mathcal{H}$) if and only if $\{\mathcal{D}_T, \mathcal{D}_{T^*}, \theta_T\}$ and $\{\mathcal{D}_R, \mathcal{D}_{R^*}, \theta_R\}$ coincide. We would like to point out here that recent works by Gelu Popescu also shows the same result in a more general setting (see [20] and [21]). Popescu has defined the characteristic function for a contractive tuple in [18]. He proved that it is a complete unitary invariant for completely non-coisometric tuples. Since he considered the non-commutative case, the characteristic function was defined as a multi analytic operator. In [20], he has shown that one can associate a constrained characteristic function to a constrained contractive tuple (a commuting contractive tuple is a particular example). Thus he also obtained the result mentioned above by compressing the multi-analytic operator to the symmetric Fock space.

This note serves the purpose of proving three basic results about the characteristic function. We show that the characteristic function is a complete unitary invariant for completely non-coisometric commuting contractive tuples. In the process, we construct a functional model for a completely non-coisometric tuple in §3. We start in §2 by showing that the characteristic function obeys a certain natural transformation rule with respect to automorphisms of the Euclidean unit ball. The automorphisms of the disk have played an important role in the model and dilation theory of single contractions. Hence it is naturally desirable to obtain a multivariable analogue. Finally, in §4, we characterize the subspaces of $H_n^2 \otimes \mathcal{E}$ reducing for the canonical shift and use this result to describe all operator-valued analytic functions which arise as characteristic functions of pure tuples. We also prove a version of the classical Beurling–Lax–Halmos theorem in that section using the characteristic function.

After completion of this note, we came to know about a recent preprint of Benhida and Timotin [7] which also studies the connection between commuting contractive tuples and automorphisms of the unit ball.

2. Transformation rule

For a fixed $a \neq 0$ in $\mathbb{B}_n$, let P_a denote the orthogonal projection of $\mathbb{C}^n$ onto the one-dimensional subspace $[a]$ generated by a, that is, $P_a z = (\langle z, a\rangle / \langle a, a\rangle) a$. Let Q_a be the orthogonal projection $I - P_a$. Define $s_a = (1 - |a|^2)^{1/2}$. Then

$$\varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle}$$

is a biholomorphic automorphism of the unit ball (see [22]). Given a commuting contractive tuple of operators T, it is easy to see that the Taylor spectrum of T is contained in the closure of the unit ball. Since φ_a is actually analytic in an open set containing the closed unit ball, we can consider the associated operator tuple

$$T_a = (1 - TA^*)^{-1}(A - P_a T - s_a Q_a T),$$

where $A = (a_1 I_{\mathcal{H}}, \ldots, a_n I_{\mathcal{H}})$, $P_a T = \frac{1}{|a|^2} T A^* A$ and $Q_a T = T - P_a T$.

Lemma 2.1. *Let T be a commuting contractive n-tuple of operators on a Hilbert space $\mathcal{H}$. Then for any $a \in \mathbb{B}_n$, we have the identity*

$$I_{\mathcal{H}} - T_a T_a^* = (1 - |a|^2)(I_{\mathcal{H}} - TA^*)^{-1}(I_{\mathcal{H}} - TT^*)(I_{\mathcal{H}} - AT^*)^{-1}.$$

Proof. Using the equality $P_a T (Q_a T)^* = 0$ and the definition of T_a, we obtain that

$$\begin{aligned} T_a T_a^* = (I - TA^*)^{-1}(A - P_a T - s_a Q_a T)(A^* - (P_a T)^* \\ - s_a (Q_a T)^*)(I - AT^*)^{-1} \end{aligned}$$

$$
\begin{aligned}
&= (I - TA^*)^{-1}\Bigg[|a|^2 I - AT^* - TA^* + \frac{1}{|a|^2} TA^*AT^* \\
&\quad + s_a^2 \left(T - \frac{1}{|a|^2} TA^*A\right)\left(T^* - \frac{1}{|a|^2} A^*AT^*\right)\Bigg](I - AT^*)^{-1} \\
&= (I - TA^*)^{-1}[|a|^2 I - AT^* - TA^* + TA^*AT^* \\
&\quad + (1 - |a|^2)TT^*](I - AT^*)^{-1} \\
&= (I - TA^*)^{-1}[(I - TA^*)(I - AT^*) - (1 - |a|^2) \\
&\quad + (1 - |a|^2)TT^*](I - AT^*)^{-1} \\
&= I - (1 - |a|^2)(I - TA^*)^{-1}(I - TT^*)(I - AT^*)^{-1}. \qquad \blacksquare
\end{aligned}
$$

The above lemma shows in particular that the commuting tuple T_a is contractive again.

COROLLARY 2.2

If T is a commuting contractive n-tuple of operators on $\mathcal{H}$, then so is T_a for any $a \in \mathbb{B}_n$.

COROLLARY 2.3

There exists a unitary operator U from $\mathcal{D}_{T_a^*}$ to $\mathcal{D}_{T^*}$ such that $UD_{T_a^*} = D_{T^*}S^*$ with $S = s_a(I - TA^*)^{-1} \in \mathcal{B}(\mathcal{H})$.

Proof. The equality

$$\|D_{T_a^*}h\|^2 = \|D_{T^*}S^*h\|^2$$

holds for each h in $\mathcal{H}$ and enables us to define an isometry $U\colon \mathcal{D}_{T_a^*} \to \mathcal{D}_{T^*}$ by $U(D_{T_a^*}h) = D_{T^*}S^*h$. Since S is invertible, this isometry is a unitary operator. $\blacksquare$

Our next aim is to show that characteristic functions behave naturally with respect to biholomorhic mappings of the unit ball.

DEFINITION 2.4

Let $\{\mathcal{E}, \mathcal{E}_*, \varphi\}$ and $\{\mathcal{F}, \mathcal{F}_*, \psi\}$ be two operator-valued bounded functions. We say that these two functions coincide weakly if there exists a unitary operator $\tau\colon \mathcal{E}_* \to \mathcal{F}_*$ such that $\psi(w)\psi(z)^* = \tau\varphi(w)\varphi(z)^*\tau^*$ for all $z, w \in \mathbb{B}_n$.

Obviously coincidence implies weak coincidence. Conversely, the bounded operator-valued functions $\{\mathcal{H}, \mathcal{H}, \varphi \equiv I_{\mathcal{H}}\}$ and $\{\mathcal{H} \oplus \mathcal{H}, \mathcal{H}, \psi \equiv (0, I_{\mathcal{H}})\}$ coincide weakly, but an elementary argument shows that they do not coincide. On the other hand, weak coincidence almost implies coincidence.

Recall that the support of a bounded operator-valued function $\{\mathcal{E}, \mathcal{E}_*, \varphi\}$, is defined as

$$\mathrm{supp}(\varphi) = \overline{\mathrm{span}} \cup \{\mathrm{Ran}\, \varphi(z)^*\colon z \in \mathbb{B}_n\} = \mathcal{E} \ominus \cap\{\ker \varphi(z)\colon z \in \mathbb{B}_n\}.$$

Lemma 2.5. *Let $\{\mathcal{E}, \mathcal{E}_*, \varphi\}$ and $\{\mathcal{F}, \mathcal{F}_*, \psi\}$ be bounded operator-valued functions.*

(1) *The above functions coincide weakly if and only if their restrictions* $\{\mathrm{supp}(\varphi), \mathcal{E}_*, \varphi|\mathrm{supp}(\varphi)\}$ *and* $\{\mathrm{supp}(\psi), \mathcal{F}_*, \psi|\mathrm{supp}(\psi)\}$ *coincide.*

(2) *The functions* $\{\mathcal{E}, \mathcal{E}_*, \varphi\}$ *and* $\{\mathcal{F}, \mathcal{F}_*, \psi\}$ *coincide if and only if they coincide weakly and* $\mathcal{E} \ominus \mathrm{supp}(\varphi)$ *is isometrically isomorphic to* $\mathcal{F} \ominus \mathrm{supp}(\psi)$.

Proof. Suppose that there is a unitary operator $\tau\colon \mathcal{E}_* \to \mathcal{F}_*$ with

$$\psi(w)\psi(z)^* = \tau\varphi(w)\varphi(z)^*\tau^*, \quad z, w \in \mathbb{B}_n.$$

Then there is a unique unitary operator $U\colon \mathrm{supp}(\varphi) \to \mathrm{supp}(\psi)$ such that

$$U(\varphi(z)^*\tau^*x) = \psi(z)^*x, \quad z \in \mathbb{B}_n, x \in \mathcal{F}_*.$$

Obviously this operator satisfies the intertwining relations

$$(\psi(z)|\mathrm{supp}(\psi))U = \tau(\varphi(z)|\mathrm{supp}(\varphi)), \quad z \in \mathbb{B}_n.$$

Conversely, suppose that there is a unitary operator $U\colon \mathrm{supp}(\varphi) \to \mathrm{supp}(\psi)$ satisfying the last intertwining relations. Then it is elementary to check that $\psi(w)\psi(z)^* = \tau\varphi(w)\varphi(z)^*\tau^*$ for $z, w \in \mathbb{B}_n$. Furthermore, if there is a unitary operator $V\colon \mathcal{E}\ominus\mathrm{supp}(\varphi) \to \mathcal{F} \ominus \mathrm{supp}(\psi)$, then obviously

$$\psi(z)(U \oplus V) = \tau\varphi(z), \quad z \in \mathbb{B}_n.$$

To complete the proof, suppose that there is a unitary operator $W\colon \mathcal{E} \to \mathcal{F}$ with

$$\psi(z)W = \tau\varphi(z), \quad z \in \mathbb{B}_n.$$

Then necessarily $W(\bigcap_{z\in\mathbb{B}_n} \ker\varphi(z)) \subset \bigcap_{z\in\mathbb{B}_n} \ker\psi(z)$, and using the same property of W^{-1}, we see that $W(\mathcal{E} \ominus \mathrm{supp}(\varphi)) = \mathcal{F} \ominus \mathrm{supp}(\psi)$. ∎

Any biholomorphic automorphism of the unit ball is of the form $u \circ \varphi_a$, where u is a unitary operator on $\mathbb{C}^n$ and $a \in \mathbb{B}_n$ (see [22]). Let (u_{ij}) be the matrix representation of u. We denote by $u(T)$ the commuting tuple $(\sum u_{1j}T_j, \ldots, \sum u_{nj}T_j)$ which is easily seen to be contractive again. The image of T under the biholomorphic automorphism $u \circ \varphi_a$, obtained by applying the anlytic functional calculus, is $u(T_a)$.

Theorem 2.6. *Let T be a commuting contractive tuple of bounded operators on $\mathcal{H}$ and let $u \circ \varphi_a$ be an arbitrary biholomorphic automorphism of $\mathbb{B}_n$. Then the operator-valued contractive analytic functions $\{\mathcal{D}_T, \mathcal{D}_{T^*}, \theta_T \circ \varphi_a \circ u^*\}$ and $\{\mathcal{D}_{u(T_a)}, \mathcal{D}_{u(T_a)^*}, \theta_{u(T_a)}\}$ coincide weakly.*

Proof. It is elementary to check that the two functions $\{\mathcal{D}_T, \mathcal{D}_{T^*}, \theta_T \circ u^*\}$ and $\{\mathcal{D}_{u(T)}, \mathcal{D}_{u(T)^*}, \theta_{u(T)}\}$ coincide. Hence we only need to prove that there is a unitary operator $U\colon \mathcal{D}_{T_a{}^*} \to \mathcal{D}_{T^*}$ such that

$$\theta_T(\varphi_a(w))\theta_T(\varphi_a(z))^* = U\theta_{T_a}(w)\theta_{T_a}(z)^*U^*$$

for $z, w \in \mathbb{B}_n$. For z in $\mathbb{B}_n$, let us abbreviate $\varphi_a(z)$ by z'. Recall that for $z, w \in \mathbb{B}_n$, the identity

$$I - \theta_T(w)\theta_T(z)^* = (1 - \langle w, z\rangle)D_{T^*}(I - WT^*)^{-1}(I - TZ^*)^{-1}D_{T^*} \quad (2.1)$$

holds (Lemma 2.2 of [8]). Using the definition of φ_a and the observation that $(P_a w)T^* = wP_a(T)^*$, we find that

$$\begin{aligned}(I - w'T^*) &= I - \frac{a - P_a w - s_a Q_a w}{1 - \langle w, a\rangle} T^* \\ &= (1 - \langle w, a\rangle)^{-1}(I - wA^* - AT^* + (P_a w)T^* + s_a(Q_a w)T^*) \\ &= (1 - \langle w, a\rangle)^{-1}[(I - AT^*) - w(A^* - (P_a T)^* - s_a(Q_a T)^*)] \\ &= (1 - \langle w, a\rangle)^{-1}[(I - AT^*) - wT_a^*(I - AT^*)] \\ &= (1 - \langle w, a\rangle)^{-1}(I - wT_a^*)(I - AT^*).\end{aligned}$$

By passing to inverses we obtain that

$$(1 - \langle w, a\rangle)^{-1}(I - w'T^*)^{-1} = (I - AT^*)^{-1}(I - wT_a^*)^{-1}. \tag{2.2}$$

Replacing w by z leads to

$$(1 - \langle a, z\rangle)^{-1}(I - Tz'^*)^{-1} = (I - T_a z^*)^{-1}(I - TA^*)^{-1}. \tag{2.3}$$

Using equation (2.1), we see that

$$\begin{aligned}\theta_T(w')\theta_T(z')^* &= D_{T^*} - (1 - w'z'^*)D_{T^*}(I - w'T^*)^{-1}(I - Tz'^*)^{-1}D_{T^*} \\ &= I - (1 - \langle\varphi_a(w), \varphi_a(z)\rangle)D_{T^*}(I - w'T^*)^{-1} \\ &\quad \times (I - Tz'^*)^{-1}D_{T^*}.\end{aligned}$$

Note that

$$1 - \langle\varphi_a(w), \varphi_a(z)\rangle = \frac{(1 - |a|^2)(1 - \langle w, z\rangle)}{(1 - \langle w, a\rangle)(1 - \langle a, z\rangle)}.$$

Therefore the last equality implies that

$$\begin{aligned}\theta_T(w')\theta_T(z')^* = I &- \frac{(1 - |a|^2)(1 - \langle w, z\rangle)}{(1 - \langle w, a\rangle)(1 - \langle a, z\rangle)} \\ &\times D_{T^*}(I - w'T^*)^{-1}(I - Tz'^*)^{-1}D_{T^*}.\end{aligned} \tag{2.4}$$

On the other hand, by (2.1) and an application of Corollary 2.3, we have

$$\begin{aligned}&U\theta_{T_a}(w)\theta_{T_a}(z)^*U^* \\ &\quad = I - (1 - \langle w, z\rangle)UD_{T_a^*}(I - WT_a^*)^{-1}(I - T_a Z^*)^{-1}D_{T_a^*}U^* \\ &\quad = I - (1 - \langle w, z\rangle)D_{T^*}S^*(I - WT_a^*)^{-1}(I - T_a Z^*)^{-1}SD_{T^*} \\ &\quad = I - (1 - \langle w, z\rangle)(1 - |a|^2)D_{T^*}(I - AT^*)^{-1}(I - WT_a^*)^{-1} \\ &\qquad \times (I - T_a Z^*)^{-1}(I - TA^*)^{-1}D_{T^*}.\end{aligned}$$

The last equality along with (2.2) and (2.3) completes the proof. ∎

3. Model and coincidence

Recall that a commuting tuple $T \in \mathcal{B}(\mathcal{H})^n$ is called a spherical isometry if the column operator $\mathcal{H} \to \mathcal{H}^n,\ x \mapsto (T_i x)_{i=1}^n$ is an isometry. We shall say that T is a co-isometry if the column operator $T^*: \mathcal{H} \to \mathcal{H}^n$ is an isometry.

DEFINITION 3.1

A commuting contractive tuple T on $\mathcal{H}$ is called completely non-coisometric (CNC) if there is no non-trivial closed joint invariant subspace $\mathcal{M}$ of $T_1^*, \ldots, T_n^*$ such that the tuple $P_{\mathcal{M}} T|_{\mathcal{M}} = (P_{\mathcal{M}} T_1|_{\mathcal{M}}, \ldots, P_{\mathcal{M}} T_n|_{\mathcal{M}})$ is a co-isometry.

Given a commuting contractive n-tuple of operators T on a Hilbert space $\mathcal{H}$, one can define a bounded operator $j: \mathcal{H} \to H_n^2(\mathcal{D}_{T^*})$ by

$$j(h) = \sum_{\alpha \in \mathbb{N}^n} \gamma_\alpha (D_{T^*} T^{*\alpha} h) z^\alpha. \tag{3.1}$$

It is well-known that $L = j^*: H_n^2(\mathcal{D}_{T^*}) \to \mathcal{H}$ is the unique continuous linear map satisfying

$$L(p \otimes \xi) = p(T) D_{T^*} \xi$$

for all $p \in \mathbb{C}[z_1, \ldots, z_n]$ and $\xi \in \mathcal{D}_{T^*}$ (Theorem 4.5 of [2]). The operator L intertwines $M_{z_i} \otimes I_{\mathcal{D}_{T^*}}$ and T_i for every $i = 1, \ldots, n$, and is closely related to the Poisson transform defined by Popescu in [19]. The following lemma gives a characterization of CNC tuples in terms of its adjoint $j = L^*$.

Lemma 3.2. *The kernel of the operator j is the largest invariant subspace for $T_1^*, \ldots, T_n^*$ such that $P_{\mathcal{M}} T|_{\mathcal{M}}$ is a co-isometry.*

Proof. Since $j T_i^* = (M_{z_i}^* \otimes I_{\mathcal{D}_{T^*}}) j$, for all i, the kernel of j is invariant for $T_1^*, \ldots, T_n^*$. From the defining formula (3.1) it follows that, for any $h \in \ker j$, we have $D_{T^*} h = 0$. Now

$$\|h\|^2 - \sum_{i=1}^n \|T_i^* h\|^2 = \left\langle \left(I - \sum_{i=1}^n T_i T_i^* \right) h, h \right\rangle = \|D_{T^*} h\|^2 = 0.$$

Therefore the tuple $P_{\ker j} T|_{\ker j}$ is a co-isometry.

If $\mathcal{M}$ is a closed subspace invariant under T_i^* for all $i = 1, \ldots, n$ such that $P_{\mathcal{M}} T|_{\mathcal{M}}$ is a co-isometry, then for all $\alpha \in \mathbb{N}^n$ and $h \in \mathcal{M}$, we have

$$\begin{aligned}\|D_{T^*} T^{*\alpha} h\|^2 &= \left\langle \left(I - \sum_{i=1}^n T_i T_i^* \right) T^{*\alpha} h, T^{*\alpha} h \right\rangle \\ &= \|T^{*\alpha} h\|^2 - \sum_{i=1}^n \|T_i^* T^{*\alpha} h\|^2 = 0.\end{aligned}$$

Hence $\mathcal{M}$ is contained in the kernel of j. ■

Let T be a commuting contractive tuple on $\mathcal{H}$. The characteristic function θ_T of T induces a contractive multiplier $M_{\theta_T}: H_n^2 \otimes \mathcal{D}_T \to H_n^2 \otimes \mathcal{D}_{T^*}$. More precisely, one can show that (Proposition 1.2 of [13])

$$(1 - \theta_T(w) \theta_T(z)^*)/(1 - \langle w, z \rangle) = k_T(w)^* k_T(z)$$

for $z, w \in \mathbb{B}_n$, where $k_T(z) = (I - Tz^*)^{-1} D_{T^*}$. The positive definiteness of the kernel on the left is equivalent to the fact that M_{θ_T} is a contractive multiplier.

It is well-known (eq. (1.11) of [3]) that the intertwining map L acts as

$$L(k(\cdot, z) \otimes x) = k_T(z)x, \quad z \in \mathbb{B}_n, x \in \mathcal{D}_{T^*},$$

where $k\colon \mathbb{B}_n \times \mathbb{B}_n \to \mathbb{C}, k(z, w) = (1 - \langle w, z\rangle)^{-1}$, is the reproducing kernel of H_n^2. By Lemma 3.2 the tuple T is completey non-coisometric if and only if

$$\mathcal{H} = \overline{\text{span}}\,\{k_T(z)x\colon z \in \mathbb{B}_n, x \in \mathcal{D}_{T^*}\}.$$

In the following we shall use that the dilation map L and the characteristic multiplier M_{θ_T} of T satisfy the relations

$$LL^* + A_T = I_{\mathcal{H}}, \tag{3.2}$$

$$L^*L + M_{\theta_T} M_{\theta_T}^* = I_{H_n^2 \otimes \mathcal{D}_{T^*}}. \tag{3.3}$$

For a proof, see [2] and [8]. In the particular case, that $T = M_z \in L(H_n^2)^n$ is the standard shift, the map L is a unitary operator and hence $\theta_{M_z} = 0$.

To construct a functional model for a given completely non-coisometric commuting contractive tuple T, let us denote by $\Delta = (I_{H_n^2 \otimes \mathcal{D}_T} - M_{\theta_T}^* M_{\theta_T})^{1/2}$ the defect operator of M_{θ_T}.

Lemma 3.3. *Let T be a CNC commuting contractive tuple on $\mathcal{H}$. Then there is a unique contractive linear operator $r\colon \mathcal{H} \to H_n^2 \otimes \mathcal{D}_T$ such that*

(1) $r(k_T(z)x) = -\Delta(k(\cdot, z) \otimes \theta_T(z)^*x)$ *for* $z \in \mathbb{B}_n$ *and* $x \in \mathcal{D}_{T^*}$;
(2) $\|h\|^2 = \|j(h)\|^2 + \|r(h)\|^2$ *for all* $h \in \mathcal{H}$;
(3) $rL = -\Delta M_{\theta_T}^*$.

Proof. By the remarks preceding the lemma the uniqueness is clear. To prove the existence, first observe that

$$\begin{aligned}
&\left\| \Delta \sum_{i=1}^m k(\cdot, z^{(i)}) \otimes \theta_T(z^{(i)})^* x_i \right\|^2 \\
&\quad = \sum_{i,j=1}^m \langle (I - M_{\theta_T}^* M_{\theta_T})(k(\cdot, z^{(i)}) \otimes \theta_T(z^{(i)})^* x_i, k(\cdot, z^{(j)}) \otimes \theta_T(z^{(j)})^* x_j\rangle \\
&\quad = \sum_{i,j=1}^m \left[\frac{\langle \theta_T(z^{(j)})\theta_T(z^{(i)})^* x_i, x_j\rangle}{1 - \langle z^{(j)}, z^{(i)}\rangle} \right. \\
&\qquad \left. - \langle (I - L^*L)^2 (k(\cdot, z^{(i)}) \otimes x_i), k(\cdot, z^{(j)}) \otimes x_j\rangle \right]
\end{aligned}$$

for $z^{(1)}, \ldots, z^{(m)} \in \mathbb{B}_n$ and $x_1, \ldots, x_m \in \mathcal{D}_{T^*}$. Using the identity

$$\langle L^*L(k(\cdot, z^{(i)}) \otimes x_i), k(\cdot, z^{(j)}) \otimes x_j\rangle = \frac{\langle 1 - \theta_T(z^{(j)})\theta_T(z^{(i)})^* x_i, x_j\rangle}{1 - \langle z^{(j)}, z^{(i)}\rangle}$$

we find that

$$\begin{aligned}
&\left\| \Delta \sum_{i=1}^{m} k(\cdot, z^{(i)}) \otimes \theta_T(z^{(i)})^* x_i \right\|^2 \\
&\quad = \sum_{i,j=1}^{m} [\langle k_T(z^{(j)})^* k_T(z^{(i)}) x_i, x_j\rangle \\
&\qquad - \langle (L^*L)^2 k(\cdot, z^{(i)}) \otimes x_i, k(\cdot, z^{(j)}) \otimes x_j\rangle \\
&\quad = \left\| \sum_{i=1}^{m} k_T(z^{(i)}) x_i \right\|^2 - \left\| L^* \left(\sum_{i=1}^{m} k_T(z^{(i)}) x_i \right) \right\|^2 .
\end{aligned}$$

Hence there is a unique contractive linear map $r \colon \mathcal{H} \to H_n^2 \otimes \mathcal{D}_T$ satisfying condition (1). The above computation shows that condition (2) holds as well. The proof is completed by the observation that

$$\begin{aligned}
rL(k(\cdot, z) \otimes x) = r(k_T(z) \otimes x) &= -\Delta(k(\cdot, z) \otimes \theta_T(z)^* x) \\
&= -\Delta M_{\theta_T}^* (k(\cdot, z) \otimes x)
\end{aligned}$$

for all $z \in \mathbb{B}_n$ and $x \in \mathcal{D}_{T^*}$. ∎

The observation (2) of the lemma above allows us to define an isometry

$$V \colon \mathcal{H} \to (H_n^2 \otimes \mathcal{D}_{T^*}) \oplus \overline{\operatorname{Ran} \Delta}, \quad Vh = jh \oplus rh.$$

Our next aim is to show that the range of V is the orthogonal complement of the range of the isometry $U \colon H_n^2 \otimes \mathcal{D}_T \to (H_n^2 \otimes \mathcal{D}_{T^*}) \oplus \overline{\operatorname{Ran} \Delta}$ defined by $U\xi = M_{\theta_T}\xi \oplus \Delta\xi$ for $\xi \in H_n^2 \otimes \mathcal{D}_T$.

Lemma 3.4. *Suppose T is a CNC commuting contractive tuple. Then the isometries U and V defined above satisfy the relation*

$$VV^* + UU^* = I_{(H_n^2 \otimes \mathcal{D}_{T^*}) \oplus \overline{\operatorname{Ran} \Delta}}.$$

Proof. Note that the block operator matrix for $VV^* + UU^*$ with respect to the decomposition $(H_n^2 \otimes \mathcal{D}_{T^*}) \oplus \overline{\operatorname{Ran} \Delta}$ is

$$\begin{bmatrix} L^*L + M_{\theta_T} M_{\theta_T}^* & L^*r^* + M_{\theta_T}\Delta \\ rL + \Delta M_{\theta_T}^* & rr^* + \Delta^2 \end{bmatrix}. \tag{3.4}$$

We know that $L^*L + M_{\theta_T} M_{\theta_T}^* = I_{H_n^2 \otimes \mathcal{D}_{T^*}}$ and $rL + \Delta M_{\theta_T}^* = 0$. So all that remains is to show that $rr^* + \Delta^2$ is the orthogonal projection onto $\overline{\operatorname{Ran} \Delta}$.

By definition, $\operatorname{Ran} r \subseteq \overline{\operatorname{Ran} \Delta}$, and therefore $\ker \Delta \subseteq \ker r^* = \ker rr^*$. Hence $rr^* + \Delta^2$ is zero on $(\operatorname{Ran} \Delta)^\perp$. Using condition (1) in Lemma 3.3, we find that

$$
\begin{aligned}
&\langle r^* \Delta(k(\cdot, z) \otimes x), k_T(w) y\rangle \\
&\quad = -\langle \Delta^2(k(\cdot, z) \otimes x), M_{\theta_T}^*(k(\cdot, w) \otimes y)\rangle \\
&\quad = -\langle M_{\theta_T}(I - M_{\theta_T}^* M_{\theta_T})(k(\cdot, z) \otimes x), k(\cdot, w) \otimes y\rangle \\
&\quad = -\langle (I - M_{\theta_T} M_{\theta_T}^*) M_{\theta_T}(k(\cdot, z) \otimes x), k(\cdot, w) \otimes y\rangle \\
&\quad = -\langle L M_{\theta_T}(k(\cdot, z) \otimes x), k_T(w) y\rangle
\end{aligned}
$$

holds for all $z, w \in \mathbb{B}_n$ and $x \in \mathcal{D}_T$, $y \in \mathcal{D}_{T^*}$. Therefore we obtain the intertwining relation

$$
r^* \Delta = -L M_{\theta_T}. \tag{3.5}
$$

But then the observation that

$$
r r^* \Delta + \Delta^3 = -r L M_{\theta_T} + \Delta^3 = \Delta M_{\theta_T}^* M_{\theta_T} + \Delta^3 = \Delta(I - \Delta^2) + \Delta^3 = \Delta
$$

suffices to complete the proof. ∎

Thus V and U are isometries with orthogonal ranges such that

$$
\operatorname{Ran} V \oplus \operatorname{Ran} U = (H_n^2 \otimes \mathcal{D}_{T^*}) \oplus \overline{\operatorname{Ran} \Delta}.
$$

According to Lemma 3.4, the isometry V induces a unitary operator between $\mathcal{H}$ and the space

$$
\mathbb{H}_T = ((H_n^2 \otimes \mathcal{D}_{T^*}) \oplus \overline{\operatorname{Ran} \Delta}) \ominus \{(M_{\theta_T} u, \Delta u) : u \in H_n^2 \otimes \mathcal{D}_T\}.
$$

To prove that the characteristic function is a complete unitary invariant we shall give a functional description of the operator tuple T with the help of the above unitary operator V. Define $\mathbb{T}_i \in \mathcal{B}(\mathbb{H}_T)$ for $i = 1, \ldots, n$ by $\mathbb{T}_i = V T_i V^*|_{\mathbb{H}_T}$. Given $(u, v) \in \mathbb{H}_T$, let h be the vector in $\mathcal{H}$ such that $(u, v) = Vh$. Then it follows that

$$
\mathbb{T}_i^*(u, v) = \mathbb{T}_i^* V h = V T_i^* h = (j T_i^* h, r T_i^* h) = ((M_{z_i}^* \otimes I_{\mathcal{D}_{T^*}}) j h, r T_i^* h). \tag{3.6}
$$

The vector $r T_i^* h$ is contained in $\overline{\operatorname{Ran} \Delta} = (\ker \Delta)^\perp$. Using (3.5) we see that

$$
\Delta r T_i^* h = -M_{\theta_T}^* L^* T_i^* h = -M_{\theta_T}^* (M_{z_i}^* \otimes I_{\mathcal{D}_{T^*}}) L^* h = (M_{z_i}^* \otimes I_{\mathcal{D}_{T^*}}) \Delta r h.
$$

So if $\Delta^{-1} \colon \operatorname{Ran} \Delta \to (\ker \Delta)^\perp = \overline{\operatorname{Ran} \Delta}$ denotes the inverse of the bijective linear map $\Delta \colon (\ker \Delta)^\perp \to \operatorname{Ran} \Delta$, then $r T_i^* h = (\Delta^{-1}(M_{z_i}^* \otimes I_{\mathcal{D}_{T^*}}) \Delta)(rh)$. Thus in view of (3.6), we have constructed the following functional model for any given completely non-coisometric commuting contractive tuple T.

Theorem 3.5. *Let T be a CNC commuting contractive tuple on a Hilbert space $\mathcal{H}$, and let the Hilbert space $\mathbb{H}_T$ be defined as above. Then T is unitarily equivalent to the tuple $\mathbb{T} \in \mathcal{B}(\mathbb{H}_T)^n$ whose action is given by*

$$
\mathbb{T}_i^*(u, v) = ((M_{z_i}^* \otimes I_{\mathcal{D}_{T^*}}) u, \Delta^{-1}(M_{z_i}^* \otimes I_{\mathcal{D}_T}) \Delta v)
$$

for $(u, v) \in \mathbb{H}_T$ and $i = 1, \ldots, n$.

As an application of the functional model constructed above, we prove that the characteristic function is a complete unitary invariant for completely non-coisometric commuting contractive tuples.

Theorem 3.6. *Suppose that $T \in \mathcal{B}(\mathcal{H})^n$ and $R \in \mathcal{B}(\mathcal{K})^n$ are CNC commuting contractive tuples. Then the characteristic functions θ_T and θ_R coincide if and only if T and R are unitarily equivalent.*

Proof. Suppose that T and R are unitarily equivalent, that is, there is a unitary operator $U\colon \mathcal{H} \to \mathcal{K}$ such that

$$UT_i = R_i U, \quad 1 \le i \le n.$$

Then it is elementary to prove that the operators $\oplus U\colon \mathcal{H}^n \to \mathcal{K}^n$ and $U\colon \mathcal{H} \to \mathcal{K}$ induce unitary operators

$$\tau = \oplus U\colon \mathcal{D}_T \to \mathcal{D}_R \quad \text{and} \quad \tau_* = U\colon \mathcal{D}_{T^*} \to \mathcal{D}_{R^*}$$

such that θ_T and θ_R coincide via τ and τ_*.

Conversely, suppose that there are unitary operators $\tau'\colon \mathcal{D}_T \to \mathcal{D}_R$ and $\tau'_*\colon \mathcal{D}_{T^*} \to \mathcal{D}_{R^*}$ with

$$\tau'_*\theta_T(z) = \theta_R(z)\tau', \quad z \in \mathbb{B}_n.$$

Then the induced operators $\tau = I \otimes \tau'\colon H_n^2 \otimes \mathcal{D}_T \to H_n^2 \otimes \mathcal{D}_R$ and $\tau_* = I \otimes \tau'_*\colon H_n^2 \otimes \mathcal{D}_{T^*} \to H_n^2 \otimes \mathcal{D}_{R^*}$ are unitary and satisfy the relations $\tau_* M_{\theta_T} = M_{\theta_R}\tau$. It follows that

$$\Delta_T = (I - M_{\theta_T}^* M_{\theta_T})^{1/2} = (I - \tau^* M_{\theta_R}^* M_{\theta_R}\tau)^{1/2} = \tau^*\Delta_R\tau.$$

Since the diagram

$$\begin{array}{ccc} H_n^2 \otimes \mathcal{D}_T & \xrightarrow{\quad\tau\quad} & H_n^2 \otimes \mathcal{D}_R \\ {\scriptstyle \begin{pmatrix} M_{\theta_T} \\ \Delta_T \end{pmatrix}} \Big\downarrow & & \Big\downarrow {\scriptstyle \begin{pmatrix} M_{\theta_R} \\ \Delta_R \end{pmatrix}} \\ (H_n^2 \otimes \mathcal{D}_{T^*}) \oplus \overline{\Delta_T(H_n^2 \otimes \mathcal{D}_T)} & \xrightarrow[\tau_* \oplus \tau]{} & (H_n^2 \otimes \mathcal{D}_{R^*}) \oplus \overline{\Delta_R(H_n^2 \otimes \mathcal{D}_R)} \end{array}$$

commutes, we obtain the unitary operator $\tau_* \oplus \tau\colon \mathbb{H}_T \to \mathbb{H}_R$ between the model spaces of T and R. We still have to prove that via this unitary operator the functional models $\mathbb{T} \in \mathcal{B}(\mathbb{H}_T)^n$ and $\mathbb{R} \in \mathcal{B}(\mathbb{H}_R)^n$ of T and R are unitarily equivalent. Thus we have to prove the identity

$$\begin{aligned} &((M_{z_i}^* \otimes I_{\mathcal{D}_{R^*}})\tau_* u, \Delta_R^{-1}(M_{z_i}^* \otimes I_{\mathcal{D}_R})\Delta_R\tau v) \\ &\quad = (\tau_*(M_{z_i}^* \otimes I_{\mathcal{D}_{T^*}})u, \tau\Delta_T^{-1}(M_{z_i}^* \otimes I_{\mathcal{D}_T})\Delta_T v) \end{aligned}$$

for all $(u, v) \in \mathbb{H}_T$ and $i = 1, \ldots, n$. However, the equality of the first components follows from the definition of τ_*. To prove the equality of the second components, denote by ξ the unique element in $(\ker \Delta_T)^\perp = \overline{\operatorname{Ran} \Delta_T}$ with

$$\Delta_T \xi = (M_{z_i}^* \otimes I_{\mathcal{D}_T}) \Delta_T v.$$

Then $\tau \Delta_T^{-1} (M_{z_i}^* \otimes I_{\mathcal{D}_T}) \Delta_T v = \tau \xi \in \overline{\operatorname{Ran} \Delta_R}$ satisfies

$$\begin{aligned} \Delta_R \tau \xi &= \tau \Delta_T \xi = \tau (M_{z_i}^* \otimes I_{\mathcal{D}_T}) \Delta_T v \\ &= (M_{z_i}^* \otimes I_{\mathcal{D}_R}) \tau \Delta_T v = (M_{z_i}^* \otimes I_{\mathcal{D}_R}) \Delta_R \tau v. \end{aligned}$$

Thus the second components also coincide.

Since both $T \in \mathcal{B}(\mathcal{H})^n$ and $R \in \mathcal{B}(\mathcal{K})^n$ are unitarily equivalent to their functional models $\mathbb{T} \in \mathcal{B}(\mathbb{H}_T)^n$ and $\mathbb{R} \in \mathcal{B}(\mathbb{H}_R)^n$, we conclude that T and R are unitarily equivalent. ■

In the one-dimensional case, Theorem 3.6 holds under the hypothesis that T and R are completely non-unitary contractions. A straightforward multivariable generalization of this notion would be to call a commuting contractive tuple $T \in \mathcal{B}(\mathcal{H})^n$ completely non-unitary if there is no non-zero reducing subspace $M \subset \mathcal{H}$ for T such that $T|M$ is a spherical unitary, that is, a normal spherical isometry. For $n \geq 2$, the non-trivial implication of Theorem 3.6 no longer hold under the weaker hypothesis that T and R are completely non-unitary. An elementary example is the following.

Let $V \in \mathcal{B}(\mathcal{H})$ be a completely non-unitary co-isometry on a complex Hilbert space $\mathcal{H} \neq 0$ (e.g., the unilateral left shift). Then the commuting pairs $T = (V, 0) \in \mathcal{B}(\mathcal{H})^2$ and $R = (0, V) \in \mathcal{B}(\mathcal{H})^2$ are completely non-unitary commuting contractive tuples which are certainly not unitarily equivalent. Since $D_T = D_V \oplus 1_{\mathcal{H}}$, $D_R = 1_{\mathcal{H}} \oplus D_V$ and $D_{T^*} = 0 = D_{R^*}$, the characteristic functions of T and R coincide.

We now relate our functional model to the model constructed by Müller and Vasilescu in [17].

PROPOSITION 3.7

Given a commuting contractive CNC tuple T on $\mathcal{H}$, there is a unique isometry φ: $\overline{\operatorname{Ran} A_T} \to \overline{\operatorname{Ran} \Delta}$ *such that $r = \varphi A_T^{1/2}$.*

Proof. Since for all $h \in \mathcal{H}$ the identity

$$\|A_T^{1/2} h\|^2 = \|h\|^2 - \|L^* h\|^2 = \|h\|^2 - \|jh\|^2 = \|rh\|^2$$

holds, there is a unique isometry φ: $\overline{\operatorname{Ran} A_T} = \overline{\operatorname{Ran} A_T^{1/2}} \to \overline{\operatorname{Ran} \Delta}$ such that $\varphi A_T^{1/2} = r$. ■

Note that, for all vectors $h \in \mathcal{H}$ the equality

$$\begin{aligned} \sum_{i=1}^n \|A_T^{1/2} T_i^* h\|^2 &= \left\langle \left(\sum_{i=1}^n T_i A_T T_i^* \right) h, h \right\rangle \\ &= \langle P_T(A_T) h, h \rangle = \langle A_T h, h \rangle = \|A_T^{1/2} h\|^2 \end{aligned}$$

holds. Hence there are bounded operators $U_i\colon \overline{\operatorname{Ran} A_T} \to \overline{\operatorname{Ran} A_T}$ such that

$$U_i(A_T^{1/2}h) = A_T^{1/2}T_i^* h$$

for $i = 1, \ldots, n$ and $h \in \mathcal{H}$. The operators U_i commute with each other and satisfy

$$\sum_{i=1}^{n} \|U_i h\|^2 = \|h\|^2, \quad h \in \overline{\operatorname{Ran} A_T}.$$

Let $W \in \mathcal{B}(\overline{\operatorname{Ran} \Delta})^n$ be a spherical isometry such that

$$W_i h = \varphi U_i \varphi^* h, \quad i = 1, \ldots, n, \ h \in \overline{\operatorname{Ran} \varphi}.$$

Using the notation introduced earlier in this section, we obtain that

$$\begin{aligned}
&[(M_{z_i}^* \otimes I_{\mathcal{D}_{T^*}}) \oplus W_i]Vh \\
&\quad = (jT_i^* h, W_i rh) = (jT_i^* h, W_i \varphi A_T^{1/2} h) \\
&\quad = (jT_i^* h, \varphi U_i A_T^{1/2} h) = (jT_i^* h, \varphi A_T^{1/2} T_i^* h) \\
&\quad = VT_i^* h = \mathbb{T}_i^*(Vh)
\end{aligned}$$

for $i = 1, \ldots, n$ and $h \in \mathcal{H}$. Therefore $\mathbb{T}^* \in \mathcal{B}(\mathbb{H}_T)^n$ is the restriction of the commuting tuple $(M_z^* \otimes I_{\mathcal{D}_{T^*}}) \oplus W$ on $(H_n^2 \otimes \mathcal{D}_{T^*}) \oplus \overline{\operatorname{Ran} \Delta}$ to the invariant subspace $\mathbb{H}_T$. Summarizing we obtain the completely non-coisometric case of a result of Müller and Vasilescu (Theorem 11 of [17]).

Theorem 3.8. *Let T be a CNC commuting contractive tuple. Then there is a spherical isometry W on $\overline{\operatorname{Ran} \Delta}$ such that T^* is unitarily equivalent to the restriction of the tuple*

$$(M_z^* \otimes I_{\mathcal{D}_{T^*}}) \oplus W \in \mathcal{B}((H_n^2 \otimes \mathcal{D}_{T^*}) \oplus \overline{\operatorname{Ran} \Delta})^n$$

to the invariant subspace $\mathbb{H}_T$.

By a result of Athavale (Proposition 2 of [4]) the spherical isometry W extends to a spherical unitary, that is, a commuting tuple $N = (N_1, \ldots, N_n)$ of normal operators satisfying the identity $\sum_{i=1}^{n} N_i N_i^* = I$.

Our final aim in this section is to show that in the class of CNC commuting contractive tuples, it is enough to consider weak coincidence of characteristic functions.

Theorem 3.9. *Let T and R be two CNC commuting contractive tuples of operators acting on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. If the two analytic operator-valued functions $\{\mathcal{D}_T, \mathcal{D}_{T^*}, \theta_T\}$ and $\{\mathcal{D}_R, \mathcal{D}_{R^*}, \theta_R\}$ coincide weakly, then they coincide.*

Proof. By definition of weak coincidence, there is a unitary $\tau\colon \mathcal{D}_{T^*} \to \mathcal{D}_{R^*}$ such that for all $z, w \in \mathbb{B}_n$, we have $\theta_R(w)\theta_R(z)^* = \tau\theta_T(w)\theta_T(z)^*\tau^*$ and hence $(I_{\mathcal{D}_{R^*}} - \theta_R(w)\theta_R(z)^*) = \tau(I_{\mathcal{D}_{T^*}} - \theta_T(w)\theta_T(z)^*)\tau^*$. Using (2.1), we get

$$\begin{aligned}
&D_{R^*}(I - WR^*)^{-1}(I - RZ^*)^{-1}D_{R^*} \\
&\quad = \tau D_{T^*}(I - WT^*)^{-1}(I - TZ^*)^{-1}D_{T^*}\tau^*, \quad \text{for all } z, w \in \mathbb{B}_n.
\end{aligned}$$

Now letting $k_T(z) = (I - TZ^*)^{-1} D_{T^*}$ for all $z \in \mathbb{B}_n$, we have

$$k_R(w)^* k_R(z) = \tau k_T(w)^* k_T(z) \tau^*, \quad \text{for all } z, w \in \mathbb{B}_n.$$

A standard uniqueness result for factorization of operator-valued positive definite maps implies now that there is a unitary

$$U \colon \overline{\text{span}}\,\{k_R(z)\xi \colon z \in \mathbb{B}_n, \xi \in \mathcal{D}_{R^*}\} \to \overline{\text{span}}\,\{k_T(z)\eta \colon z \in \mathbb{B}_n, \eta \in \mathcal{D}_{T^*}\}$$

which satisfies

$$U k_R(z)\xi = k_T(z)\tau^* \xi. \tag{3.7}$$

Now note that $k_R(z)\xi = L_R(k(\cdot, z) \otimes \xi)$ so that we get from (3.7)

$$U L_R = L_T(I \otimes \tau^*).$$

Invoking the CNC assumption, we see that $\mathcal{H} = \overline{\text{Ran}}\ L_T = \overline{\text{span}}\,\{k_T(z)\eta \colon z \in \mathbb{B}_n, \eta \in \mathcal{D}_{T^*}\}$ and $\mathcal{K} = \overline{\text{Ran}}\ L_R = \overline{\text{span}}\,\{k_R(z)\xi \colon z \in \mathbb{B}_n, \xi \in \mathcal{D}_{R^*}\}$ so that U is a unitary from $\mathcal{K}$ to $\mathcal{H}$. We shall show that $UR_i = T_i U$ for all $i = 1, 2, \ldots, n$. It is enough to show that $UR_i L_R = T_i U L_R$ for all $i = 1, 2, \ldots, n$. But

$$\begin{aligned} UR_i L_R &= UL_R(M_{z_i} \otimes I_{\mathcal{D}_{R^*}}) = L_T(I \otimes \tau^*)(M_{z_i} \otimes I_{\mathcal{D}_{R^*}}) \\ &= L_T(M_{z_i} \otimes I_{\mathcal{D}_{R^*}})(I \otimes \tau^*) = T_i L_T(I \otimes \tau^*) \\ &= T_i U L_R. \end{aligned}$$

Hence the proof is complete. ■

4. A Beurling–Lax–Halmos theorem and characteristic functions

A function $\varphi \in M(\mathcal{E}, \mathcal{E}_*)$ is called purely contractive if $\|\varphi(0)\eta\| < \|\eta\|$ for all non-zero $\eta \in \mathcal{E}$, and it is called inner if M_φ is a partial isometry. The characteristic function is always purely contractive. It is inner when the tuple is pure. The last assertion follows from (3.3). Our first aim in this section is to prove the following version of the classical Beurling–Lax–Halmos theorem (see [16]).

Theorem 4.1. *Let $\mathcal{E}$ be a Hilbert space. Then a closed subspace $\mathcal{M}$ of $H_n^2 \otimes \mathcal{E}$ is invariant under $M_z \otimes I_{\mathcal{E}}$ if and only if it is of the form*

$$\mathcal{M} = (H_n^2 \otimes \mathcal{X}) \oplus \mathcal{Y},$$

where $\mathcal{X}$ is a closed subspace of $\mathcal{E}$ and $\mathcal{Y}$ is a closed subspace of $H_n^2 \otimes \mathcal{E}$ which is invariant under $M_z \otimes I_{\mathcal{E}}$ and contains no reducing subspace of $M_z \otimes I_{\mathcal{E}}$. Moreover, there is a Hilbert space $\mathcal{F}$ and a purely contractive, inner function $\varphi \in \mathcal{M}(\mathcal{F}, \mathcal{E})$ such that $\mathcal{Y} = M_\varphi(H_n^2 \otimes \mathcal{F})$.

To prove Theorem 4.1, we need some preparations.

Lemma 4.2. *A closed subspace $\mathcal{M}$ of $H_n^2 \otimes \mathcal{E}$ is reducing for the multiplication tuple $M_z \otimes I_{\mathcal{E}}$ if and only if there exists a closed subspace $\mathcal{L}$ of $\mathcal{E}$ such that $\mathcal{M} = H_n^2 \otimes \mathcal{L}$.*

Proof. Obviously every subspace of the form $H_n^2 \otimes \mathcal{L}$, where $\mathcal{L}$ is a closed subspace of $\mathcal{E}$, is reducing for $M_z \otimes I_{\mathcal{E}}$.

Conversely, let $\mathcal{M}$ be a reducing subspace. Denote by $P_{\mathcal{M}}$ the orthogonal projection onto $\mathcal{M}$ and by $P_{\mathcal{E}} \in \mathcal{B}(H_n^2 \otimes \mathcal{E})$ the orthogonal projection onto the subspace of all constant $\mathcal{E}$-valued functions. Then

$$P_{\mathcal{E}} = \left(I_{H_n^2} - \sum_{i=1}^{n} M_{z_i} M_{z_i}^* \right) \otimes I_{\mathcal{E}}$$

and hence $P_{\mathcal{M}} P_{\mathcal{E}} = P_{\mathcal{E}} P_{\mathcal{M}}$. Define $\mathcal{L} = P_{\mathcal{M}} \mathcal{E}$. Then $\mathcal{L} = \mathcal{M} \cap \mathcal{E} \subset \mathcal{E}$ is a closed subspace with

$$H_n^2 \otimes \mathcal{L} = \overline{\operatorname{span}}\, \{z^k \otimes h\colon k \in \mathbb{N}^n \text{ and } h \in \mathcal{L}\} \subset \mathcal{M}.$$

To show the opposite inclusion, let $f = \sum_{k \in \mathbb{N}^n} z^k \otimes \eta_k \in \mathcal{M}$ with $\eta_k \in \mathcal{E}$. Then the proof is completed by the observation that

$$\begin{aligned} f = P_{\mathcal{M}} f &= \sum_{k \in \mathbb{N}^n} P_{\mathcal{M}}(z^k \otimes \eta_k) = \sum_{k \in \mathbb{N}^n} P_{\mathcal{M}}(M_z^k \otimes I_{\mathcal{E}})(1 \otimes \eta_k) \\ &= \sum_{k \in \mathbb{N}^n} (M_z^k \otimes I_{\mathcal{E}}) P_{\mathcal{M}}(1 \otimes \eta_k) \in H_n^2 \otimes \mathcal{L}. \end{aligned}$$

■

Lemma 4.3. *Let $\mathcal{N}$ be an invariant subspace for the tuple $M_z \otimes I_{\mathcal{E}}$ on $H_n^2 \otimes \mathcal{E}$. Then there is a Hilbert space $\mathcal{F}$ and a purely contractive inner function $\varphi \in \mathcal{M}(\mathcal{F}, \mathcal{E})$ such that $\mathcal{N} = M_\varphi(H_n^2 \otimes \mathcal{F})$ if and only if $\mathcal{N}$ does not contain any non-trivial reducing subspace of $M_z \otimes I_{\mathcal{E}}$.*

Proof. First suppose that $\mathcal{N}$ does not contain any non-trivial reducing subspace. Define T to be the compression of $M_z \otimes I_{\mathcal{E}}$ to the subspace $\mathcal{N}^\perp$, that is,

$$T_i = P_{\mathcal{N}^\perp}(M_{z_i} \otimes I_{\mathcal{E}})|_{\mathcal{N}^\perp} \quad \text{for } i = 1, 2, \ldots, n.$$

Then T is a pure commuting contractive tuple.

Since the C^*-subalgebra of $\mathcal{B}(H_n^2)$ generated by M_z is of the form

$$C^*(M_z) = \overline{\operatorname{span}}\, \{M_z^k M_z^{*j}\colon k, j \in \mathbb{N}^n\} \tag{4.1}$$

(Theorem 5.7 of [2]), the space

$$\mathcal{M} = \overline{\operatorname{span}} \bigcup \{(M_z^k \otimes I_{\mathcal{E}})\mathcal{N}^\perp\colon k \in \mathbb{N}^n\}$$

is a reducing subspace for $M_z \otimes I_{\mathcal{E}}$ which contains $\mathcal{N}^\perp$. Therefore $\mathcal{N}$ contains the reducing subspace $\mathcal{M}^\perp$. Thus by hypothesis $\mathcal{M}^\perp$ is $\{0\}$ and hence $\mathcal{M} = H_n^2 \otimes \mathcal{E}$.

On the other hand, it is elementary to check that

$$H_n^2 \otimes \mathcal{D}_{T^*} = \overline{\operatorname{span}} \bigcup \{(M_z^k \otimes I_{\mathcal{D}_{T^*}}) j \mathcal{N}^\perp\colon k \in \mathbb{N}^n\},$$

where $j\colon \mathcal{N}^\perp \to H_n^2 \otimes \mathcal{D}_{T^*}$ is the isometry associated with the pure commuting contractive tuple T according to formula (3.1).

Using (4.1) one can easily show that there is a unique unitary operator $U\colon H_n^2 \otimes \mathcal{D}_{T^*} \to H_n^2 \otimes \mathcal{E}$ such that

$$U(M_z^k \otimes I_{\mathcal{D}_{T^*}})(jh) = (M_z^k \otimes I_{\mathcal{E}})h$$

for all $k \in \mathbb{N}^n$ and $h \in \mathcal{N}^\perp$. In particular, $U(\text{Ran } j) = \mathcal{N}^\perp$. But then U is a unitary operator that intertwines $M_z \otimes I_{\mathcal{D}_{T^*}}$ and $M_z \otimes I_{\mathcal{E}}$. By a well-known commutant lifting theorem (Theorem 5.1 of [6]), there is a multiplier $u \in \mathcal{M}(\mathcal{D}_{T^*}, \mathcal{E})$ with $U = M_u$. A standard argument shows that u has to be of the form $u \equiv \tau$ for some unitary operator $\tau\colon \mathcal{D}_{T^*} \to \mathcal{E}$. Then $\varphi(z) = \tau\theta_T(z)$ defines a purely contractive inner multiplier $\varphi \in \mathcal{M}(\mathcal{D}_T, \mathcal{E})$ with

$$\mathcal{N} = [(I \otimes \tau)\text{Ran } j]^\perp = (I \otimes \tau)(\text{Ran } j)^\perp = \text{Ran } M_\varphi.$$

Conversely, let $\varphi \in \mathcal{M}(\mathcal{F}, \mathcal{E})$ be a purely contractive inner multiplier, and let $\mathcal{L} \subset \mathcal{E}$ be a closed subspace such that $H_n^2 \otimes \mathcal{L} \subset \text{Ran } M_\varphi$. Then, for $\eta \in \mathcal{L}$, we obtain that

$$1 \otimes \eta = P_{\mathcal{E}}(1 \otimes \eta) = P_{\mathcal{E}} M_\varphi M_\varphi^*(1 \otimes \eta) = \varphi(0)\varphi(0)^*\eta.$$

Since φ is purely contractive, it follows that $\mathcal{L} = \{0\}$. ∎

As a particular case of the above lemma we obtain the following result for characteristic multpliers.

COROLLARY 4.4

Let T be a pure commuting contractive tuple of operators on some Hilbert space $\mathcal{H}$. Then Ran M_{θ_T} *contains no non-trivial reducing subspace for the multiplication tuple $M_z \otimes I_{\mathcal{D}_{T^*}}$.*

Proof of Theorem 4.1. Let $\mathcal{M}$ be a closed subspace of $H_n^2 \otimes \mathcal{E}$ invariant for the tuple $M_z \otimes I_{\mathcal{E}}$. By Lemma 4.2, any reducing subspace of $H_n^2 \otimes \mathcal{E}$ for the multiplication tuple $M_z \otimes I_{\mathcal{E}}$ is of the form $H_n^2 \otimes \mathcal{L}$ for some closed subspace $\mathcal{L}$ of $\mathcal{E}$. Define

$$\mathcal{C}(\mathcal{M}) = \{\mathcal{L}\colon \mathcal{L} \text{ is a closed subspace of } \mathcal{E} \text{ and } H_n^2 \otimes \mathcal{L} \subseteq \mathcal{M}\}$$

and

$$\mathcal{X} = \overline{\text{span}} \cup \{\mathcal{L}\colon \mathcal{L} \in \mathcal{C}(\mathcal{M})\}, \quad \mathcal{Y} = \mathcal{M} \ominus (H_n^2 \otimes \mathcal{X}).$$

Clearly $\mathcal{Y}$ is an invariant subspace for $M_z \otimes I_{\mathcal{E}}$ which does not contain any non-zero reducing subspace. To complete the proof, it suffices to apply Lemma 4.3. ∎

It was shown in [8] that the characteristic function of a pure commuting contractive tuple is purely contractive and inner. We end this note with the converse.

Theorem 4.5. *Let $\mathcal{E}$ and $\mathcal{E}_*$ be Hilbert spaces and let $\theta \in \mathcal{M}(\mathcal{E}, \mathcal{E}_*)$ be purely contractive and inner. Then there is a Hilbert space $\mathcal{H}$ and a pure commuting contractive tuple of operators T on $\mathcal{H}$ such that the function $\{\mathcal{E}, \mathcal{E}_*, \theta\}$ coincides weakly with $\{\mathcal{D}_T, \mathcal{D}_{T^*}, \theta_T\}$. Furthermore, the tuple $T \in \mathcal{B}(\mathcal{H})^n$ is uniquely determined up to unitary equivalence.*

Proof. Define $\mathcal{N} = \text{Ran } M_\theta$ and $T = P_{\mathcal{N}^\perp} M_z|\mathcal{N}^\perp$. As shown in the proof of Lemma 4.3, there is a unitary operator $U = I \otimes \tau\colon H_n^2 \otimes \mathcal{D}_{T^*} \to H_n^2 \otimes \mathcal{E}_*$ such that

$$U \circ j\colon \mathcal{N}^\perp \to H_n^2 \otimes \mathcal{E}_*$$

is the inclusion mapping. Here we have used the notations established in Lemma 4.3. Using eq. (3.2) we find that

$$UM_{\theta_T}M_{\theta_T}^*U^* = I - Ujj^*U^* = P_{\mathcal{N}} = M_\theta M_\theta^*.$$

Applying both sides on vectors of the form $k(\cdot, w) \otimes x$ and forming scalar product with $k(\cdot, z) \otimes y$, one obtains that

$$\tau\theta_T(w)\theta_T(z)^*\tau^* = \theta(w)\theta(z)^*$$

for all $z, w \in \mathbb{B}_n$.

Suppose that $R \in \mathcal{B}(\mathcal{K})^n$ is a pure commuting contractive tuple such that $\{\mathcal{E}, \mathcal{E}_*, \theta\}$ and $\{\mathcal{D}_R, \mathcal{D}_{R^*}, \theta_R\}$ coincide weakly. By definition there is a unitary operator $\sigma\colon \mathcal{D}_{R^*} \to \mathcal{E}_*$ such that

$$\sigma\theta_R(w)\theta_R(z)^*\sigma^* = \theta(w)\theta(z)^*, \quad z, w \in \mathbb{B}_n.$$

By reversing the arguments from the previous paragraph we find that

$$VM_{\theta_R}M_{\theta_R}^*V^* = M_\theta M_\theta^*,$$

where $V = I_{H_n^2} \otimes \sigma$. Hence V induces a unitary operator

$$V\colon (\mathrm{Ran}\ M_{\theta_R})^\perp \to (\mathrm{Ran}\ M_\theta)^\perp$$

intertwining the compressions of $M_z \otimes I_{\mathcal{D}_{R^*}}$ and $M_z \otimes I_{\mathcal{E}_*}$ on both spaces. Hence R and T are unitarily equivalent. ∎

The above theorem shows that up to weak coincidence each purely contractive inner function $\theta \in \mathcal{M}(\mathcal{E}, \mathcal{E}_*)$ is the characteristic function of a uniquely determined pure commuting contractive tuple T. It would be desirable to decide when $\{\mathcal{E}, \mathcal{E}_*, \theta\}$ and $\{\mathcal{D}_T, \mathcal{D}_{T^*}, \theta_T\}$ strongly coincide. Lemma 2.5 gives at least a first answer to this question.

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