CONTRACTIVE REPRESENTATIONS OF ODOMETER SEMIGROUP

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Abstract. Given a natural number $n \geq 1$, the odometer semigroup O_n, also known as the adding machine or the Baumslag-Solitar monoid with two generators, is a well-known object in group theory. This paper examines the odometer semigroup in relation to representations of bounded linear operators. We focus on noncommutative operators and prove that contractive representations of O_n always admit to nicer representations of O_n. We give a complete description of representations of O_n on the Fock space and relate it to the odometer lifting and subrepresentations of O_n. Along the way, we also classify Nica covariant representations of O_n.

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1. Introduction

Let $n \geq 1$ be a natural number. The *odometer semigroup* O_n is a semigroup formed by n generators $\{v_1, \ldots, v_n\}$ along with a generator w that satisfies the conditions

$$wv_i = \begin{cases} v_{i+1} & \text{if } i = 1, \ldots, n-1 \\ v_1 w & \text{if } i = n. \end{cases}$$

This semigroup is frequently referred to as the adding machine or the Baumslag-Solitar monoid with two generators [11]. This can also be seen as a Zappa-Szép product of the free semigroup with n generators. Researchers working in group theory have widely recognized the importance of this object; for instance, see [1, 5] and the references therein for more advancement. This is also useful for understanding algebraic-analytic structures, including Toeplitz algebras [4], von Neumann algebras

2020 *Mathematics Subject Classification.* 47A13, 46L05, 47A20, 30H10, 20E08, 32A35.

Key words and phrases. Odometer semigroup, row contractions, invariant subspaces, isometric representations, dilations, Nica covariant representations, subrepresentations.

[8], C^*-algebras [21], semigroup C^*-algebras [2, 13, 14, 15], and numerous others. Our main goal in this paper is to enhance it with representations of bounded linear operators on Hilbert spaces.

Let $\mathcal{H}$ be a Hilbert space. In this paper, all Hilbert spaces are separable and over $\mathbb{C}$. Let $\mathcal{H}^n$ denote the Hilbert space of the n-direct sum of $\mathcal{H}$ with itself. By an *n-row operator*, or simply a *row operator* (as n will be clear from the context), we mean a bounded linear operator

$$T := (T_1, \ldots, T_n) : \mathcal{H}^n \to \mathcal{H},$$

where $T_i \in \mathcal{B}(\mathcal{H})$ for all $i = 1, \ldots, n$. Here, given Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, we set the space of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ by $\mathcal{B}(\mathcal{H}, \mathcal{K})$. If $\mathcal{H} = \mathcal{K}$, we simply write this space as $\mathcal{B}(\mathcal{H})$. Given a row operator T acting on $\mathcal{H}$ as above and $W \in \mathcal{B}(\mathcal{H})$, we say that (W, T) is a *representation of O_n* if the following conditions are satisfied:

$$WT_i = \begin{cases} T_{i+1} & \text{if } i = 1, \ldots, n-1 \\ T_1 W & \text{if } i = n. \end{cases} \tag{1.1}$$

Without a doubt, the general representations of O_n are a too-wide problem that might not yield much unless certain meaningful restrictions are put on the row operators. This is where row contractions enter into our theory. A *row contraction* on $\mathcal{H}$ is defined as a row operator $T = (T_1, \ldots, T_n)$ that meets the following condition

$$\sum_{i=1}^{n} T_i T_i^* \leq I.$$

Equivalently, the row operator $T \in \mathcal{B}(\mathcal{H}^n, \mathcal{H})$ is a contraction. The row contraction T is said to be *pure* if

$$\lim_{m \to \infty} \sum_{|\mu| = m, \mu \in F_n^+} \|T_\mu^* f\|^2 = 0, \tag{1.2}$$

for all $f \in \mathcal{H}$. Here F_n^+ denotes the free semigroup generated by n alphabets $\{g_1, \ldots, g_n\}$ with identity g_0, and for each word $\mu = g_{\mu_1} \cdots g_{\mu_k} \in F_n^+$, $k \geq 1$, we write $T_\mu = T_{\mu_1} \cdots T_{\mu_k}$, and $T_{g_0} = I_{\mathcal{H}}$. The above limit condition is well-accepted and has broad applicability that aligns seamlessly with our context, which we shall elaborate on shortly. In general, we remark that the theory of noncommutative pure row contractions is renowned for its abundance, creativity, and ability to integrate a number of established theories [16, 17, 18], such as the Sz.-Nagy and Foias model theory, interpolation problem, Toeplitz operators, and C^*-algebras. The effect of the noncommutative dilation theory will also be heavily reflected in this paper. At this point, we introduce the central notion of this investigation.

Definition 1.1. *A contractive representation of O_n is a representation (W, T) of O_n such that T is a pure contraction.*

The goal of this paper is to present a complete picture of representations of O_n. Along the way, we offer a precise depiction of the operator W, the other generator of representations of O_n. The latter direction enhances the existing noncommutative dilation theory, which also relates to joint invariant subspaces (or subrepresentations) of certain natural contractive representations of O_n. Before we get into more details about the main results of this paper, we need to set up a notion that also provides natural examples of contractive representations of O_n.

A row contraction T is called *row isometry* if T_i is isometry for all $i = 1, \ldots, n$. Note that a row contraction T is a row isometry if and only if $T_i^* T_j = \delta_{ij} I$ for all $i, j = 1, \ldots, n$. Creation operators on the (full) Fock space provides concrete examples of row isometries: Denote by $\mathcal{F}_n^2$ the *Fock space* over $\mathbb{C}^n$, that is

$$\mathcal{F}_n^2 = \mathbb{C}\Omega \oplus \bigoplus_{k=1}^{\infty} (\mathbb{C}^n)^{\otimes k},$$

where Ω is a unit vector called *vacuum state*. Let $\{e_1, \ldots, e_n\}$ be the standard orthonormal basis of $\mathbb{C}^n$. For each $i = 1, \ldots, n$, the i-th *creation* or *left shift operator* S_i on $\mathcal{F}_n^2$ is defined by

$$S_i f = e_i \otimes f \qquad (f \in \mathcal{F}_n^2).$$

A simple computation shows that $(S_1, \ldots, S_n)$ is a pure row isometry. More generally, if $\mathcal{E}$ is a Hilbert space, then

$$S^{\mathcal{E}} := (S_1 \otimes I_{\mathcal{E}}, \ldots, S_n \otimes I_{\mathcal{E}}),$$

on $\mathcal{F}_n^2 \otimes \mathcal{E}$ is a pure row isometry. We furthermore emphasize that the creation operators on vector-valued Fock spaces are highly significant in noncommutative operator theory and are utilized in free analytic models [17]. For instance, the noncommutative Wold type decomposition theorem (see [9, Theorem 2] and [18, Theorem 1.3]) says that up to unitary equivalence, a pure row isometry is of the form $S^{\mathcal{E}}$ for some Hilbert space $\mathcal{E}$.

Our theory will center on representations of O_n that correspond to row isometries on vector-valued Fock spaces. We introduce them formally as follows:

Definition 1.2. *Given a Hilbert space $\mathcal{E}$ and an operator $W \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E})$, we refer to $(W, S^{\mathcal{E}})$ as a Fock representation of O_n, or simply a Fock representation. If, in addition, W is an isometry (unitary/contraction) operator, then we call it an isometric (unitary/contractive) Fock representation.*

We need to fix a notation. For each word $\mu \in F_n^+$, we write

$$e_\mu = \begin{cases} e_{\mu_1} \otimes \cdots \otimes e_{\mu_k} & \text{if } \mu = g_{\mu_1} g_{\mu_2} \ldots g_{\mu_k} \\ \Omega & \text{if } \mu = g_0. \end{cases}$$

It follows that the set $\{e_\mu : \mu \in F_n^+\}$ is an orthonormal basis for $\mathcal{F}_n^2$. As already pointed out, Fock representations will play an important role in our analysis. Our first result yields a complete description as well as a classification of Fock representations (see Theorem 2.2).

Theorem 1.3. *Let $\mathcal{E}$ be a Hilbert space, and let $W \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E})$. Then $(W, S^{\mathcal{E}})$ is a Fock representation if and only if there exists $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$ such that*

$$W = W_L,$$

where $W_L \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E})$ is defined by

$$W_L(\Omega \otimes \eta) = L\eta,$$

for all $\eta \in \mathcal{E}$, and for all $e_\mu = e_{\mu_1} \otimes \cdots \otimes e_{\mu_m} \in F_n^+$, $\mu \neq g_0$, define

$$W_L(e_\mu \otimes \eta) = \begin{cases} e_{\mu_1+1} \otimes e_{\mu_2} \otimes \cdots \otimes e_{\mu_m} \otimes \eta & \text{if } \mu_1 \neq n \\ e_1 \otimes e_{\mu_2+1} \otimes e_{\mu_3} \otimes \cdots \otimes e_{\mu_m} \otimes \eta & \text{if } \mu_1 = n, \mu_2 \neq n \\ e_1^{\otimes 2} \otimes e_{\mu_3+1} \otimes e_{\mu_4} \otimes \cdots \otimes e_{\mu_m} \otimes \eta & \text{if } \mu_1 = \mu_2 = n, \mu_3 \neq n \\ \vdots & \vdots \\ e_1^{\otimes m} \otimes L\eta & \text{if } \mu_1 = \cdots = \mu_m = n. \end{cases}$$

Moreover, if $W = W_L$ for some $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$, then L is unique.

Such a map W_L is referred to as an *odometer map* with the symbol $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$. It is a very curious fact that the odometer maps are defined for all $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$, and subsequently, the theme of symbols becomes comparable to the noncommutative Toeplitz operators [16] (see Remark 2.4 for more on this).

A special form of the above operator W_L was introduced by Li [12] in a restricted context, namely when W_L is unitary. He proved, in the language at hand, that $(W, S^{\mathcal{E}})$ is a unitary Fock representation if and only if there exists $L \in \mathcal{B}(\mathcal{E})$ such that $W = W_L$ [12, Corollary 3.6]. In our general case, we needed to adopt a more finer notion of the model operator W acting on vector-valued Fock spaces. Ultimately, we effectively improved Li's notion, which amounts to extending the idea of symbols from $\mathcal{B}(\mathcal{E})$ to $\mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$. We have applied this to the definition of W_L in the above theorem. This paper will also retrieve Li's result using a completely different methodology (see Theorem 4.4). In Remark 2.4, we will discuss more about the odometer maps and corresponding symbols.

In Theorem 3.1, we present a complete classification of isometric Fock representations. To explain the result, for each $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$, we define the closed subspace (see (3.2))

$$\mathcal{E}_L = \overline{\text{span}}\{e_1^{\otimes m} \otimes \eta : m \in \mathbb{Z}_+, \eta \in \mathcal{E}\} \ominus \overline{\text{span}}\{e_1^{\otimes p} \otimes L\zeta : p \geq 1, \zeta \in \mathcal{E}\}.$$

We shall also designate W_L as the odometer map that corresponds to the symbol L, as defined in Theorem 1.3. We have the following classification of isometric Fock representations:

Theorem 1.4. *Let $\mathcal{E}$ be a Hilbert space, and let $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$. Then $(W_L, S^{\mathcal{E}})$ is an isometric Fock representation if and only if the following conditions hold:*

(1) *L is an isometry.*
(2) *$L\mathcal{E} \subseteq \mathcal{E}_L$.*

In the scalar case (that is, $\mathcal{E} = \mathbb{C}$), odometer maps are precisely given by W_ξ, where ξ is in $\mathcal{F}_n^2$. In this case, the above result becomes even more concrete (see Corollary 3.2): Let $\xi \in \mathcal{F}_n^2$. Then W_ξ on $\mathcal{F}_n^2$ is an isometry if and only if

$$\xi = \sum_{p=0}^{\infty} c_p e_1^{\otimes p},$$

where the sequence of scalars $\{c_p\}_{p=0}^{\infty}$ satisfies the conditions $\sum_{p=0}^{\infty} |c_p|^2 = 1$, and $\sum_{p=0}^{\infty} c_{p+r}\overline{c_p} = 0$ for all $r \geq 1$.

The existence of vectors in $\mathcal{F}_n^2$ that satisfy the above conditions may appear counterintuitive. In this regard, we will present numerous constructions of such ξ in $\mathcal{F}_n^2$ in the final section. There, we will also present several examples of general isometric Fock representations.

Clearly, the evaluation of the conjugates of odometer maps is a basic question. How nice the general formula could be is not apparent. However, the conjugate operator formula is precise and significant for isometric odometer maps (see Proposition 4.1). This is a technique we applied to understand Nica-covariant representations. In fact, this is a specific category of Fock representations of O_n that is of interest (see [12, page 742] and also see [2, 14, 15]):

Definition 1.5. *Let $\mathcal{E}$ be a Hilbert space, and let $(W, S^{\mathcal{E}})$ be an isometric Fock representation of O_n. We say that $(W, S^{\mathcal{E}})$ is a Nica-covariant representation of O_n if*

$$W^*(S_1 \otimes I_{\mathcal{E}}) = (S_n \otimes I_{\mathcal{E}})W^*.$$

The classification of Nica covariant representations of O_n is rather clean (see Theorem 4.2):

Theorem 1.6. *Let $(W_L, S^{\mathcal{E}})$ be an isometric Fock representation. Then $(W_L, S^{\mathcal{E}})$ is a Nica covariant representation of O_n if and only if*

$$L\mathcal{E} \subseteq \Omega \otimes \mathcal{E}.$$

We will refer to a symbol that satisfies the above criteria as a *constant symbol.* Simply put, constant symbols precisely implement Nica covariant representations of O_n. In the scalar case, the above further simplifies to the following (see Theorem 4.3):

Theorem 1.7. *Let $\xi \in \mathcal{F}_n^2$, and suppose (W_ξ, S) is an isometric Fock representation. Then the following are equivalent:*

1. *(W_ξ, S) is a Nica covariant representation of O_n.*
2. *There exists $c \in \mathbb{T}$ such that $\xi = c\Omega$.*
3. *W_ξ is unitary.*

As a result of the classification of general Nica covariant representations of O_n along with some computations, we recover the structure of unitary Fock representations that were previously obtained in [12, Corollary 3.6]. Pursuant to the terminology used in our present paper, it states (see Theorem 4.4):

Theorem 1.8. *Let $\mathcal{E}$ be a Hilbert space, and let $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$ be an isometry. Then W_L is unitary if and only if*

$$L\mathcal{E} = \Omega \otimes \mathcal{E}.$$

Now we turn to the noncommutative dilation theory, and connect it with representations of O_n. Two row operators $T = (T_1, \dots, T_n)$ on $\mathcal{H}$ and $R = (R_1, \dots, R_n)$ on $\mathcal{K}$ are considered *unitarily equivalent* if there exists a unitary $U \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $UT_i = R_iU$ for all $i = 1, \dots, n$. This is abbreviated as

$$T \cong R.$$

The defect space $\mathcal{D}_{T^*}$ of a row contraction T on $\mathcal{H}$ is defined by

$$\mathcal{D}_{T^*} = \overline{\operatorname{ran}}\Big(I_{\mathcal{H}} - \sum_{j=1}^{n} T_jT_j^*\Big).$$

The noncommutative dilation theorem [18, Theorem 2.1] states the following (the present language is adapted to our current context): There exists an isometry $\Pi_T : \mathcal{H} \to \mathcal{F}_n^2 \otimes \mathcal{D}_{T^*}$ such that

$$\Pi_T T_i^* = (S_i \otimes I_{\mathcal{D}_{T^*}})^* \Pi_T,$$

for all $i = 1, \dots, n$. Moreover, $\mathcal{Q}_T$ is a joint $(S_1^* \otimes I_{\mathcal{D}_{T^*}}, \dots, S_n^* \otimes I_{\mathcal{D}_{T^*}})$-invariant subspace of $\mathcal{F}_n^2 \otimes \mathcal{D}_{T^*}$, where $\mathcal{Q}_T$ is the *model space* defined by

$$\mathcal{Q}_T = \Pi_T \mathcal{H},$$

and

$$T \cong (P_{\mathcal{Q}_T}(S_1 \otimes I_{\mathcal{D}_{T^*}})|_{\mathcal{Q}_T}, \dots, P_{\mathcal{Q}_T}(S_n \otimes I_{\mathcal{D}_{T^*}})|_{\mathcal{Q}_T}).$$

Now, in addition, we assume that (W, T) is a contractive representation of O_n. In Theorem 5.4 and Corollary 5.5, we prove the following dilation (or odometer lift on the model space) theorem:

Theorem 1.9. *Let (W, T) be a contractive representation of O_n. Then there exists a symbol $L \in \mathcal{B}(\mathcal{D}_{T^*}, \mathcal{F}^2 \otimes \mathcal{D}_{T^*})$ such that*

$$\Pi_T W^* = W_L^* \Pi_T.$$

Moreover, $(W, T) \cong (P_{\mathcal{Q}_T} W_L|_{\mathcal{Q}_T}, P_{\mathcal{Q}_T}(S_1 \otimes I_{\mathcal{D}_{T^*}})|_{\mathcal{Q}_T}, \dots, P_{\mathcal{Q}_T}(S_n \otimes I_{\mathcal{D}_{T^*}})|_{\mathcal{Q}_T})$.

We now turn to invariant subspaces of Fock representations. Invariant subspaces of any natural operator are of interest, and in our case, concrete representations of invariant subspaces would also induce sub-representations of O_n. Let $\mathcal{E}$ be a Hilbert space. By an *invariant subspace of* $\mathcal{F}_n^2 \otimes \mathcal{E}$, we mean a closed subspace $\mathcal{S}$ such that

$$(S_i \otimes I_{\mathcal{E}})\mathcal{S} \subseteq \mathcal{S},$$

for all $i = 1, \dots, n$. In addition, if $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$ is a symbol, and

$$W_L \mathcal{S} \subseteq \mathcal{S},$$

then we say that $\mathcal{S}$ is an *invariant subspace of the Fock representation* $(W_L, S^{\mathcal{E}})$. Clearly, if $\mathcal{S}$ is an invariant subspace of $(W_L, S^{\mathcal{E}})$, then $(W_L|_{\mathcal{S}}, S^{\mathcal{E}}|_{\mathcal{S}})$ on $\mathcal{S}$ is a contractive representation of O_n, where

$$S^{\mathcal{E}}|_{\mathcal{S}} = ((S_1 \otimes I_{\mathcal{E}})|_{\mathcal{S}}, \dots, (S_n \otimes I_{\mathcal{E}})|_{\mathcal{S}}).$$

We call $(W_L|_{\mathcal{S}}, S^{\mathcal{E}}|_{\mathcal{S}})$ a *subrepresentation* of the Fock representation $(W_L, S^{\mathcal{E}})$. From this vantage point, it is therefore natural to ask for a thorough description of subrepresentations of Fock representations. We will provide a clear answer to this question, and to clarify what we mean, we need to discuss the invariant subspaces of vector-valued Fock spaces.

First, we recall the noncommutative Beurling-Lax Halmos theorem [17, Theorem 2.2]: Let $\mathcal{S} \subseteq \mathcal{F}_n^2 \otimes \mathcal{E}$ be a closed subspace. Then $\mathcal{S}$ is an invariant subspace if and only if there exist a Hilbert space $\mathcal{E}_*$ and an inner multi-analytic operator $\Phi \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E}_*, \mathcal{F}_n^2 \otimes \mathcal{E})$ such that

$$\mathcal{S} = \Phi(\mathcal{F}_n^2 \otimes \mathcal{E}_*).$$

Recall that an operator $\Phi \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E}_*, \mathcal{F}_n^2 \otimes \mathcal{E})$ is inner multi-analytic if Φ is an isometry and

$$\Phi(S_i \otimes I_{\mathcal{E}_*}) = (S_i \otimes I_{\mathcal{E}})\Phi,$$

for all $i = 1, \dots, n$. We use standard techniques to reveal more information about Φ: If $\mathcal{S} = \Phi(\mathcal{F}_n^2 \otimes \mathcal{E}_*)$ is an invariant subspace of $\mathcal{F}_n^2 \otimes \mathcal{E}$ as above, then there exists a unitary $\Pi : \mathcal{F}_n^2 \otimes \mathcal{E}_* \to \mathcal{S}$ such that

$$\Phi = i_{\mathcal{S}} \circ \Pi,$$

where $i_{\mathcal{S}} : \mathcal{S} \hookrightarrow \mathcal{F}_n^2 \otimes \mathcal{E}$ is the isometric embedding. Therefore, the following diagram commutes:

$$\begin{tikzcd}
& \mathcal{F}_n^2 \otimes \mathcal{E}_* \arrow[dl, "\Pi"'] \arrow[d, "\Phi"] \\
\mathcal{S} \arrow[r, "i_{\mathcal{S}}"'] & \mathcal{F}_n^2 \otimes \mathcal{E}
\end{tikzcd}$$

This additional information about the factorization of inner multi-analytic operator plays a crucial role in proving the following (see Theorem 6.4):

Theorem 1.10. *Let $(W_L, S^{\mathcal{E}})$ be a Fock representation, and let $\mathcal{S} \subseteq \mathcal{F}_n^2 \otimes \mathcal{E}$ be a closed subspace. Then $\mathcal{S}$ is invariant under $(W_L, S^{\mathcal{E}})$ if and only if there exist a Hilbert space $\mathcal{E}_*$, an inner multi-analytic operator $\Phi \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E}_*, \mathcal{F}_n^2 \otimes \mathcal{E})$, and a Fock representation $(W_{L_*}, S^{\mathcal{E}_*})$ for some symbol*

$L_* \in \mathcal{B}(\mathcal{E}_*, \mathcal{F}_n^2 \otimes \mathcal{E}_*)$ *such that*

$$\mathcal{S} = \Phi(\mathcal{F}_n^2 \otimes \mathcal{E}_*),$$

and

$$W_L \Phi = \Phi W_{L_*}.$$

In the setting of this theorem, the existence of the inner multi-analytic operator Φ is due to Popescu. However, the fact that Φ admits the factorization $\Phi = i_{\mathcal{S}} \circ \Pi$ is new. Now, given $\mathcal{S}$ as an invariant subspace with the inner function Φ as above, if we also consider the odometer map W_L, saying that $\mathcal{S}$ is invariant under W_L becomes equivalent to finding a unique solution $X \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E}_*)$ to the operator equation (see the discussion preceding Theorem 6.4)

$$W_L \Phi = \Phi X.$$

Theorem 6.4 establishes that this condition occurs exclusively when X is an odometer map W_{L_*} with a unique symbol $L_* \in \mathcal{B}(\mathcal{E}_*, \mathcal{F}_n^2 \otimes \mathcal{E}_*)$. Furthermore, the operator L_* is defined as

$$L_* = \Pi^* W_L|_{\mathcal{E}_*}.$$

As a special case of the invariant subspace theorem, we have the following answer to the question concerning subrepresentations of Fock representations (see Corollary 6.5 for more details):

Corollary 1.11. *Let $\mathcal{S}$ be an invariant subspace of a Fock representation $(W_L, S^{\mathcal{E}})$. Then the subrepresentation $(W_L|_{\mathcal{S}}, S^{\mathcal{E}}|_{\mathcal{S}})$ is also a Fock representation.*

We finally remark that the representations of O_n yield pairs of commuting isometries when $n = 1$. Moreover, pairs of doubly commuting isometries arise from Nica-covariant representations of O_n.

The remaining sections of the paper are organized as follows: Section 2 is the backbone of this paper, where we completely characterize Fock representations. Section 3 provides an extensive classification of isometric Fock representations. Next, in Section 4, we present a complete description of Nica-covariant representations. This has been made possible by using explicit representations of conjugates of isometric odometer operators, which are also computed in this section. Section 5 connects dilation theory and Fock representations. More specifically, we prove that every contractive representation of O_n admits a dilation (or odometer lifting) in the sense of Fock representations. In Section 6, we classify invariant subspaces of Fock representations and essentially prove that subrepresentations of Fock representations are also Fock representations. The final section, Section 6, presents examples of various contractive representations of O_n.

2. Fock representations

Let $\mathcal{E}$ be a Hilbert space. Recall that

$$S^{\mathcal{E}} = (S_1 \otimes I_{\mathcal{E}}, \ldots, S_n \otimes I_{\mathcal{E}}),$$

denotes the n-tuple of creation operators on the $\mathcal{E}$-valued Fock space $\mathcal{F}_n^2 \otimes \mathcal{E}$. In this section, we aim at classifying Fock representations of O_n (see Definition 1.2). In the following, we introduce a new class of operators that depend on symbols that are operators in $\mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$. Recall that $\{e_\mu : \mu \in F_n^+\}$ is an orthonormal basis for $\mathcal{F}_n^2$.

Definition 2.1. *Let $\mathcal{E}$ be a Hilbert space and let $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$. The odometer map $W_L : \mathcal{F}_n^2 \otimes \mathcal{E} \to \mathcal{F}_n^2 \otimes \mathcal{E}$ with symbol L is defined by*

$$W_L(\Omega \otimes \eta) = L\eta,$$

for all $\eta \in \mathcal{E}$, *and for each* $e_\mu = e_{\mu_1} \otimes \cdots \otimes e_{\mu_m}$, $m \geq 1$, *define*

$$W_L(e_\mu \otimes \eta) = \begin{cases} e_{\mu_1+1} \otimes e_{\mu_2} \otimes \cdots \otimes e_{\mu_m} \otimes \eta & \text{if } \mu_1 \neq n \\ e_1 \otimes e_{\mu_2+1} \otimes e_{\mu_3} \otimes \cdots \otimes e_{\mu_m} \otimes \eta & \text{if } \mu_1 = n, \mu_2 \neq n \\ e_1^{\otimes 2} \otimes e_{\mu_3+1} \otimes e_{\mu_4} \otimes \cdots \otimes e_{\mu_m} \otimes \eta & \text{if } \mu_1 = \mu_2 = n, \mu_3 \neq n \\ \vdots & \vdots \\ e_1^{\otimes m} \otimes L\eta & \text{if } \mu_1 = \cdots = \mu_m = n. \end{cases}$$

A couple of comments are in order. Since $\{e_\mu : \mu \in F_n^+\}$ is an orthonormal basis for $\mathcal{F}_n^2$, it follows that $W_L \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E})$. Second, the unitary odometer maps were introduced by Li in [12]. The fundamental difference in his case is that the symbol L is a self-map, that is, $L \in \mathcal{B}(\mathcal{E})$ (instead of $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$). Moreover, in the case of unitary odometer maps, Li proved that the symbols become unitary. This will be observed again in Theorem 4.4. Before we go any further, we introduce the notion of the length of words. The *length* of the word $\mu \in F_n^+$ is defined by

$$|\mu| = \begin{cases} k & \text{if } \mu = g_{\mu_1} g_{\mu_2} \cdots g_{\mu_k} \\ 0 & \text{if } \mu = g_0. \end{cases}$$

Now we are ready for the general classification and representations of odometer maps:

Theorem 2.2. *Let* $\mathcal{E}$ *be a Hilbert space, and let* $W \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E})$. *Then* $(W, S^{\mathcal{E}})$ *is a Fock representation if and only if there is a unique symbol* $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$ *such that*

$$W = W_L.$$

Proof. Suppose $(W, S^{\mathcal{E}})$ is a Fock representation. Define $L : \mathcal{E} \to \mathcal{F}_n^2 \otimes \mathcal{E}$ by

$$L\eta = W(\Omega \otimes \eta),$$

for all $\eta \in \mathcal{E}$. Uniqueness of L is clear by this definition. We claim that $W = W_L$. To this end, fix $\eta \in \mathcal{E}$ and $\mu \in F_n^+$. Then for any $k < n$, we have $W(S_k \otimes I_{\mathcal{E}}) = S_{k+1} \otimes I_{\mathcal{E}}$ and hence

$$W(e_k \otimes e_\mu \otimes \eta) = W(S_k \otimes I_{\mathcal{E}})(e_\mu \otimes \eta) = (S_{k+1} \otimes I_{\mathcal{E}})(e_\mu \otimes \eta),$$

which yields

$$W(e_k \otimes e_\mu \otimes \eta) = e_{k+1} \otimes e_\mu \otimes \eta.$$

On the other hand, by the definition of W_L, we know that $W_L(e_k \otimes e_\mu \otimes \eta) = e_{k+1} \otimes e_\mu \otimes \eta$, and consequently

$$W(e_k \otimes e_\mu \otimes \eta) = W_L(e_k \otimes e_\mu \otimes \eta). \tag{2.1}$$

Now, let $m \geq 1$, and assume that $\mu_1 \neq n$. Since $W(S_n^m \otimes I_{\mathcal{E}}) = (S_1^m \otimes I_{\mathcal{E}})W$, it follows that

$$W(e_n^{\otimes m} \otimes e_\mu \otimes \eta) = W(S_n^m \otimes I_{\mathcal{E}})(e_\mu \otimes \eta) = e_1^{\otimes m} \otimes W(e_\mu \otimes \eta).$$

We have two cases to consider: Let $|\mu| = 0$, that is, $\mu = g_0$. Then $W(e_\mu \otimes \eta) = W(\Omega \otimes \eta) = L\eta$. By the definition of W_L, we have that

$$W(e_n^{\otimes m} \otimes \eta) = W_L(e_n^{\otimes m} \otimes \eta).$$

For the next case, assume that $|\mu| > 0$, and let $e_\mu = e_{\mu_1} \otimes \cdots \otimes e_{\mu_k}$, where $k \geq 1$. By (2.1), we know that

$$W(e_\mu \otimes \eta) = W_L(e_\mu \otimes \eta).$$

Combining these two cases, we finally conclude that

$$W(e_n^{\otimes m} \otimes e_\mu \otimes \eta) = W_L(e_n^{\otimes m} \otimes e_\mu \otimes \eta),$$

for all $\mu \in F_n^+$, which completes the proof of the fact that $W = W_L$.

For the converse, assuming that $W = W_L$ for some symbol $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$, we want to prove that $(W, S^{\mathcal{E}})$ is a Fock representation, that is, to prove that $W_L(S_k \otimes I_{\mathcal{E}}) = S_{k+1} \otimes I_{\mathcal{E}}$ for all $k < n$, and $W_L(S_n \otimes I_{\mathcal{E}}) = (S_1 \otimes I_{\mathcal{E}})W_L$. Fix $\eta \in \mathcal{E}$ and $\mu \in F_n^+$. For any $k < n$, since $W_L(e_k \otimes \eta) = e_{k+1} \otimes \eta$, we have

$$W_L(S_k \otimes I_{\mathcal{E}})(e_\mu \otimes \eta) = W_L(e_k \otimes e_\mu \otimes \eta) = e_{k+1} \otimes e_\mu \otimes \eta = (S_{k+1} \otimes I_{\mathcal{E}})(e_\mu \otimes \eta),$$

which means

$$W_L(S_k \otimes I_{\mathcal{E}}) = S_{k+1} \otimes I_{\mathcal{E}},$$

for all $k < n$. Now we prove that

$$W_L(S_n \otimes I_{\mathcal{E}}) = (S_1 \otimes I_{\mathcal{E}})W_L.$$

Observe that

$$(S_1 \otimes I_{\mathcal{E}})W_L(e_\mu \otimes \eta) = e_1 \otimes W_L(e_\mu \otimes \eta) = W_L(e_n \otimes e_\mu \otimes \eta) = W_L(S_n \otimes I_{\mathcal{E}})(e_\mu \otimes \eta).$$

Summarizing, we have $W_L(S_n \otimes I_{\mathcal{E}}) = (S_1 \otimes I_{\mathcal{E}})W_L$. This along with $W_L(S_k \otimes I_{\mathcal{E}}) = S_{k+1} \otimes I_{\mathcal{E}}$ for all $k < n$, implies that $(W_L, S^{\mathcal{E}})$ is a Fock representation. □

In particular, in the scalar case, we have the following:

Corollary 2.3. *Let $W \in \mathcal{B}(\mathcal{F}_n^2)$. Then (W, S) is a Fock representation if and only if there is unique $\xi \in \mathcal{F}_n^2$ such that $W = W_\xi$.*

Proof. The proof is immediate, given that a bounded linear operator $L : \mathbb{C} \to \mathcal{F}_n^2$ can be represented as $L1 = \xi$ for some $\xi \in \mathcal{F}_n^2$. □

A symbol $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$ is said to be *constant* if

$$L\mathcal{E} \subseteq \Omega \otimes \mathcal{E}.$$

From the above theorem, it is clear that L is a constant symbol if and only if the odometer map W_L satisfies the following condition:

$$W_L(\Omega \otimes \mathcal{E}) \subseteq \Omega \otimes \mathcal{E}.$$

In the following sections, we will focus on more specific representations within the category of Fock representations. For instance, we will discuss the matter of constant unitary symbols and revisit the flavor of Li's original finding of unitary odometer maps.

Remark 2.4. *It should be noted that odometer maps are clearly defined for any operator (which we call a symbol) belonging to $\mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$. This bears a resemblance to Popescu's concept of noncommutative analytic Toepliz operators* [16]. *However, it is crucial to note that noncommutative analytic Toepliz operators are not defined for all symbols in $\mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$. The odometer maps display a clear contrast when viewed from this perspective. We also have additional features about these maps. Let $\mathcal{E}$ be a Hilbert space. Define $\pi : \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E}) \to \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E})$ by*

$$\pi(L) = W_L,$$

for all $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$. Then π is an isometry. Indeed, by the definition of W_L, we know that $\|W_L\| \leq \|L\|$. Next, for $\eta \in \mathcal{E}$, we have

$$\|L\eta\| = \|W_L(\Omega \otimes \eta)\| \leq \|W_L\|\|\eta\|,$$

which implies that $\|L\| \leq \|W_L\|$. *Therefore, we conclude that*

$$\|\pi(L)\| = \|L\|,$$

that is, $\|W_L\| = \|L\|$ *for all* $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$.

3. Isometric Fock representations

In this section, we offer a thorough classification of isometric Fock representations. Recall that an isometric Fock representation is a Fock representation $(W, S^{\mathcal{E}})$ where $W \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E})$ is an isometry.

We note a simple fact that will be significant in the subsequent background computation. We partition the orthonormal basis $\{e_\mu : \mu \in F_n^+\}$ into two disjoint sets as follows:

$$\{e_\mu : \mu \in F_n^+\} = \{e_1^{\otimes m} : m \in \mathbb{Z}_+\} \bigsqcup \{e_\mu : \mu \in F_n^+, |\mu| \geq 1, \mu_i \neq 1 \text{ for some } i\}. \tag{3.1}$$

Throughout this section, we fix a Hilbert space $\mathcal{E}$ and an orthonormal basis $\{h_p\}_{p\in\Lambda}$ for $\mathcal{E}$. Note that Λ is either a finite or a countably infinite set. Clearly, $\{e_\mu \otimes h_p : \mu \in F_n^+, p \in \Lambda\}$ is an orthonormal basis for $\mathcal{F}_n^2 \otimes \mathcal{E}$, and hence

$$\mathcal{F}_n^2 \otimes \mathcal{E} = \overline{\text{span}}\{e_\mu \otimes h_p : \mu \in F_n^+, p \in \Lambda\}.$$

Moreover, given $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$, we define the closed subspace

$$\mathcal{E}_L = \overline{\text{span}}\{e_1^{\otimes m} \otimes \eta : m \in \mathbb{Z}_+, \eta \in \mathcal{E}\} \ominus \overline{\text{span}}\{e_1^{\otimes p} \otimes L\zeta : p \geq 1, \zeta \in \mathcal{E}\}. \tag{3.2}$$

We are now ready for the classification of isometric Fock representations.

Theorem 3.1. *Let* $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$. *Then* $(W_L, S^{\mathcal{E}})$ *is an isometric Fock representation if and only if the following conditions hold:*

1. L *is an isometry.*
2. $L\mathcal{E} \subseteq \mathcal{E}_L$.

Proof. Suppose W_L is an isometry, and let $\eta \in \mathcal{E}$. Since

$$\|L\eta\| = \|W_L(\Omega \otimes \eta)\| = \|\Omega \otimes \eta\| = \|\eta\|,$$

it follows that L is an isometry. For (2), we apply the partition in (3.2). Define a closed subspace $\mathcal{M}$ of $\mathcal{F}_n^2 \otimes \mathcal{E}$ as

$$\mathcal{M} = \overline{\text{span}}\{e_\mu \otimes \eta : \eta \in \mathcal{E}, \mu \in F_n^+, |\mu| \geq 1, \mu_i \neq 1 \text{ for some } i\}.$$

Clearly, $\mathcal{M}^\perp = \overline{\text{span}}\{e_1^{\otimes m} \otimes \eta : m \in \mathbb{Z}_+, \eta \in \mathcal{E}\}$. Pick $\eta, \gamma \in \mathcal{E}$, and $e_\mu = e_1^{\otimes m} \otimes e_{\mu_1} \otimes \cdots \otimes e_{\mu_k}$. Assume that $\mu_1 \neq 1$ and $m \in \mathbb{Z}_+$, so that $e_\mu \otimes \gamma \in \mathcal{M}$. Since W_L is an isometry, it follows that

$$\langle L\eta, e_\mu \otimes \gamma\rangle = \langle W_L(\Omega \otimes \eta), W_L(e_n^{\otimes m} \otimes e_{\mu_1 - 1} \otimes \cdots \otimes e_{\mu_k} \otimes \gamma)\rangle = 0,$$

and hence $L\mathcal{E} \subseteq \mathcal{M}^\perp = \overline{\text{span}}\{e_1^{\otimes m} \otimes \eta : m \geq 0, \eta \in \mathcal{E}\}$. Next, for any $p \geq 1$, the fact that W_L is an isometry again implies that

$$\langle L\eta, e_1^{\otimes p} \otimes L\gamma\rangle = \langle W_L(\Omega \otimes \eta), W_L(e_n^{\otimes p} \otimes \gamma)\rangle = 0,$$

and hence

$$L\mathcal{E} \perp \overline{\text{span}}\{e_1^{\otimes p} \otimes L\zeta : p \geq 1, \zeta \in \mathcal{E}\}.$$

This, along with $L\mathcal{E} \subseteq \mathcal{M}^\perp$, implies that $L\mathcal{E} \subseteq \mathcal{E}_L$.

For the converse direction, again, in view of the partition in (3.1), we write $\mathcal{F}_n^2 = \overline{\text{span}}(\mathcal{M}_1 \cup \mathcal{M}_2)$, where

$$\mathcal{M}_1 = \{e_n^{\otimes m} : m \geq 0\},$$

and

$$\mathcal{M}_2 = \{e_\mu : \mu \in F_n^+, |\mu| \geq 1, \mu_i \neq n \text{ for some } i\}.$$

Pick $e_\mu \otimes \eta \in \mathcal{F}_n^2 \otimes \mathcal{E}$. If $e_\mu = e_n^{\otimes p} \in \mathcal{M}_1$, then

$$\|W_L(e_n^{\otimes p} \otimes \eta)\| = \|e_1^{\otimes p} \otimes L\eta\| = \|L\eta\| = \|\eta\| = \|e_n^{\otimes p} \otimes \eta\|.$$

If $e_\mu = e_n^{\otimes m} \otimes e_{\mu_1} \otimes \cdots \otimes e_{\mu_k} \in \mathcal{M}_2$, $\mu_1 \neq n$, then

$$\|W_L(e_\mu \otimes \eta)\| = \|e_1^{\otimes m} \otimes e_{\mu_1+1} \otimes e_{\mu_2} \otimes \cdots \otimes e_{\mu_k} \otimes \eta\| = \|\eta\| = \|e_\mu \otimes \eta\|.$$

As a result, we proved that W_L is isometric while acting on every basis vector. To conclude that W_L is isometry, it clearly remains to prove that

$$\langle W_L(e_\mu \otimes \eta), W_L(e_\lambda \otimes \zeta)\rangle = 0, \tag{3.3}$$

whenever $\langle e_\mu \otimes \eta, e_\lambda \otimes \zeta\rangle = 0$ for $\mu, \lambda \in F_n^+$ and $\eta, \zeta \in \mathcal{E}$. The latter condition is equivalent to $\langle e_\mu, e_\lambda\rangle = 0$ or $\langle \eta, \zeta\rangle = 0$. Suppose $\langle e_\mu, e_\lambda\rangle = 0$. Let $|\mu| = p, |\lambda| = q$. We have the following three cases:

Case I: Let $e_\mu, e_\lambda \in \mathcal{M}_1$. Then $e_\mu = e_n^{\otimes p}$ and $e_\lambda = e_n^{\otimes q}$. Since $\langle e_\mu, e_\lambda\rangle = 0$, we conclude that $p \neq q$. Assume, without any loss of generality, that $p < q$. Then

$$\begin{aligned}
\langle W_L(e_\mu \otimes \eta), W_L(e_\lambda \otimes \zeta)\rangle &= \langle W_L(e_n^{\otimes p} \otimes \eta), W_L(e_n^{\otimes q} \otimes \zeta)\rangle \\
&= \langle e_1^{\otimes p} \otimes L\eta, e_1^{\otimes q} \otimes L\zeta\rangle \\
&= \langle L\eta, e_1^{\otimes (q-p)} \otimes L\zeta\rangle \\
&= 0,
\end{aligned}$$

as $L\mathcal{E} \subseteq \mathcal{E}_L$. This proves (3.3).

Case II: Let $e_\mu \in \mathcal{M}_i$ and $e_\lambda \in \mathcal{M}_j$, where $i \neq j$. Without loss of generality, we assume that $e_\mu \in \mathcal{M}_1$ and $e_\lambda \in \mathcal{M}_2$ (as other cases will follow similarly). Then $e_\mu = e_n^{\otimes p}$ and $e_\lambda = e_n^{\otimes m} \otimes e_{\mu_1} \otimes \cdots \otimes e_{\mu_k}$ and $\mu_i \neq n$ for some i. Then $W_L(e_\lambda \otimes \zeta)$ has one component different from e_1, whereas $L\mathcal{E} \subseteq \mathcal{E}_L$ implies that

$$W_L(e_\mu \otimes \eta) \in \overline{\text{span}}\{e_1^{\otimes m} \otimes \eta : m \in \mathbb{Z}_+, \eta \in \mathcal{E}\}.$$

Therefore, (3.3) holds.

Case III: Let $e_\mu, e_\lambda \in \mathcal{M}_2$. Here, we have two subcases to deal with: $p = q$ and $p < q$.

If $p = q$, then there exists a minimum i such that $\mu_i \neq \lambda_i$. Therefore, (3.3) holds because the definition of W_L implies that they will differ at the i^{th} component. If $p < q$, then again, the definition of W_L implies that $W_L(e_\mu \otimes \eta) \in (\mathbb{C}^n)^{\otimes p} \otimes \mathcal{E}$ and $W_L(e_\lambda \otimes \zeta) \in (\mathbb{C}^n)^{\otimes q} \otimes \mathcal{E}$, proving (3.3).

Finally, assume that $\langle e_\mu, e_\lambda\rangle \neq 0$. Then $\langle \eta, \zeta\rangle = 0$ and $\mu = \lambda$. We have the following two cases:

Case A: $e_\mu \in \mathcal{M}_1$: Let $e_\mu = e_n^{\otimes p}$. Then

$$\langle W_L(e_\mu \otimes \eta), W_L(e_\mu \otimes \zeta)\rangle = \langle e_1^{\otimes p} \otimes L\eta, e_1^{\otimes p} \otimes L\zeta\rangle = \|e_1^{\otimes p}\|^2 \langle L\eta, L\zeta\rangle = 0,$$

where the last equality follows from the fact that L is an isometry.

Case B: $e_\mu \in \mathcal{M}_2$: Let $e_\mu = e_n^{\otimes m} \otimes e_{\mu_1} \otimes \cdots \otimes e_{\mu_k}$, where $\mu_1 \neq n$. Then

$$\begin{aligned}\langle W_L(e_\mu \otimes \eta), W_L(e_\mu \otimes \zeta)\rangle &= \langle e_1^{\otimes m} \otimes e_{\mu_1+1} \otimes \cdots \otimes e_{\mu_k} \otimes \eta, e_1^{\otimes m} \otimes e_{\mu_1+1} \otimes \cdots \otimes e_{\mu_k} \otimes \zeta\rangle \\ &= \left\| e_1^{\otimes m} \otimes e_{\mu_1+1} \otimes \cdots \otimes e_{\mu_k}\right\|^2 \langle \eta, \zeta\rangle \\ &= 0.\end{aligned}$$

The above two cases also assert that $\langle \eta, \zeta\rangle = 0$ implies (3.3), which consequently completes the proof of the theorem. □

Let $(W_L, S^{\mathcal{E}})$ be a Fock representation. By condition (2) of the above theorem, we know that $L\mathcal{E} \subseteq \mathcal{E}_L$. We claim that this implies the following identity:

$$\sum_{p\in\mathbb{Z}_+, q\in\Lambda} c^\eta_{p+r,q}\overline{c^\zeta_{p,q}} = 0, \tag{3.4}$$

for all $r \geq 1$, where

$$L\eta = \sum_{s\in\mathbb{Z}_+, t\in\Lambda} c^\eta_{s,t} e_1^{\otimes s} \otimes h_t, \text{ and } L\zeta = \sum_{p\in\mathbb{Z}_+, q\in\Lambda} c^\zeta_{p,q} e_1^{\otimes p} \otimes h_q,$$

and $\{h_p\}_{p\in\Lambda}$ is an orthonormal basis for $\mathcal{E}$. Indeed, for every $r \geq 1$, we compute

$$\begin{aligned}\left\langle L\eta, e_1^{\otimes r} \otimes L\zeta\right\rangle &= \left\langle \sum_{s\in\mathbb{Z}_+, t\in\Lambda} c^\eta_{s,t} e_1^{\otimes s} \otimes h_t, e_1^{\otimes r} \otimes \sum_{p\in\mathbb{Z}_+, q\in\Lambda} c^\zeta_{p,q} e_1^{\otimes p} \otimes h_q\right\rangle \\ &= \left\langle \sum_{s\in\mathbb{Z}_+, t\in\Lambda} c^\eta_{s,t} e_1^{\otimes s} \otimes h_t, \sum_{p\in\mathbb{Z}_+, q\in\Lambda} c^\zeta_{p,q} e_1^{\otimes (p+r)} \otimes h_q\right\rangle \\ &= \sum_{p\in\mathbb{Z}_+, q\in\Lambda} c^\eta_{p+r,q}\overline{c^\zeta_{p,q}}\, \|h_q\|^2 \\ &= \sum_{p\in\mathbb{Z}_+, q\in\Lambda} c^\eta_{p+r,q}\overline{c^\zeta_{p,q}},\end{aligned}$$

as $\{h_q\}_{q\in\Lambda}$ forms an orthonormal basis for $\mathcal{E}$. Therefore, $\langle L\eta, e_1^{\otimes r} \otimes L\zeta\rangle = 0$ for all $r \geq 1$ and all $\eta, \zeta \in \mathcal{E}$ implies (3.4).

The characterization of isometric Fock representations obtained in Theorem 3.1 is more explicit in the scalar case. Recall that the odometer maps on $\mathcal{F}^2_n$ are of the form W_ξ, $\xi \in \mathcal{F}^2_n$. In this case, we will represent $c^\eta_{p,q}$ in (3.4) simply as c_p.

Corollary 3.2. *Let $\xi \in \mathcal{F}^2_n$. Then W_ξ is an isometry if and only if*

$$\xi = \sum_{p=0}^{\infty} c_p e_1^{\otimes p},$$

where $\sum_{p=0}^\infty |c_p|^2 = 1$ and $\sum_{p=0}^\infty c_{p+r}\overline{c_p} = 0$ for all $r \geq 1$.

Proof. Let W_ξ be an isometry. Theorem 3.1 implies that $\xi \in \overline{\text{span}}\{e_1^{\otimes m} : m \in \mathbb{Z}_+\}$. There exists a sequence of scalars $\{c_p\}_{p\in\mathbb{Z}_+}$ such that

$$\xi = W_\xi(\Omega) = \sum_{p\in\mathbb{Z}_+} c_p e_1^{\otimes p}.$$

Also, W_ξ is an isometry, which implies

$$1 = \|\Omega\|^2 = \|W_\xi(\Omega)\|^2 = \|\xi\|^2 = \sum_{p\in\mathbb{Z}_+} |c_p|^2.$$

The remaining condition directly follows from the scalar case of (3.4). We can derive the converse direction from the vector setting of Theorem 3.1 by considering the map $L : \mathbb{C} \to \mathcal{F}_n^2$ defined by $L1 = \xi$. $\square$

A number of questions arise about odometer maps. Within this section, our accomplishments were limited to the classification of isometric Fock representations. While this may be adequate for our needs, it can be challenging to find out the overall structure of odometer maps. Calculating the conjugate of an odometer map may appear to be a simple question, yet the solution is unclear. In the following section, however, we will compute the conjugate of isometric odometer maps.

4. Nica covariant representations

This section aims at classifying Nica-covariant representations of O_n. Recall from Definition 1.5 that a Nica-covariant representation of O_n is an isometric Fock representation $(W, S^{\mathcal{E}})$ such that

$$W^*(S_1 \otimes I_{\mathcal{E}}) = (S_n \otimes I_{\mathcal{E}})W^*.$$

Let $\mathcal{E}$ be a Hilbert space. Suppose $(W, S^{\mathcal{E}})$ on $\mathcal{F}_n^2 \otimes \mathcal{E}$ is a Nica covariant representation of O_n. By Theorem 2.2, W is an odometer map, that is, there is a unique symbol $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$ such that $W = W_L$. Therefore, the revised goal of this section is to classify the symbol L such that the representation $(W_L, S^{\mathcal{E}})$ is a Nica covariant representation. At this stage, we need representations of the adjoints of isometric odometer maps.

In what follows, given a Hilbert space $\mathcal{E}$, we always assume that $\{h_p\}_{p\in\Lambda}$ is an orthonormal basis for $\mathcal{E}$. Moreover, for all $q \in \Lambda$, we write the Fourier series representation

$$Lh_q = \sum_{r\in\mathbb{Z}_+, s\in\Lambda} c_{r,s}^{h_q} e_1^{\otimes r} \otimes h_s. \tag{4.1}$$

It is also worth recalling the partition of the orthogonal basis $\{e_\mu : \mu \in F_n^+\}$ for $\mathcal{F}_n^2$, as in (3.1).

Proposition 4.1. *Let $\mathcal{E}$ be a Hilbert space, and let $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$. If the odometer map W_L is an isometry, then the adjoint of W_L is given by*

$$W_L^* f = \begin{cases} \displaystyle\sum_{p=0}^{m} \sum_{q\in\Lambda} \overline{c_{m-p,l}^{h_q}} (e_n^{\otimes p} \otimes h_q) & \text{if } f = e_1^{\otimes m} \otimes h_l \\ e_n^{\otimes m} \otimes e_{\mu_1 - 1} \otimes e_{\mu_2} \cdots \otimes e_{\mu_k} \otimes h_l & \text{if } f = e_1^{\otimes m} \otimes e_{\mu_1} \otimes \cdots \otimes e_{\mu_k} \otimes h_l \text{ and } \mu_1 > 1, \end{cases}$$

for all $m \in \mathbb{Z}_+$ and $l \in \Lambda$.

Proof. Fix $m \in \mathbb{Z}_+$, $l \in \Lambda$, and let $\mu = g_{\mu_1} \cdots g_{\mu_k}$, $k \geq 1$. For all $q \in \Lambda$, we have

$$\langle W_L^*(e_1^{\otimes m} \otimes h_l), e_\mu \otimes h_q\rangle = \langle e_1^{\otimes m} \otimes h_l, W_L(e_\mu \otimes h_q)\rangle = 0,$$

whenever $\mu_i \neq n$ for some i or $|\mu| \geq m+1$. Next, assume that $|\mu| \leq m$. For any $0 \leq p \leq m$, we have

$$\langle W_L^*(e_1^{\otimes m} \otimes h_l), e_n^{\otimes p} \otimes h_q\rangle = \langle e_1^{\otimes m} \otimes h_l, W_L(e_n^{\otimes p} \otimes h_q)\rangle = \langle e_1^{\otimes m} \otimes h_l, e_1^{\otimes p} \otimes Lh_q\rangle.$$

Substituting the value of Lh_q yields (see (4.1))

$$\begin{aligned}\langle e_1^{\otimes m} \otimes h_l, e_1^{\otimes p} \otimes Lh_q\rangle &= \langle e_1^{\otimes m} \otimes h_l, e_1^{\otimes p} \otimes \sum_{r\in\mathbb{Z}_+, s\in\Lambda} c_{r,s}^{h_q} e_1^{\otimes r} \otimes h_s\rangle \\ &= \langle e_1^{\otimes m} \otimes h_l, \sum_{r\in\mathbb{Z}_+, s\in\Lambda} c_{r,s}^{h_q} e_1^{\otimes (r+p)} \otimes h_s\rangle \\ &= \langle e_1^{\otimes m} \otimes h_l, \sum_{r=p}^{\infty}\sum_{s\in\Lambda} c_{r-p,s}^{h_q} e_1^{\otimes r} \otimes h_s\rangle \\ &= \overline{c_{m-p,l}^{h_q}},\end{aligned}$$

which implies

$$W_L^*(e_1^{\otimes m} \otimes h_l) = \sum_{p=0}^{m}\sum_{q\in\Lambda} \langle W_L^*(e_1^{\otimes m} \otimes h_l), e_n^{\otimes p} \otimes h_q\rangle (e_n^{\otimes p} \otimes h_q) = \sum_{p=0}^{m}\sum_{q\in\Lambda} \overline{c_{m-p,l}^{h_q}} (e_n^{\otimes p} \otimes h_q).$$

Finally, assume that $f = e_1^{\otimes m} \otimes e_{\mu_1} \otimes \cdots \otimes e_{\mu_k} \otimes \eta$, where $\mu_1 > 1$ and $\eta \in \mathcal{E}$. As $W_L^* W_L = I$, we have

$$\begin{aligned}W_L^* f &= W_L^*(e_1^{\otimes m} \otimes e_{\mu_1} \otimes \cdots \otimes e_{\mu_k} \otimes \eta) \\ &= W_L^* W_L(e_n^{\otimes m} \otimes e_{\mu_1 - 1} \otimes e_{\mu_2} \otimes \cdots \otimes e_{\mu_k} \otimes \eta) \\ &= e_n^{\otimes m} \otimes e_{\mu_1 - 1} \otimes e_{\mu_2} \otimes \cdots \otimes e_{\mu_k} \otimes \eta,\end{aligned}$$

which completes the proof of the proposition. $\square$

Recall from Corollary 3.2 that if $W_\xi : \mathcal{F}_n^2 \to \mathcal{F}_n^2$ is an isometry for some $\xi \in \mathcal{F}_n^2$, then, in particular, we have

$$\xi = \sum_{p=0}^{\infty} c_p e_1^{\otimes p}.$$

In this case, the above proposition yields the adjoint formula of W_ξ as (note that here $\xi = L1$, and hence (4.1) is comparable)

$$W_\xi^* f = \begin{cases} \overline{c_0}\Omega & \text{if } f = \Omega \\ \sum_{p=0}^{m} \overline{c_{m-p}} e_n^{\otimes p} & \text{if } f = e_1^{\otimes m}, m \geq 1 \\ e_n^{\otimes m} \otimes e_{\mu_1 - 1} \otimes e_{\mu_2} \cdots \otimes e_{\mu_k} & \text{if } f = e_1^{\otimes m} \otimes e_{\mu_1} \otimes \cdots \otimes e_{\mu_k}, \mu_1 > 1, \text{and } m \geq 0. \end{cases}$$

Now we are ready to characterize Nica covariant representations of O_n:

Theorem 4.2. *Let $\mathcal{E}$ be a Hilbert space, and let $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$. Assume that $(W_L, S^{\mathcal{E}})$ is an isometric Fock representation. Then $(W_L, S^{\mathcal{E}})$ is a Nica covariant representation of O_n if and only if*

$$L\mathcal{E} \subseteq \Omega \otimes \mathcal{E}.$$

Proof. Note that $(W_L, S^{\mathcal{E}})$ is an isometric Fock representation. Following (4.1), for each $q \in \Lambda$, we write

$$Lh_q = \sum_{r\in\mathbb{Z}_+, s\in\Lambda} c_{r,s}^{h_q} e_1^{\otimes r} \otimes h_s,$$

Suppose $(W_L, S^{\mathcal{E}})$ is a Nica covariant representation, that is, $W_L^*(S_1 \otimes I_{\mathcal{E}}) = (S_n \otimes I_{\mathcal{E}})W_L^*$. For each $m \in \mathbb{Z}_+$ and $l \in \Lambda$, the adjoint formula in Proposition 4.1 yields

$$W_L^*(S_1 \otimes I_{\mathcal{E}})(e_1^{\otimes m} \otimes h_l) = W_L^*(e_1^{\otimes (m+1)} \otimes h_l) = \sum_{p=0}^{m+1} \sum_{q \in \Lambda} \overline{c_{(m+1-p),l}^{h_q}}(e_n^{\otimes p} \otimes h_q).$$

Rearranging the terms corresponding to $p = 0$, we get

$$W_L^*(S_1 \otimes I_{\mathcal{E}})(e_1^{\otimes m} \otimes h_l) = \sum_{q \in \Lambda} \overline{c_{m+1,l}^{h_q}}(\Omega \otimes h_q) + \sum_{p=1}^{m+1} \sum_{q \in \Lambda} \overline{c_{m+1-p,l}^{h_q}}(e_n^{\otimes p} \otimes h_q). \tag{4.2}$$

Again, by Proposition 4.1, we have

$$\begin{aligned}(S_n \otimes I_{\mathcal{E}})W_L^*(e_1^{\otimes m} \otimes h_l) &= (S_n \otimes I_{\mathcal{E}})(\sum_{p=0}^{m} \sum_{q \in \Lambda} \overline{c_{m-p,l}^{h_q}}(e_n^{\otimes p} \otimes h_q)) \\ &= \sum_{p=0}^{m} \sum_{q \in \Lambda} \overline{c_{m-p,l}^{h_q}}(e_n^{\otimes (p+1)} \otimes h_q) \\ &= \sum_{p=1}^{m+1} \sum_{q \in \Lambda} \overline{c_{m+1-p,l}^{h_q}}(e_n^{\otimes p} \otimes h_q).\end{aligned}$$

Since $W_L^*(S_1 \otimes I_{\mathcal{E}}) = (S_n \otimes I_{\mathcal{E}})W_L^*$, the above comparing with (4.2) gives

$$\sum_{q \in \Lambda} \overline{c_{m+1,l}^{h_q}}(\Omega \otimes h_q) = 0.$$

But $\{\Omega \otimes h_q\}_{q \in \Lambda}$ is an orthonormal set, and hence $c_{m+1,l}^{h_q} = 0$ for all $m \in \mathbb{Z}_+$ and $q, l \in \Lambda$. Therefore

$$Lh_q = \sum_{r \in \mathbb{Z}_+, s \in \Lambda} c_{r,s}^{h_q}(e_1^{\otimes r} \otimes h_s) = \sum_{s \in \Lambda} c_{0,s}^{h_q}(\Omega \otimes h_s).$$

This proves that $L\mathcal{E} \subseteq \Omega \otimes \mathcal{E}$. For the reverse direction, it is enough to prove that the identity $W_L^*(S_1 \otimes I_{\mathcal{E}}) = (S_n \otimes I_{\mathcal{E}})W_L^*$ holds on basis vectors. For each $m \in \mathbb{Z}_+$ and $l \in \Lambda$, Proposition 4.1 gives

$$W_L^*(e_1^{\otimes m} \otimes h_l) = \sum_{p=0}^{m} \sum_{q \in \Lambda} \overline{c_{m-p,l}^{h_q}}(e_n^{\otimes p} \otimes h_q).$$

Since $L\mathcal{E} \subseteq \Omega \otimes \mathcal{E}$, we have $Lh_q = \sum_{s \in \Lambda} c_s^{h_q}(\Omega \otimes h_s)$, $q \in \Lambda$. A similar computation as in the first part of the proof of Proposition 4.1 implies that the term in the above series corresponding to $p = m$ will remain, and hence

$$W_L^*(e_1^{\otimes m} \otimes h_l) = \sum_{q \in \Lambda} \overline{c_l^{h_q}}(e_n^{\otimes m} \otimes h_q).$$

For all $m \in \mathbb{Z}_+$, we have

$$\begin{aligned} W_L^*(S_1 \otimes I)(e_1^{\otimes m} \otimes h_l) &= W_L^*(e_1^{\otimes (m+1)} \otimes h_l) \\ &= \sum_{q \in \Lambda} \overline{c_l^{h_q}} (e_n^{\otimes (m+1)} \otimes h_q) \\ &= e_n \otimes (\sum_{q \in \Lambda} \overline{c_l^{h_q}} (e_n^{\otimes m} \otimes h_q)) \\ &= e_n \otimes W_L^*(e_1^{\otimes m} \otimes h_l) \\ &= (S_n \otimes I_{\mathcal{E}}) W_L^*(e_1^{\otimes m} \otimes h_l). \end{aligned}$$

On the other hand, for each $\eta \in \mathcal{E}$ and $\mu \in F_n^+$ with $\mu_1 > 1$, Proposition 4.1 implies

$$\begin{aligned} W_L^*(S_1 \otimes I_{\mathcal{E}})(e_1^{\otimes m} \otimes e_{\mu_1} \otimes \cdots \otimes e_{\mu_k} \otimes \eta) &= W_L^*(e_1^{\otimes (m+1)} \otimes e_{\mu_1} \otimes \cdots \otimes e_{\mu_k} \otimes \eta) \\ &= e_n^{\otimes (m+1)} \otimes e_{\mu_1 - 1} \otimes e_{\mu_2} \cdots \otimes e_{\mu_k} \otimes \eta \\ &= (S_n \otimes I_{\mathcal{E}})(e_n^{\otimes m} \otimes e_{\mu_1 - 1} \otimes e_{\mu_2} \cdots \otimes e_{\mu_k} \otimes \eta) \\ &= (S_n \otimes I_{\mathcal{E}}) W_L^*(e_1^{\otimes m} \otimes e_{\mu_1} \otimes e_{\mu_2} \cdots \otimes e_{\mu_k} \otimes \eta), \end{aligned}$$

and hence, $W_L^*(S_1 \otimes I_{\mathcal{E}}) = (S_n \otimes I_{\mathcal{E}}) W_L^*$ for each basis element, proving our claim. ☐

When we consider the scalar case (that is, $\mathcal{E} = \mathbb{C}$), Theorem 4.2 simplifies to:

Theorem 4.3. *Let (W_ξ, S) be an isometric Fock representation for some $\xi \in \mathcal{F}_n^2$. Then the following are equivalent:*

(1) *(W_ξ, S) is a Nica covariant representation of O_n.*
(2) *There exists $c \in \mathbb{T}$ such that $\xi = c\Omega$.*
(3) *W_ξ is unitary.*

Proof. (1) $\Rightarrow$ (2): By Theorem 4.2, there is a scalar $c \in \mathbb{C}$ such that $L1 = c\Omega$. Also, we know that $L1 = \xi$, and hence $\xi = c\Omega$. Since L is an isometry, we also have that $|c| = 1$.
(2) $\Rightarrow$ (3): Suppose $\xi = c\Omega$ for some $c \in \mathbb{T}$. Then $W_\xi(\Omega) = c\Omega$, and by the definition of odometer map, it follows that

$$W_\xi(e_{\mu_1} \otimes \cdots \otimes e_{\mu_m}) = \begin{cases} e_{\mu_1+1} \otimes e_{\mu_2} \otimes \cdots \otimes e_{\mu_m} & \text{if } \mu_1 \neq n \\ e_1 \otimes e_{\mu_2+1} \otimes e_{\mu_3} \otimes \cdots \otimes e_{\mu_m} & \text{if } \mu_1 = n, \mu_2 \neq n \\ e_1^{\otimes 2} \otimes e_{\mu_3+1} \otimes e_{\mu_4} \otimes \cdots \otimes e_{\mu_m} & \text{if } \mu_1 = \mu_2 = n, \mu_3 \neq n \\ \vdots & \vdots \\ c e_1^{\otimes m} & \text{if } \mu_1 = \cdots = \mu_m = n. \end{cases}$$

Let $\mathcal{H}_m = \text{span}\{e_\alpha : |\alpha| = m\}$. Then the above identity implies that $W_\xi|_{\mathcal{H}_m} : \mathcal{H}_m \to \mathcal{H}_m$ is a unitary for each $m \in \mathbb{Z}_+$. Since

$$\mathcal{F}_n^2 = \bigoplus_{m=0}^{\infty} \mathcal{H}_m,$$

it follows that W_ξ is a unitary on $\mathcal{F}_n^2$.
(3) $\Rightarrow$ (1): Since W_ξ is a unitary and also $W_\xi S_n = S_1 W_\xi$, it follows that $W_\xi^* S_1 = S_n W_\xi^*$. Therefore, (W_ξ, S) is a Nica covariant representation. ☐

Given an isometry W_L, the following theorem highlights the condition on L under which W_L becomes a unitary operator. This subsequently recovers the unitary Fock representations previously obtained by Li in [12, Corollary 3.6]. We note that the following assumes L to be an isometry, which is a necessary condition for W_L to be an isometry (see Theorem 3.1).

Theorem 4.4. *Let $\mathcal{E}$ be a Hilbert space, and let $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$ be an isometry. Then W_L is unitary if and only if*

$$L\mathcal{E} = \Omega \otimes \mathcal{E}.$$

Proof. Suppose $L\mathcal{E} = \Omega \otimes \mathcal{E}$. We intend to apply Theorem 3.1 to the odometer map W_L. We already know that L is an isometry. Moreover, in this case, the space $\mathcal{E}_L$, defined by (3.2), simplifies to

$$\mathcal{E}_L = \Omega \otimes \mathcal{E}.$$

Therefore, by Theorem 3.1, W_L is an isometry. Now, to show that W_L is onto, it suffices to find the preimage of orthonormal basis elements for $\mathcal{F}_n^2 \otimes \mathcal{E}$. Equivalently, it is enough to find preimage of $e_\mu \otimes \eta$ for all $\mu \in F_n^+$ and $\eta \in \mathcal{E}$. This follows at once, looking at the definition of the map W_L and the fact that $L\mathcal{E} = \Omega \otimes \mathcal{E}$. However, for the convenience of general readers, we provide complete details. Fix $\eta \in \mathcal{E}$. As $L\mathcal{E} = \Omega \otimes \mathcal{E}$, there exists $\gamma \in \mathcal{E}$ such that

$$L\gamma = \Omega \otimes \eta.$$

By the definition of W_L, we know $W_L(\Omega \otimes \gamma) = \Omega \otimes \eta$, and in general, for all $m \geq 1$, we have

$$W_L(e_n^{\otimes m} \otimes \gamma) = e_1^{\otimes m} \otimes \eta.$$

Next, assume that $e_\mu = e_1^{\otimes m} \otimes e_{\mu_1} \otimes \cdots \otimes e_{\mu_k}$, and $\mu_1 > 1$. Again, by the definition of W_L, we have that

$$W_L(e_n^{\otimes m} \otimes e_{\mu_1 - 1} \otimes e_{\mu_2} \otimes \cdots \otimes e_{\mu_k} \otimes \eta) = e_\mu \otimes \eta.$$

It now follows that W_L is onto and hence a unitary. For the converse, assume that W_L is unitary. In view of Proposition 4.1, $m = 0$ implies that

$$W_L^*(\Omega \otimes h_l) = \sum_{q \in \Lambda} \overline{c_l^{h_q}} (\Omega \otimes h_q) \in \Omega \otimes \mathcal{E},$$

and hence $W_L^*(\Omega \otimes \mathcal{E}) \subseteq \Omega \otimes \mathcal{E}$. Now W_L is unitary makes $(W_L, S^{\mathcal{E}})$ a Nica covariant representation of O_n, and hence, by Theorem 4.2, we have $L\mathcal{E} \subseteq \Omega \otimes \mathcal{E}$. By the definition of W_L, we now know that $W_L(\Omega \otimes \mathcal{E}) \subseteq \Omega \otimes \mathcal{E}$. Therefore, the closed subspace $\Omega \otimes \mathcal{E}$ reduces W_L, and hence

$$W_L|_{\Omega \otimes \mathcal{E}} : \Omega \otimes \mathcal{E} \to \Omega \otimes \mathcal{E},$$

is a unitary operator. Then $L\mathcal{E} = W_L(\Omega \otimes \mathcal{E}) = \Omega \otimes \mathcal{E}$ proves the necessary part of the theorem. $\square$

Li's proof [12, Corollary 3.6] of the above theorem employs distinct methodologies, such as Wold decompositions of odometer semigroups.

5. Odometer lifting

This section aims to draw a connection between noncommutative dilations and representations of O_n, which will be referred to as the lifting of representations of O_n or odometer lifting. We begin by discussing the principle of noncommutative isometric dilations of pure row contraction, a theory that is largely attributed to Popescu [18] (also see Bunce [3], Frazho [7], and Durszt and Sz.-Nagy

[10]). Let $\mathcal{H}$ be a Hilbert space and let $T = (T_1, \ldots, T_n)$ be a pure row contraction on $\mathcal{H}$; that is, T is a row contraction and satisfies the limit condition in (1.2). Define the *defect space* $\mathcal{D}_{T^*}$ as

$$\mathcal{D}_{T^*} = \overline{\text{ran}}\Big(I_{\mathcal{H}} - \sum_{j=1}^{n} T_j T_j^*\Big).$$

The following is known as the noncommutative dilation of T [18, Theorem 2.1]. The language is slightly different and adapted to our current context. Let $\mathcal{S}$ be a closed subspace of a Hilbert space $\mathcal{H}$. Throughout, we denote $P_{\mathcal{S}}$ as the orthogonal projection onto $\mathcal{S}$.

Theorem 5.1 (Dilations of pure row contractions)**.** *Let T be a pure row contraction on $\mathcal{H}$. Then there exists an isometry $\Pi_T : \mathcal{H} \to \mathcal{F}_n^2 \otimes \mathcal{D}_{T^*}$ such that*

$$\Pi_T T_i^* = (S_i \otimes I_{\mathcal{D}_{T^*}})^* \Pi_T,$$

for all $i = 1, \ldots, n$. Moreover, if we set

$$\mathcal{Q}_T = \Pi_T \mathcal{H},$$

then $\mathcal{Q}_T$ is a joint invariant subspace of $(S_1^* \otimes I_{\mathcal{D}_{T^*}}, \ldots, S_n^* \otimes I_{\mathcal{D}_{T^*}})$, and

$$T \cong (P_{\mathcal{Q}_T}(S_1 \otimes I_{\mathcal{D}_{T^*}})|_{\mathcal{Q}_T}, \ldots, P_{\mathcal{Q}_T}(S_n \otimes I_{\mathcal{D}_{T^*}})|_{\mathcal{Q}_T}).$$

Moreover, $\mathcal{F}_n^2 \otimes \mathcal{D}_{T^*} = \overline{span}\{S_\mu \mathcal{Q}_T : \mu \in F_n^+\}$.

One can write down Π_T explicitly. However, for our purposes, we do not need such representations. The final identity is referred to as the minimality property, a distinguished property that makes Π_T unique up to unitary equivalence.

Definition 5.2. *We refer to the pair $(\Pi_T, \mathcal{F}_n^2 \otimes \mathcal{D}_{T^*})$ as T's minimal dilation. The closed subspace $\mathcal{Q}_T$ will be referred to as a model space.*

In other words, tuples of creation operators on vector-valued Fock spaces are universal models for pure row contractions [18]. This section aims to show that isometric Fock representations are universal models for contractive representations of O_n. To give greater clarity, we now introduce the following notion:

Definition 5.3. *Let (W, T) defined on $\mathcal{H}$ be a contractive representation of O_n, and let $L \in \mathcal{B}(\mathcal{D}_{T^*}, \mathcal{F}_n^2 \otimes \mathcal{D}_{T^*})$. Then $(W_L, S^{\mathcal{D}_{T^*}})$ is said to be an odometer lift of (W, T) if*

$$\Pi_T W^* = W_L^* \Pi_T,$$

where $(\Pi_T, \mathcal{F}_n^2 \otimes \mathcal{D}_{T^*})$ is the minimal dilation of T.

Suppose $(W_L, S^{\mathcal{D}_{T^*}})$ is an odometer lift of (W, T) for some symbol $L \in \mathcal{B}(\mathcal{D}_{T^*}, \mathcal{F}_n^2 \otimes \mathcal{D}_{T^*})$. In view of Theorem 5.1, we have

$$\Pi_T T_i^* = (S_i \otimes I_{\mathcal{D}_{T^*}})^* \Pi_T,$$

for all $i = 1, \ldots, n$. Then, with the additional property that $\Pi_T W^* = W_L^* \Pi_T$, it follows that the model space $\mathcal{Q}_T (= \Pi_T \mathcal{H})$ is jointly invariant under $(W_L^*, S_1^* \otimes I_{\mathcal{D}_{T^*}}, \ldots, S_n^* \otimes I_{\mathcal{D}_{T^*}})$. Therefore, we can say that (W, T) dilates to $(W_L, S^{\mathcal{D}_{T^*}})$. Also, in particular, $\mathcal{Q}_T$ is jointly invariant under $n+1$ tuple $(W_L, S^{\mathcal{D}_{T^*}})$. With this arrangement, we are now ready to prove our dilation result concerning the representations of O_n.

Theorem 5.4. *A contractive representation of O_n always admits an odometer lift.*

Proof. Let (W,T) defined on $\mathcal{H}$ be a contractive representation of O_n. First, we establish two essential identities (namely, (5.1) and (5.2) as below) that will play a significant part in the main body of the proof of this theorem. Since Π_T is an isometry and $P_{\mathcal{Q}_T} = \Pi_T\Pi_T^*$, it follows that

$$\Pi_T^* = \Pi_T^* P_{\mathcal{Q}_T}.$$

For each $k < n$, we compute

$$W\Pi_T^*(S_k \otimes I_{\mathcal{D}_{T^*}}) = WT_k\Pi_T^* = T_{k+1}\Pi_T^*,$$

where the final identity follows from the odometer property that $WT_k = T_{k+1}$. Since $T_{k+1}\Pi_T^* = \Pi_T^*(S_{k+1} \otimes I_{\mathcal{D}_{T^*}})$, it follows that

$$W\Pi_T^*(S_k \otimes I_{\mathcal{D}_{T^*}}) = \Pi_T^*(S_{k+1} \otimes I_{\mathcal{D}_{T^*}}), \tag{5.1}$$

for all $k = 1, \ldots, n-1$. This proves the first set of essential identities. For the final one, similar to the above, we compute

$$W\Pi_T^*(S_n \otimes I_{\mathcal{D}_{T^*}}) = WT_n\Pi_T^* = T_1W\Pi_T^* = T_1\Pi_T^*\Pi_TW\Pi_T^*.$$

Then $T_1\Pi_T^* = \Pi_T^*(S_1 \otimes I_{\mathcal{D}_{T^*}})$ implies

$$W\Pi_T^*(S_n \otimes I_{\mathcal{D}_{T^*}}) = \Pi_T^*(S_1 \otimes I_{\mathcal{D}_{T^*}})\Pi_TW\Pi_T^*. \tag{5.2}$$

Now we turn to construct the required odometer map W_L on $\mathcal{F}_n^2 \otimes \mathcal{D}_{T^*}$ so that $(W_L, S^{\mathcal{D}_{T^*}})$ is an odometer lift of (W,T). Define $L : \mathcal{D}_{T^*} \to \mathcal{F}_n^2 \otimes \mathcal{D}_{T^*}$ by

$$L\eta = (\Pi_TW\Pi_T^*)(\Omega \otimes \eta) \qquad (\eta \in \mathcal{D}_{T^*}),$$

where $(\Pi_T, \mathcal{F}_n^2 \otimes \mathcal{D}_{T^*})$ is the minimal dilation of T. Clearly, $L \in \mathcal{B}(\mathcal{D}_{T^*}, \mathcal{F}_n^2 \otimes \mathcal{D}_{T^*})$ is a symbol. Consider the odometer map W_L corresponding to L. We claim that $\Pi_TW^* = W_L^*\Pi_T$. It is enough to prove this identity on $\{e_\mu \otimes \eta : \mu \in F_n^+, \eta \in \mathcal{D}_{T^*}\}$. To this end, fix $\eta \in \mathcal{D}_{T^*}$ and $\mu \in F_n^+$. Suppose $k < n$. By (5.1), it follows that

$$\begin{aligned}
W\Pi_T^*(e_k \otimes e_\mu \otimes \eta) &= W\Pi_T^*(S_k \otimes I_{\mathcal{D}_{T^*}})(e_\mu \otimes \eta) \\
&= \Pi_T^*(S_{k+1} \otimes I_{\mathcal{D}_{T^*}})(e_\mu \otimes \eta) \\
&= \Pi_T^*(e_{k+1} \otimes e_\mu \otimes \eta) \\
&= \Pi_T^*W_L(e_k \otimes e_\mu \otimes \eta).
\end{aligned}$$

Now we need to focus on vectors of the form $e_n \otimes e_\mu \otimes \eta$. First, we consider the case of $e_n \otimes \eta$. In this case, we use the identity (5.2) to derive

$$W\Pi_T^*(e_n \otimes \eta) = W\Pi_T^*(S_n \otimes I_{\mathcal{D}_{T^*}})(\Omega \otimes \eta) = \Pi_T^*(S_1 \otimes I_{\mathcal{D}_{T^*}})\Pi_TW\Pi_T^*(\Omega \otimes \eta).$$

But $(\Pi_TW\Pi_T^*)(\Omega \otimes \eta) = L\eta$, and hence

$$\Pi_T^*(S_1 \otimes I_{\mathcal{D}_{T^*}})\Pi_TW\Pi_T^*(\Omega \otimes \eta) = \Pi_T^*(e_1 \otimes L\eta).$$

Since, $W_L(e_n \otimes \eta) = e_1 \otimes L\eta$, by the definition of the odometer map W_L, it follows that

$$W\Pi_T^*(e_n \otimes \eta) = \Pi_T^*W_L(e_n \otimes \eta).$$

Finally, we consider the general case of $e_n \otimes e_\mu \otimes \eta$. Again, using the identity (5.2), we see

$$\begin{aligned}
W\Pi_T^*(e_n \otimes e_\mu \otimes \eta) &= W\Pi_T^*(S_n \otimes I_{\mathcal{D}_{T^*}})(e_\mu \otimes \eta) \\
&= \Pi_T^*(S_1 \otimes I_{\mathcal{D}_{T^*}})\Pi_TW\Pi_T^*(e_\mu \otimes \eta) \\
&= \Pi_T^*(e_1 \otimes \Pi_TW\Pi_T^*(e_\mu \otimes \eta)).
\end{aligned}$$

Using the preceding two cases repeatedly along with the identities $\Pi_T^* = \Pi_T^* P_{\mathcal{Q}_T}$ and $P_{\mathcal{Q}_T}(S_k \otimes I_{\mathcal{D}_{T^*}}) = P_{\mathcal{Q}_T}(S_k \otimes I_{\mathcal{D}_{T^*}}) P_{\mathcal{Q}_T}$ for all k, we conclude that

$$W\Pi_T^*(e_n \otimes e_\mu \otimes \eta) = \Pi_T^* W_L(e_n \otimes e_\mu \otimes \eta).$$

This completes the proof of the theorem. □

From the standpoint of model spaces, Theorem 5.4 and the discussion that came before it yield the following:

Corollary 5.5. *Let (W,T) be a contractive representation of O_n. Then there exist a symbol $L \in \mathcal{B}(\mathcal{D}_{T^*}, \mathcal{F}_n^2 \otimes \mathcal{D}_{T^*})$ and a closed subspace $\mathcal{Q}_T \subseteq \mathcal{F}_n^2 \otimes \mathcal{D}_{T^*}$ such that $(S_j \otimes I_{\mathcal{D}_{T^*}})^* \mathcal{Q}_T \subseteq \mathcal{Q}_T$ for all $j = 1, \dots, n$, and $W_L^* \mathcal{Q}_T \subseteq \mathcal{Q}_T$, and*

$$(W,T) \cong (P_{\mathcal{Q}_T} W_L|_{\mathcal{Q}_T}, P_{\mathcal{Q}_T}(S_1 \otimes I_{\mathcal{D}_{T^*}})|_{\mathcal{Q}_T}, \dots, P_{\mathcal{Q}_T}(S_n \otimes I_{\mathcal{D}_{T^*}})|_{\mathcal{Q}_T}).$$

This result is, on the one hand, dilation for contractive representations of O_n in the sense of Sz.-Nagy and Foias [22], and Popescu [17]. On the other hand, it shares some (but not all) resemblance to Sarason's commutant lifting theorem [19] (and also to the noncommutative commutant lifting theorem). More specifically, in the present scenario, we are lifting the odometer maps acting on Hilbert spaces to the odometer maps acting on the Fock spaces (the minimal isometric dilation spaces). The sole element absent, in contrast to the classical commutant lifting theorem, is the preservation of the norm of the lifted operators. We address this particular shortcoming in the subsequent observation:
Let (W,T) be a contractive representation of O_n. By Theorem 5.4 and its proof, we know that $(W_L, S^{\mathcal{D}_{T^*}})$ is an odometer lift of (W,T), where the symbol $L \in \mathcal{B}(\mathcal{D}_{T^*}, \mathcal{F}_n^2 \otimes \mathcal{D}_{T^*})$ is given by

$$L\eta = \Pi_T W \Pi_T^*(\Omega \otimes \eta),$$

for all $\eta \in \mathcal{D}_{T^*}$. As $W = \Pi_T^* W_L \Pi_T$ and Π_T is an isometry, it follows that $\|W\| \le \|W_L\| = \|L\|$. On the other hand, since $\|L\| = \sup\{\|L\eta\| : \eta \in \mathcal{E}, \|\eta\| = 1\}$, we compute

$$\begin{aligned}
\|W_L\| &= \|L\| \\
&= \sup\{\|\Pi_T W \Pi_T^*(\Omega \otimes \eta)\| : \eta \in \mathcal{E}, \|\eta\| = 1\} \\
&\le \|\Pi_T\| \, \|W\| \, \|\Pi_T^*\| \sup\{\|\Omega \otimes \eta\| : \eta \in \mathcal{E}, \|\eta\| = 1\} \\
&= \|W\|,
\end{aligned}$$

as $\|\Pi_T\| = 1$. This implies that $\|W\| = \|W_L\|$. This and Theorem 5.4 then implies the following:

Corollary 5.6. *Let (W,T) be a contractive representation of O_n. Then $(W_L, S^{\mathcal{D}_{T^*}})$ is an odometer lift of (W,T), where the symbol $L \in \mathcal{B}(\mathcal{D}_{T^*}, \mathcal{F}_n^2 \otimes \mathcal{D}_{T^*})$ is defined by*

$$L\eta = \Pi_T W \Pi_T^*(\Omega \otimes \eta),$$

for all $\eta \in \mathcal{D}_{T^*}$, and $(\Pi_T, \mathcal{F}_n^2 \otimes \mathcal{D}_{T^*})$ is the minimal isometric dilation of T. Moreover

$$\|W\| = \|W_L\| = \|L\|.$$

6. Subrepresentations

Now we turn to joint invariant subspaces of Fock representations, which will also be referred to as subrepresentations of the odometer semigroup. First, we introduce the concept of invariant subspaces in the traditional context and then extend it to invariant subspaces of Fock representations (or subrepresentations of O_n) in the present context.

Definition 6.1. *Let $(W_L, S^{\mathcal{E}})$ be a Fock representation. A closed subspace $\mathcal{S} \subseteq \mathcal{F}_n^2 \otimes \mathcal{E}$ is referred to as an invariant subspace if*

$$(S_j \otimes I_{\mathcal{E}})\mathcal{S} \subseteq \mathcal{S},$$

for all $j = 1, \dots, n$. If, in addition, $W_L\mathcal{S} \subseteq \mathcal{S}$, then we call $\mathcal{S}$ invariant under the Fock representation $(W_L, S^{\mathcal{E}})$. We also say in this instance that $(W_L|_{\mathcal{S}}, S^{\mathcal{E}}|_{\mathcal{S}})$ is a subrepresentation of the Fock representation $(W_L, S^{\mathcal{E}})$.

In the definition above, the symbol $S^{\mathcal{E}}|_{\mathcal{S}}$ denotes the n-tuple on $\mathcal{S}$, which is defined as follows:

$$S^{\mathcal{E}}|_{\mathcal{S}} = ((S_1 \otimes I_{\mathcal{E}})|_{\mathcal{S}}, \cdots, (S_n \otimes I_{\mathcal{E}})|_{\mathcal{S}}).$$

Recall that a bounded linear operator $\Phi : \mathcal{F}_n^2 \otimes \mathcal{E}_* \to \mathcal{F}_n^2 \otimes \mathcal{E}$ is *multi-analytic* [16] if

$$\Phi(S_j \otimes I_{\mathcal{E}_*}) = (S_j \otimes I_{\mathcal{E}})\Phi, \tag{6.1}$$

for all $j = 1, \dots, n$. Moreover, a multi-analytic operator Φ, as above, is *inner* if Φ is an isometry. Popescu's result [17], commonly referred to as the noncommutative Beurling-Lax-Halmos theorem, links up invariant subspaces with multi-analytic inner functions. To facilitate the subsequent construction, we will now offer a sketch of the proof of the result. The technique is standard and simply follows the lines of [9, Theorem 2] and [17, Theorem 2.2]. Here the only distinction is in our emphasis on the factorizations of the multi-analytic operators, which will yield fresh perspectives that we will elaborate on shortly.

Theorem 6.2. *Let $\mathcal{S} \subseteq \mathcal{F}_n^2 \otimes \mathcal{E}$ be a closed subspace. Then $\mathcal{S}$ is an invariant subspace if and only if there exist a Hilbert space $\mathcal{E}_*$ and an inner multi-analytic operator $\Phi : \mathcal{F}_n^2 \otimes \mathcal{E}_* \to \mathcal{F}_n^2 \otimes \mathcal{E}$ such that*

$$\mathcal{S} = \Phi(\mathcal{F}_n^2 \otimes \mathcal{E}_*).$$

Moreover, there exists a unitary $\Pi : \mathcal{F}_n^2 \otimes \mathcal{E}_* \to \mathcal{S}$ such that $\Pi(S_i \otimes I_{\mathcal{E}_*}) = (S_i \otimes I_{\mathcal{E}})\Pi$ for all $i = 1, \dots, n$, and

$$\Phi = i_{\mathcal{S}} \circ \Pi.$$

Proof. If $\mathcal{S} = \Phi(\mathcal{F}_n^2 \otimes \mathcal{E}_*)$, then (6.1) immediately implies that $\mathcal{S}$ is an invariant subspace. For the reverse direction, for each $j = 1, \dots, n$, set $V_j = (S_j \otimes I_{\mathcal{E}})|_{\mathcal{S}}$. Since $\mathcal{S}$ is an invariant subspace, for all $i, j \in \{1, \dots, n\}$, it follows that $V_i^* V_j = \delta_{ij} I_{\mathcal{S}}$. Moreover, for all $\{f_i\}_{i=1}^n \subseteq \mathcal{S}$, we have

$$\|\sum_{i=1}^n V_i f_i\|^2 = \|\sum_{i=1}^n (S_i \otimes I_{\mathcal{E}}) f_i\|^2 \leq \sum_{i=1}^n \|f_i\|^2.$$

Consequently, $(V_1, \dots, V_n)$ is a row isometry defined on $\mathcal{S}$. Furthermore, since $V_\mu \mathcal{S} = (S_\mu \otimes I_{\mathcal{E}})\mathcal{S} \subseteq (S_\mu \otimes I_{\mathcal{E}})(\mathcal{F}_n^2 \otimes \mathcal{E})$, it follows that

$$\bigcap_{m=0}^{\infty} \Big(\bigoplus_{\mu \in F_n^+, |\mu| = m} V_\mu \mathcal{S} \Big) \subseteq \bigcap_{m=0}^{\infty} \Big(\bigoplus_{\mu \in F_n^+, |\mu| = m} (S_\mu \otimes I_{\mathcal{E}})(\mathcal{F}_n^2 \otimes \mathcal{E}) \Big) = \{0\}.$$

By the noncommutative Wold decomposition ([9, Theorem 2] or [18, Theorem 1.3]), $(V_1, \dots, V_n)$ is a pure row isometry on $\mathcal{S}$, and we have the orthogonal decomposition

$$\mathcal{S} = \bigoplus_{\mu \in F_n^+} V_\mu \mathcal{E}_* = \bigoplus_{\mu \in F_n^+} (S_\mu \otimes I_{\mathcal{E}}) \mathcal{E}_*,$$

where

$$\mathcal{E}_* = \mathcal{S} \ominus \sum_{i=1}^{n} (S_i \otimes I_{\mathcal{E}})\mathcal{S}.$$

Then

$$\Pi(e_\mu \otimes \eta) = (S_\mu \otimes I_{\mathcal{E}})\eta \qquad (\mu \in F_n^+, \eta \in \mathcal{E}_*),$$

defines a unitary $\Pi : \mathcal{F}_n^2 \otimes \mathcal{E}_* \to \mathcal{S}$. Moreover, it is easy to see that $\Pi(S_i \otimes I_{\mathcal{E}_*}) = (S_i \otimes I_{\mathcal{E}})\Pi$ for all $i = 1, \ldots, n$. Now we consider the inclusion map $i_{\mathcal{S}} : \mathcal{S} \hookrightarrow \mathcal{F}_n^2 \otimes \mathcal{E}$, and define

$$\Pi_{\mathcal{S}} = i_{\mathcal{S}} \circ \Pi \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E}_*, \mathcal{F}_n^2 \otimes \mathcal{E}).$$

Since $i_{\mathcal{S}}$ is an isometry, it is evident that $\Pi_{\mathcal{S}}$ is an isometry and $\Pi_{\mathcal{S}}\Pi_{\mathcal{S}}^* = P_{\mathcal{S}}$. An easy computation shows that $\Pi_{\mathcal{S}}(S_i \otimes I_{\mathcal{E}_*}) = (S_i \otimes I_{\mathcal{E}})\Pi_{\mathcal{S}}$ for all $i = 1, \ldots, n$, and consequently, there exists an inner multi-analytic operator $\Phi \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E}_*, \mathcal{F}_n^2 \otimes \mathcal{E})$ such that

$$\Phi = \Pi_{\mathcal{S}} = i_{\mathcal{S}} \circ \Pi \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E}_*, \mathcal{F}_n^2 \otimes \mathcal{E}),$$

and $\mathcal{S} = \Phi(\mathcal{F}_n^2 \otimes \mathcal{E}_*)$. $\square$

The operator Φ is unique in the following sense: If $\mathcal{S} = \Phi(\mathcal{F}_n^2 \otimes \mathcal{E}_*)$ is an invariant subspace of $\mathcal{F}_n^2 \otimes \mathcal{E}$ as above, and if $\mathcal{S} = \tilde{\Phi}(\mathcal{F}_n^2 \otimes \tilde{\mathcal{E}}_*)$ for some Hilbert space $\tilde{\mathcal{E}}_*$ and inner multi-analytic operator $\tilde{\Phi} : \mathcal{F}_n^2 \otimes \tilde{\mathcal{E}}_* \to \mathcal{F}_n^2 \otimes \mathcal{E}$, then there exists a unitary operator $\tau \in \mathcal{B}(\tilde{\mathcal{E}}_*, \mathcal{E}_*)$ such that

$$\tilde{\Phi} = \Phi(I_{\mathcal{F}_n^2} \otimes \tau).$$

We again stress the factorization $\Phi = i_{\mathcal{S}} \circ \Pi$ in the above theorem, which, in terms of the commutative diagram, yields the following:

$$\begin{tikzcd}
& \mathcal{F}_n^2 \otimes \mathcal{E}_* \arrow[dl, "\Pi"'] \arrow[d, "\Phi"] \\
\mathcal{S} \arrow[r, "i_{\mathcal{S}}"'] & \mathcal{F}_n^2 \otimes \mathcal{E}
\end{tikzcd}$$

The way we factorize Φ will be significant in what comes next. And this is where our use of the noncommutative Beurling-Lax-Halmos theorem will differ from how it was used earlier. For instance:

Lemma 6.3. *In the setting of Theorem 6.2, suppose $\mathcal{S} = \Phi(\mathcal{F}_n^2 \otimes \mathcal{E}_*)$. For each $\mu \in F_n^+$ and $\eta \in \mathcal{E}_*$, we have the following:*

(1) $\Phi(e_\mu \otimes \eta) = (S_\mu \otimes I_{\mathcal{E}})\eta$.
(2) $\Phi^*(S_\mu \otimes I_{\mathcal{E}})\eta = e_\mu \otimes \eta$.

Proof. By the factorization part of Theorem 6.2, we know that $\Phi = i_{\mathcal{S}} \circ \Pi$. Since $\mathcal{E}_* = \mathcal{S} \ominus \sum_{i=1}^n (S_i \otimes I_{\mathcal{E}})\mathcal{S}$, we have $\eta \in \mathcal{E}_* \subseteq \mathcal{S}$, so that

$$(S_\mu \otimes I_{\mathcal{E}})\eta \in \mathcal{S},$$

as $\mathcal{S}$ is an invariant subspace. Then

$$\Phi(e_\mu \otimes \eta) = i_{\mathcal{S}} \circ \Pi(e_\mu \otimes \eta) = i_{\mathcal{S}}(S_\mu \otimes I_{\mathcal{E}})\eta = (S_\mu \otimes I_{\mathcal{E}})\eta.$$

The second identity can be easily derived from the fact that Φ is isometric. $\square$

Now we are ready for the invariant subspaces of Fock representations. However, we must lay the right foundation. Let $\mathcal{S} \subseteq \mathcal{F}_n^2 \otimes \mathcal{E}$ be a closed subspace. First, assume that $\mathcal{S}$ is an invariant subspace. By Theorem 6.2, there exist a Hilbert space $\mathcal{E}_*$ and an inner multi-analytic operator $\Phi \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E}_*, \mathcal{F}_n^2 \otimes \mathcal{E})$ such that

$$\mathcal{S} = \Phi(\mathcal{F}_n^2 \otimes \mathcal{E}_*).$$

Next, consider a Fock representation $(W_L, S^{\mathcal{E}})$. According to Douglas' range inclusion theorem [6], $\mathcal{S}$ is invariant under $(W_L, S^{\mathcal{E}})$ if and only if there is $C \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E}_*)$ such that

$$W_L \Phi = \Phi C.$$

If such a C exists, it would be unique, and this simply follows from the fact that

$$C = \Phi^* W_L \Phi. \tag{6.2}$$

The remaining task, therefore, along with its existence, is to determine the representation of the map C. This is what we do in the following theorem, which in particular asserts that C is an odometer map and subsequently classifies invariant subspaces of Fock representations.

Theorem 6.4. *Let $\mathcal{E}$ be a Hilbert space, $(W_L, S^{\mathcal{E}})$ be a Fock representation, and let $\mathcal{S} \subseteq \mathcal{F}_n^2 \otimes \mathcal{E}$ be a closed subspace. Then $\mathcal{S}$ is invariant under $(W_L, S^{\mathcal{E}})$ if and only if there exist a Hilbert space $\mathcal{E}_*$, an inner multi-analytic operator $\Phi \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E}_*, \mathcal{F}_n^2 \otimes \mathcal{E})$, and a Fock representation $(W_{L_*}, S^{\mathcal{E}_*})$ such that*

$$\mathcal{S} = \Phi(\mathcal{F}_n^2 \otimes \mathcal{E}_*),$$

and

$$W_L \Phi = \Phi W_{L_*}.$$

Proof. Following Theorem 6.2 and what we have discussed preceding the statement of this theorem, all that is left is to show that there is an $L_* \in \mathcal{B}(\mathcal{E}_*, \mathcal{F}_n^2 \otimes \mathcal{E}_*)$ such that $W_L \Phi = \Phi W_{L_*}$, where $\mathcal{S}$ is the invariant subspace of $\mathcal{F}_n^2 \otimes \mathcal{E}$ admitting inner multi-analytic representation $\mathcal{S} = \Phi(\mathcal{F}_n^2 \otimes \mathcal{E}_*)$ and $W_L \mathcal{S} \subseteq \mathcal{S}$. By (6.2), we already know that there is a $C \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E}_*)$ such that

$$C = \Phi^* W_L \Phi.$$

We need to prove that $C = W_{L_*}$ for some $L_* \in \mathcal{B}(\mathcal{E}_*, \mathcal{F}_n^2 \otimes \mathcal{E}_*)$. We now start to build the map L_*. It is crucial to recall from the proof of Theorem 6.2 that

$$\mathcal{E}_* = \mathcal{S} \ominus \sum_{i=1}^{n} (S_i \otimes I_{\mathcal{E}}) \mathcal{S}.$$

Recall also that the inner multi-analytic operator Φ factors as (see (6.2))

$$\Phi = i_{\mathcal{S}} \circ \Pi \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E}_*, \mathcal{F}_n^2 \otimes \mathcal{E}),$$

where $i_{\mathcal{S}} : \mathcal{S} \hookrightarrow \mathcal{F}_n^2 \otimes \mathcal{E}$ is the inclusion map, and $\Pi : \mathcal{F}_n^2 \otimes \mathcal{E}_* \to \mathcal{S}$ is the unitary operator defined by

$$\Pi(e_\mu \otimes \eta) = (S_\mu \otimes I_{\mathcal{E}})\eta,$$

for all $\mu \in F_n^+$ and $\eta \in \mathcal{E}_*$. Also recall that $W_L \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E})$. As

$$\mathcal{E}_* \subseteq \mathcal{S} \subseteq \mathcal{F}_n^2 \otimes \mathcal{E},$$

it follows that $W_L|_{\mathcal{E}_*} : \mathcal{E}_* \to \mathcal{F}_n^2 \otimes \mathcal{E}$ is a well-defined bounded linear operator. Moreover, since $W_L \mathcal{S} \subseteq \mathcal{S}$, we conclude that

$$W_L|_{\mathcal{E}_*} : \mathcal{E}_* \to \mathcal{S}.$$

Finally, we define the symbol $L_* \in \mathcal{B}(\mathcal{E}_*, \mathcal{F}_n^2 \otimes \mathcal{E}_*)$ by

$$L_* = \Pi^* W_L|_{\mathcal{E}_*}.$$

Therefore, $(W_{L_*}, S^{\mathcal{E}_*})$ on $\mathcal{F}_n^2 \otimes \mathcal{E}_*$ is a Fock representation. It is now enough to prove that

$$\Phi^* W_L \Phi = W_{L_*}.$$

To this end, fix $\eta \in \mathcal{E}_*$ and $\mu \in F_n^+$. Assume that $k < n$. By applying Lemma 6.3 twice, we obtain

$$\begin{aligned}\Phi^* W_L \Phi(e_k \otimes e_\mu \otimes \eta) &= \Phi^* W_L (S_k \otimes I_{\mathcal{E}_*})(S_\mu \otimes I_{\mathcal{E}_*})\eta \\ &= \Phi^* (S_{k+1} \otimes I_{\mathcal{E}_*})(S_\mu \otimes I_{\mathcal{E}_*})\eta \\ &= e_{k+1} \otimes e_\mu \otimes \eta.\end{aligned}$$

Moreover, since $W_{L_*}(e_k \otimes e_\mu \otimes \eta) = e_{k+1} \otimes e_\mu \otimes \eta$, we conclude that

$$\Phi^* W_L \Phi(e_k \otimes e_\mu \otimes \eta) = W_{L_*}(e_k \otimes e_\mu \otimes \eta),$$

for all $k < n$. Next, consider the basis element $e_n \otimes \eta$. Again, Lemma 6.3 implies

$$\Phi^* W_L \Phi\,(e_n \otimes \eta) = \Phi^* W_L (S_n \otimes I_{\mathcal{E}})\eta = \Phi^* (S_1 \otimes I_{\mathcal{E}}) W_L(\eta).$$

As $\eta \in \mathcal{E}_* \subseteq \mathcal{S}$ and $W_L \mathcal{S} \subseteq \mathcal{S}$, it follows that $W_L(\eta) \in \mathcal{S}$. By using the orthogonal direct sum decomposition of $\mathcal{S}$, we write

$$W_L(\eta) = \sum_\alpha c_\alpha (S_\alpha \otimes I_{\mathcal{E}})\eta_\alpha,$$

where $\eta_\alpha \in \mathcal{E}_* \subseteq \mathcal{S}$ for all $\alpha \in F_n^+$. Then $\Pi^* W_L(\eta) = \sum_\alpha c_\alpha e_\alpha \otimes \eta_\alpha$, and consequently

$$\begin{aligned}\Phi^*(S_1 \otimes I_{\mathcal{E}}) W_L(\eta) &= \Phi^*(S_1 \otimes I_{\mathcal{E}}) \Big(\sum_\alpha c_\alpha (S_\alpha \otimes I_{\mathcal{E}})\eta_\alpha \Big) \\ &= e_1 \otimes \sum_\alpha c_\alpha e_\alpha \otimes \eta_\alpha \\ &= e_1 \otimes \Pi^* W_L(\eta),\end{aligned}$$

which, along with $e_1 \otimes \Pi^* W_L(\eta) = W_{L_*}(e_n \otimes \eta)$ implies that

$$\Phi^* W_L \Phi(e_n \otimes \eta) = W_{L_*}(e_n \otimes \eta).$$

Finally, we consider the basis vector $e_n \otimes e_\mu \otimes \eta \in \mathcal{F}_n^2 \otimes \eta_*$. Lemma 6.3 again implies that

$$\Phi^* W_L \Phi(e_n \otimes e_\mu \otimes \eta) = \Phi^* W_L (S_n \otimes I_{\mathcal{E}})(S_\mu \otimes I_{\mathcal{E}})\eta = \Phi^*(S_1 \otimes I_{\mathcal{E}}) W_L (S_\mu \otimes I_{\mathcal{E}})\eta.$$

Depending on the symbol μ, the definition of odometer relations and Lemma 6.3 implies that

$$\Phi^* W_L \Phi(e_n \otimes e_\mu \otimes \eta) = W_{L_*}(e_n \otimes e_\mu \otimes \eta).$$

This says $\Phi^* W_L \Phi$ and W_{L_*} agree on the orthonormal basis vectors, and hence $\Phi^* W_L \Phi = W_{L_*}$, completing the proof of the theorem. □

The formulation of the problem of invariant subspaces of Fock representations is comparable to a similar problem in a different context, which is described in [2] in answer to a question by J. Agler and N. Young.

The following corollary is now easy and follows the lines of the classical Beurling theorem:

Corollary 6.5. *Let $\mathcal{S}$ be an invariant subspace of a Fock representation $(W_L, S^{\mathcal{E}})$. Then there exists a Fock representation $(W_{L_*}, S^{\mathcal{E}_*})$ such that*

$$(W_L|_{\mathcal{S}}, S^{\mathcal{E}}|_{\mathcal{S}}) \cong (W_{L_*}, S^{\mathcal{E}_*}).$$

Proof. We know that there is a Hilbert space $\mathcal{E}_*$, an inner multi-analytic operator $\Phi \in \mathcal{B}(\mathcal{F}_n^2 \otimes \mathcal{E}_*, \mathcal{F}_n^2 \otimes \mathcal{E})$, and a symbol $L_* \in \mathcal{B}(\mathcal{E}_*, \mathcal{F}_n^2 \otimes \mathcal{E}_*)$ such that $\mathcal{S} = \Phi(\mathcal{F}_n^2 \otimes \mathcal{E}_*)$ and $W_L \Phi = \Phi W_{L_*}$. Then

$$U(f \otimes \eta) = \Phi(f \otimes \eta) \qquad (f \in \mathcal{F}_n^2, \eta \in \mathcal{E}_*),$$

defines a unitary $U : \mathcal{F}_n^2 \otimes \mathcal{E}_* \to \mathcal{S}$. Evidently, $UW_{L_*} = W_L|_{\mathcal{S}} U$ and $U(S_i \otimes I_{\mathcal{E}_*}) = (S_i \otimes I_{\mathcal{E}})|_{\mathcal{S}} U$ for all $i = 1, \ldots, n$. This completes the proof of the corollary. $\square$

Therefore, up to unitary equivalence, subrepresentations of Fock representations are also Fock representations.

7. Examples

In this concluding section, we aim to present some examples of representations of O_n and some simple C^*-algebras that are generated by isometric odometer operators.

7.1. Nica covariant.

We begin with a simple example of Nica covariant representation of O_n.

Example 7.1. *Let $\mathcal{H}$ be Hilbert space, $\{e_n : n \in \mathbb{Z}_+\}$ be an orthonormal basis for $\mathcal{H}$, and let $q \in \mathbb{T}$. Define isometries V_1, V_2 and W on $\mathcal{H}$ as follows:*

$$V_1(e_k) = \overline{q}^{2k} e_{2k}, \; V_2(e_k) = \overline{q}^{2k+1} e_{2k+1},$$

and

$$W(e_k) = \overline{q} e_{k+1},$$

for all $k \in \mathbb{Z}_+$. It is easy to check that $V = (V_1, V_2)$ is a row isometry. Moreover, $WV_1 = V_2$ and $W^* V_1 = \overline{q} V_2 W^*$. If $q = 1$, then (W, V) becomes a Nica covariant representation of O_n.

Next, we present an example of an isometric Fock representation. In other words, in the following, given a Hilbert space $\mathcal{E}$, we will provide an explicit example of a map $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$ such that W_L is an isometry on $\mathcal{F}_n^2 \otimes \mathcal{E}$.

Example 7.2. *Let $\mathcal{E}$ be a Hilbert space, and let $\{h_p\}_{p \in \Lambda}$ be an orthonormal basis for $\mathcal{E}$. Define $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$ by*

$$L h_m = e_1^{\otimes m} \otimes h_m \qquad (m \in \Lambda).$$

Given the notation in (3.4)*, the Fourier coefficients of Lh_m satisfy the identity*

$$c_{p,q}^{h_m} = 0,$$

for all $(p, q) \neq (m, m)$. Therefore, for any $m, t \in \Lambda$, we have

$$\sum_{p \in \mathbb{Z}_+, q \in \Lambda} c_{p+r,q}^{h_m} \overline{c_{p,q}^{h_t}} = 0 \qquad (r \geq 1).$$

Clearly, L is an isometry and $ranL \subseteq \mathcal{E}_L$. Then, Theorem 3.1 itself guarantees that W_L is an isometry on $\mathcal{F}_n^2 \otimes \mathcal{E}$.

The example above suggests that Li's construction [12] cannot be extended beyond unitary Fock representations of O_n. In the latter scenario, the symbol represents a mapping from $\mathcal{E}$ to itself.

In the following example, we highlight, in the context of Theorem 4.2, that Nica covariant representations $(W_L, S^{\mathcal{E}})$ exist such that W_L is isometric but not unitary:

Example 7.3. *Let $\mathcal{E}$ be a Hilbert space and let $\{h_p\}_{p\in\mathbb{Z}_+}$ be an orthonormal basis for $\mathcal{E}$. Define $L \in \mathcal{B}(\mathcal{E}, \mathcal{F}_n^2 \otimes \mathcal{E})$ by*

$$L(h_p) = \Omega \otimes h_{p+1} \qquad (p \in \mathbb{Z}_+).$$

Clearly, L is an isometry (in fact, it is a kind of shift operator). It is easy to see that $(W_L, S^{\mathcal{E}})$ is a Nica covariant representation. However, W_L is not a unitary.

7.2. **Fock represenations.** It makes sense to wonder if there are examples of nontrivial sequences that meet the conditions of Corollary 3.2. In order to address this, we first consider a sequence of scalars $\{c_p : p \in \mathbb{Z}_+\}$ that satisfies the conditions of Corollary 3.2. Assuming $\{c_p : p \in \mathbb{Z}_+\}$ is a finite sequence of nonzero numbers, by an obvious deduction, there exists $m \in \mathbb{Z}_+$ such that

$$c_p = 0 \qquad (p \neq m).$$

Therefore, we need to focus on nontrivial sequences. With this view in mind, now we illustrate Corollary 3.2 with a concrete example.

Example 7.4. *Consider the quadratic equation*

$$x^2 - x - 1 = 0.$$

Clearly

$$\omega = \frac{1-\sqrt{5}}{2},$$

is a solution to the above equation. Construct a sequence $\{c_p\}_{p\in\mathbb{Z}_+}$ by defining

$$c_p = \begin{cases} \sqrt{\frac{2}{\sqrt{5}+3}} & \textit{if } p = 0 \\ \sqrt{\frac{2}{\sqrt{5}+3}}\,\omega^{p-1} & \textit{if } p \geq 1. \end{cases}$$

First, we compute

$$\begin{aligned}
\sum_{p=0}^{\infty} |c_p|^2 &= \frac{2}{\sqrt{5}+3}\Big(1 + \sum_{p=1}^{\infty} \omega^{2(p-1)}\Big) \\
&= \frac{2}{\sqrt{5}+3}\Big(1 + \frac{1}{1-\omega^2}\Big) \\
&= \frac{2}{\sqrt{5}+3}\Big(\frac{2-\omega^2}{1-\omega^2}\Big) \\
&= \frac{2}{\sqrt{5}+3}\Big(\frac{1-\omega}{-\omega}\Big) \qquad (\textit{as } 1+\omega-\omega^2 = 0) \\
&= \frac{2}{\sqrt{5}+3}\Big(\frac{\sqrt{5}+1}{\sqrt{5}-1}\Big) \\
&= 1.
\end{aligned}$$

Moreover, for each $r \geq 1$, we have

$$\begin{aligned}\sum_{p=0}^{\infty} c_{p+r}\overline{c_p} &= \frac{2}{\sqrt{5}+3}\Big(\omega^{r-1} + \sum_{n=1}^{\infty} \omega^{n+r-1}\omega^{n-1}\Big)\\ &= \frac{2}{\sqrt{5}+3}\Big(\omega^{r-1} + \sum_{p=1}^{\infty} \omega^{2n+r-2}\Big)\\ &= \frac{2}{\sqrt{5}+3}\Big(\omega^{r-1} + \omega^{r}\sum_{p=1}^{\infty} \omega^{2n-2}\Big)\\ &= \frac{2}{\sqrt{5}+3}\Big(\omega^{r-1} + \frac{\omega^{r}}{1-\omega^2}\Big)\\ &= \frac{2}{\sqrt{5}+3}\omega^{r-1}\Big(\frac{1-\omega^2+\omega}{1-\omega^2}\Big).\end{aligned}$$

Since $\omega^2 - \omega - 1 = 0$, we conclude that

$$\sum_{p=0}^{\infty} c_{p+r}\overline{c_p} = 0,$$

for all $r \geq 1$. This is condition (3.2). *If we define*

$$\xi = \sum_{p=0}^{\infty} c_p e_1^{\otimes p},$$

then W_ξ is an isometry, as the sequence $\{c_p\}_{p\in\mathbb{Z}_+}$ meets all of Corollary 3.2's specifications. Hence there are nontrivial examples of isometric Fock representations of O_n.

The preceding example indicates once more that the symbol of non-unitary Fock representations of O_n, even at the level of the scalar case, does not have to be a self-map, unlike how it appeared in Li's construction [12] of unitary Fock representations of O_n.

7.3. $C^*(W_{c\Omega})$**.** As stated in Theorem 4.3, Nica covariant representations of O_n are exactly determined by the circle group. In other words, the only Nica covariant representations of O_n are (W_ξ, S), where

$$\xi = c\Omega \qquad (c \in \mathbb{T}).$$

Our aim is to investigate the C^*-algebra generated by $W_{c\Omega}$. It is important to mention that $W_{c\Omega}$ is unitary, according to Theorem 4.3. This has led us to be interested in finding out the operator $W_{c\Omega}$'s spectrum. For each $m \in \mathbb{Z}_+$, define $\mathcal{H}_m = \text{span}\{e_\mu : |\mu| = m\}$. Then

$$\mathcal{F}_n^2 = \bigoplus_{m=0}^{\infty} \mathcal{H}_m.$$

Clearly, $\mathcal{B}_m$ is an orthonormal basis for the finite dimensional subspace $\mathcal{H}_m$, where $\mathcal{B}_m = \{e_\mu : |\mu| = m\}$. Now, we calculate the spectrum of the unitary operator $W_{c\Omega}$, $c \in \mathbb{T}$.

Proposition 7.5. $\sigma(W_{c\Omega}) = \mathbb{T}$ *for all $c \in \mathbb{T}$.*

Proof. Fix $c \in \mathbb{T}$. From the definition of odometer maps, it is clear that $\mathcal{H}_m$ reduces $W_{c\Omega}$, and hence $W_{c\Omega}|\mathcal{H}_m : \mathcal{H}_m \to \mathcal{H}_m$ is a unitary for all $m \in \mathbb{Z}_+$. Then the matrix representation of $W_{c\Omega}|_{\mathcal{H}_m}$ with respect to the ordered basis $\mathcal{B}_m$ is given by

$$[W_{c\Omega}|_{\mathcal{H}_m}]_{\mathcal{B}_m} = P_m(c),$$

where

$$P_m(c) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & c \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}_{n^m \times n^m}.$$

Observe that $P_m(c)$ is an $n^m \times n^m$ matrix satisfying the identity

$$(P_m(c))^{n^m} = cI.$$

Therefore, the spectrum of the matrix $P_m(c)$ is given by

$$\sigma(P_m(c)) = \{\lambda \in \mathbb{C} : \lambda^{n^m} - c = 0\}.$$

Moreover, $W_{c\Omega} : \mathcal{F}_n^2 \to \mathcal{F}_n^2$ is unitarily equivalent to the direct sum operator

$$\bigoplus_{m=0}^{\infty} W_{c\Omega}|_{\mathcal{H}_m} : \bigoplus_{m=0}^{\infty} \mathcal{H}_m \to \bigoplus_{m=0}^{\infty} \mathcal{H}_m.$$

Therefore

$$\sigma\Big(\bigoplus_{m=0}^{\infty} W_{c\Omega}|_{\mathcal{H}_m}\Big) = \overline{\bigcup_{m=0}^{\infty} \sigma(W_{c\Omega}|_{\mathcal{H}_m})} = \overline{\bigcup_{m=0}^{\infty} \{\lambda \in \mathbb{C} : \lambda^{n^m} - c = 0\}} = \mathbb{T}.$$

Since $W_{c\Omega}$ is unitarily equivalent to $\bigoplus_{m=0}^{\infty} W_{c\Omega}|_{\mathcal{H}_m}$, it follows that $\sigma(W_{c\Omega}) = \mathbb{T}$. ☐

Notice that if $c = 1$, then the $n^m \times n^m$ matrix in the proof of the above theorem given by

$$P_m(1) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}_{n^m \times n^m},$$

is the permutation matrix of order n^m and $P^{n^m} = 1$.

Denote by $C^*(W_{c\Omega})$ the C^*-algebra generated by $W_{c\Omega}$. That is, $C^*(W_{c\Omega})$ is the smallest C^*-algebra containing $W_{c\Omega}$. Since $W_{c\Omega}$ is unitary and $\sigma(W_{c\Omega}) = \mathbb{T}$, the following is now immediate:

Theorem 7.6. *$C^*(W_{c\Omega}) \cong C(\mathbb{T})$ for all $c \in \mathbb{T}$.*

In closing, we again reiterate Remark 2.4 that the concept of odometer maps relies on symbols, which can vary all over the Fock space. This is as opposed to the noncommutative Toeplitz operators, which are by themselves a fascinating subject of research. This calls for an in-depth investigation of odometer maps acting on vector-valued Fock spaces. Moreover, with regard to the structure of C^*-algebras generated by Fock representations of O_n, we expect that the results of this paper—more precisely, the explicit description of the isometric Fock representations—will be helpful.

Acknowledgement: The research of the third named author is supported in part by TARE (TAR/2022/000063) by SERB, Department of Science & Technology (DST), Government of India.

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