AN INTRODUCTION TO HILBERT MODULE APPROACH TO MULTIVARIABLE OPERATOR THEORY

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Abstract. Let $\{T_1, \ldots, T_n\}$ be a set of n commuting bounded linear operators on a Hilbert space $\mathcal{H}$. Then the n-tuple $(T_1, \ldots, T_n)$ turns $\mathcal{H}$ into a module over $\mathbb{C}[z_1, \ldots, z_n]$ in the following sense:

$$\mathbb{C}[z_1, \ldots, z_n] \times \mathcal{H} \to \mathcal{H}, \qquad (p, h) \mapsto p(T_1, \ldots, T_n)h,$$

where $p \in \mathbb{C}[z_1, \ldots, z_n]$ and $h \in \mathcal{H}$. The above module is usually called the Hilbert module over $\mathbb{C}[z_1, \ldots, z_n]$. Hilbert modules over $\mathbb{C}[z_1, \ldots, z_n]$ (or natural function algebras) were first introduced by R. G. Douglas and C. Foias in 1976. The two main driving forces were the algebraic and complex geometric views to multivariable operator theory.

This article gives an introduction of Hilbert modules over function algebras and surveys some recent developments. Here the theory of Hilbert modules is presented as combination of commutative algebra, complex geometry and the geometry of Hilbert spaces and its applications to the theory of n-tuples ($n \geq 1$) of commuting operators. The topics which are studied include: model theory from Hilbert module point of view, Hilbert modules of holomorphic functions, module tensor products, localizations, dilations, submodules and quotient modules, free resolutions, curvature and Fredholm Hilbert modules. More developments in the study of Hilbert module approach to operator theory can be found in a companion paper, "Applications of Hilbert Module Approach to Multivariable Operator Theory".

Contents

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1. Introduction

One of the most important areas of investigation in operator theory is the study of n-tuples of commuting bounded linear operators on Hilbert spaces, or Hilbert modules over natural function algebras. A Hilbert module $\mathcal{H}$ over $\mathbb{C}[z_1, \dots, z_n]$ is the Hilbert space $\mathcal{H}$ equipped with n module maps, that is, with an n-tuple of commuting bounded linear operators on $\mathcal{H}$.

The origins of Hilbert modules, in fact, lie in classical linear operators on finite dimensional vector spaces. For instance, let T be a linear operator on an n-dimensional vector space $\mathcal{H}$. Then $\mathcal{H}$ is a module over $\mathbb{C}[z]$ in the following sense:

$$\mathbb{C}[z] \times \mathcal{H} \to \mathcal{H}, \qquad (p, h) \mapsto p(T)h, \qquad (p \in \mathbb{C}[z], h \in \mathcal{H})$$

where for $p = \sum_{k \geq 0} a_k z^k \in \mathbb{C}[z]$, $p(T)$ is the natural functional calculus given by $p(T) = \sum_{k \geq 0} a_k T^k$. Since $\mathbb{C}[z]$ is a principle ideal domain, the ideal $\{p \in \mathbb{C}[z] : p(T) = 0\}$ is generated by a non-zero polynomial. Such a polynomial is called a *minimal polynomial* for T. The existence of a minimal polynomial is a key step in the classification of linear operators in finite dimensional Hilbert spaces [Fuh12]. More precisely, the existence of the Jordan form follows from the structure theorem for finitely-generated modules over principle ideal domains.

The idea of viewing a commuting tuple of operators on a Hilbert space as Hilbert module over a natural function algebra goes back to Ronald G. Douglas in the middle of 1970. Perhaps, the main motivations behind the Douglas approach to Hilbert modules were the elucidating role of Brown-Douglas-Fillmore theory (1973), complex geometric interpretation of the Cowen-Douglas class (1978), Hormandar's algebraic approach, in the sense of Koszul complex, to corona problem (1967) and later, Taylor's notion of joint spectrum (1970), again in the sense of Koszul complex, in operator theory and function theory.

Historically, the first ideas leading to Hilbert modules can be traced back to the unpublished manuscript [DoFo76], in which Douglas and Foias proposed an algebraic approach to dilation theory. Then in [Do86] and [Do88], the notion of Hilbert modules became refined.

A systematic study of Hilbert modules only really started in 1989 with the work of Douglas, Paulsen and Yan [DoPaY89] and the monograph by Douglas and Paulsen [DoPa89].
Since then, this approach has become one of the essential tools of multivariable operator theory. This field now has profound connections to various areas of mathematics including commutative algebra, complex geometry and topology (see [DoY92], [DoPaSY95], [Ya92], [DoKKSa14], [DoKKSa12], [DoSa08], [DoSa11], [DoMiSa12]).

The purpose of this survey article is to present a (Hilbert) module approach to multivariable operator theory. The topics and results covered here are chosen to complement the existing monographs by Douglas and Paulsen [DoPa89] and Chen and Guo [XCGu03] and surveys by Douglas [Do14] and [Do09] though some overlap will be unavoidable. Many interesting results, open problems and references can be found in the monograph and the surveys mentioned above.

In view of time and space constraints, the present survey will not cover many interesting aspects of Hilbert module approach to multivariable operator theory, including the case of single operators. A few of these are: (1) The classification program for reducing subspaces of multiplication by Blaschke products on the Bergman space, by Zhu, Guo, Douglas, Sun, Wang, Putinar. (see [DoSuZ11], [DoPuW12], [GuH11]). (2) Extensions of Hilbert modules by Carlson, Clark, Foias, Guo, Didas and Eschmeier (see [DiEs06], [CaCl95], [CaCl97], [Gu99]). (3) K_0-group and similarity classification by Jiang, Wang, Ji, Guo (see [JJ07], [JWa06], [JGuJ05]. (4) Classification programme of homogeneous operators by Clark, Bagchi, Misra, Sastry and Koranyi (see [MiS90], [BaMi03], [KMi11] and [KMi09]). (4) Sheaf-theoretic techniques by Eschmeier, Albrecht, Putinar, Taylor and Vasilescu (see [EsP96], [Va82]).

Finally, although the main guiding principle of this development is the correspondence

$$\text{commuting}\, n\text{-tuples} \quad \longleftrightarrow \quad \text{Hilbert modules over } \mathbb{C}[z_1, \dots, z_n],$$

it is believed that the Hilbert module approach is a natural way to understand the subject of multivariable operator theory.

Outline of the paper: The paper has seven sections besides this introduction. Section 2 begins with a brief introduction of Hardy module which is a well-established procedure to pass from the function theory to the one variable operator theory. This section also includes basics of Hilbert modules over function algebras, localizations and dilations. The third section is centered around those aspects of operator theory that played an important role in the development of Hilbert modules. In particular, the third section introduce three basic notions which are directly formulated with the required structures, namely, algebraic, analytic and geometric. Section 4 is devoted to the study of contractive Hilbert modules over $\mathbb{C}[z]$. Section 5 describes the relationship of von Neumann-Wold decomposition with the structure of submodules of the Hardy (general function Hilbert) module(s). Section 6 introduces the notion of unitarily equivalence submodules of Hilbert modules of holomorphic functions. Section 7 sets the homological framework for Hilbert modules and Section 8 introduces the theory of Drury-Arveson module.

Notations and Conventions: (i) $\mathbb{N}$ = Set of all natural numbers including 0. (ii) $n \in \mathbb{N}$ and $n \geq 1$, unless specifically stated otherwise. (iii) $\mathbb{N}^n = \{\boldsymbol{k} = (k_1, \ldots, k_n) : k_i \in \mathbb{N}, i = 1, \ldots, n\}$. (iv) $\mathbb{C}^n$ = the complex n-space. (v) Ω : Bounded domain in $\mathbb{C}^n$. (vi) $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$. (vii) $z^{\boldsymbol{k}} = z_1^{k_1} \cdots z_n^{k_n}$. (viii) $\mathcal{H}, \mathcal{K}, \mathcal{E}, \mathcal{E}_*$: Hilbert spaces. (ix) $\mathcal{B}(\mathcal{H}, \mathcal{K})$ = the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. (x) $T = (T_1, \ldots, T_n)$, n-tuple of commuting operators. (xi) $T^{\boldsymbol{k}} = T_1^{k_1} \cdots T_n^{k_n}$. (xii) $\mathbb{C}[\boldsymbol{z}] = \mathbb{C}[z_1, \ldots, z_n]$. (xiii) $\mathbb{D}^n = \{\boldsymbol{z} : |z_i| < 1, i = 1, \ldots, n\}$, $\mathbb{B}^n = \{\boldsymbol{z} : \|\boldsymbol{z}\|_{\mathbb{C}^n} < 1\}$. (xiv) $H^2_{\mathcal{E}}(\mathbb{D})$: $\mathcal{E}$-valued Hardy space over $\mathbb{D}$.

Throughout this note all Hilbert spaces are over the complex field and separable. Also for a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, the orthogonal projection of $\mathcal{H}$ onto $\mathcal{S}$ will be denoted by $P_{\mathcal{S}}$.

2. Hilbert modules

The purpose of this section is to give some of the essential background for Hilbert modules. The first subsection is devoted to set up the notion of Hilbert modules over the polynomial algebra. The third subsection deals with Hilbert modules over function algebras. Basic concepts and classical definitions are summarized in the subsequent subsections.

Before proceeding to the detailed development, it is more convenient to introduce a brief overview of the Hardy space over the unit disc $\mathbb{D}$. Results based on the Hardy space and the multiplication operator on the Hardy space play an important role in both operator theory and function theory. More precisely, for many aspects of geometric and analytic intuition, the Hardy space techniques play a fundamental role in formulating problems in operator theory and function theory both in one and several variables.

The *Hardy space* $H^2(\mathbb{D})$ over $\mathbb{D}$ is the set of all power series

$$f = \sum_{m=0}^{\infty} a_m z^m, \qquad (a_m \in \mathbb{C})$$

such that

$$\|f\|_{H^2(\mathbb{D})} := (\sum_{m=0}^{\infty} |a_m|^2)^{\frac{1}{2}} < \infty.$$

Let $f = \sum_{m=0}^{\infty} a_m z^m \in H^2(\mathbb{D})$. It is obvious that $\sum_{m=0}^{\infty} |w|^m < \infty$ for each $w \in \mathbb{D}$. This and $\sum_{m=0}^{\infty} |a_m|^2 < \infty$ readily implies that

$$\sum_{m=0}^{\infty} a_m w^m$$

converges absolutely for each $w \in \mathbb{D}$. In other words, $f = \sum_{m=0}^{\infty} a_m z^m$ is in $H^2(\mathbb{D})$ if and only if f is a square summable holomorphic function on $\mathbb{D}$.
Now, for each $w \in \mathbb{D}$ one can define a complex-valued function $\mathbb{S}(\cdot, w) : \mathbb{D} \to \mathbb{C}$ by

$$(\mathbb{S}(\cdot, w))(z) = \sum_{m=0}^{\infty} \bar{w}^m z^m. \qquad (z \in \mathbb{D})$$

Since

$$\sum_{m=0}^{\infty} |\bar{w}^m|^2 = \sum_{m=0}^{\infty} (|w|^2)^m = \frac{1}{1 - |w|^2},$$

it follows that

$$\mathbb{S}(\cdot, w) \in H^2(\mathbb{D}), \qquad (w \in \mathbb{D})$$

and

$$\|\mathbb{S}(\cdot, w)\|_{H^2(\mathbb{D})} = \frac{1}{(1 - |w|^2)^{\frac{1}{2}}}. \qquad (w \in \mathbb{D})$$

Moreover, if $f = \sum_{m=0}^{\infty} a_m z^m \in H^2(\mathbb{D})$ and $w \in \mathbb{D}$, then

$$f(w) = \sum_{m=0}^{\infty} a_m w^m = \langle \sum_{m=0}^{\infty} a_m z^m, \sum_{m=0}^{\infty} \bar{w}^m z^m \rangle_{H^2(\mathbb{D})} = \langle f, \mathbb{S}(\cdot, w) \rangle_{H^2(\mathbb{D})}.$$

Therefore, the vector $\mathbb{S}(\cdot, w) \in H^2(\mathbb{D})$ *reproduces* (cf. Subsection 3.1) the value of $f \in H^2(\mathbb{D})$ at $w \in \mathbb{D}$. In particular,

$$(\mathbb{S}(\cdot, w))(z) = \langle \mathbb{S}(\cdot, w), \mathbb{S}(\cdot, z) \rangle_{H^2(\mathbb{D})} = \sum_{m=0}^{\infty} z^m \bar{w}^m = (1 - z\bar{w})^{-1}. \qquad (z, w \in \mathbb{D})$$

The function $\mathbb{S} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ defined by

$$\mathbb{S}(z, w) = (1 - z\bar{w})^{-1}, \qquad (z, w \in \mathbb{D})$$

is called the *Szegő* or *Cauchy-Szegő* kernel of $\mathbb{D}$. Consequently, $H^2(\mathbb{D})$ is a *reproducing kernel Hilbert space* with kernel function $\mathbb{S}$ (see Subsection 3.1).

The next goal is to show that the set $\{\mathbb{S}(\cdot, w) : w \in \mathbb{D}\}$ is *total* in $H^2(\mathbb{D})$, that is,

$$\overline{\text{span}}\{\mathbb{S}(\cdot, w) : w \in \mathbb{D}\} = H^2(\mathbb{D}).$$

To see this notice that the reproducing property of the Szegő kernel yields $f(w) = \langle f, \mathbb{S}(\cdot, w) \rangle_{H^2(\mathbb{D})}$ for all $f \in H^2(\mathbb{D})$ and $w \in \mathbb{D}$. Now the result follows from the fact that

$$f \perp \mathbb{S}(\cdot, w),$$

for $f \in H^2(\mathbb{D})$ and for all $w \in \mathbb{D}$ if and only if

$$f = 0.$$

It also follows that for each $w \in \mathbb{D}$, the *evaluation map* $ev_w : H^2(\mathbb{D}) \to \mathbb{C}$ defined by

$$ev_w(f) = f(w), \qquad (f \in H^2(\mathbb{D}))$$

is continuous.

The next task is to recall some of the most elementary properties of the multiplication operator on $H^2(\mathbb{D})$. Observe first that

$$\langle z(z^k), z(z^l)\rangle_{H^2(\mathbb{D})} = \langle z^{k+1}, z^{l+1}\rangle_{H^2(\mathbb{D})} = \delta_{k,l} = \langle z^k, z^l\rangle_{H^2(\mathbb{D})}. \qquad (k,l \in \mathbb{N})$$

Using the fact that the set $\{z^m : m \in \mathbb{N}\}$ is total in $H^2(\mathbb{D})$, the previous equality implies that the multiplication operator M_z on $H^2(\mathbb{D})$ defined by

$$(M_z f)(w) = wf(w), \qquad (f \in H^2(\mathbb{D}), w \in \mathbb{D})$$

is an isometric operator, that is,

$$M_z^* M_z = I_{H^2(\mathbb{D})}.$$

Moreover,

$$\langle M_z^* z^k, z^l\rangle_{H^2(\mathbb{D})} = \langle z^k, z^{l+1}\rangle_{H^2(\mathbb{D})} = \delta_{k,l+1} = \delta_{k-1,l} = \langle z^{k-1}, z^l\rangle_{H^2(\mathbb{D})},$$

for all $k \geq 1$ and $l \in \mathbb{N}$. Also it follows that $\langle M_z^* 1, z^l\rangle_{H^2(\mathbb{D})} = 0$. Consequently,

$$M_z^* z^k = \begin{cases} z^{k-1} & \text{if } k \geq 1; \\ 0 & \text{if } k = 0. \end{cases}$$

It also follows that

$$\begin{aligned}\langle (I_{H^2(\mathbb{D})} - M_z M_z^*)\mathbb{S}(\cdot, w), \mathbb{S}(\cdot, z)\rangle_{H^2(\mathbb{D})} &= \langle \mathbb{S}(\cdot, w), \mathbb{S}(\cdot, z)\rangle_{H^2(\mathbb{D})} - \langle M_z^*\mathbb{S}(\cdot, w), M_z^*\mathbb{S}(\cdot, z)\rangle_{H^2(\mathbb{D})} \\ &= \mathbb{S}(z, w) - z\bar{w}\mathbb{S}(z, w) = 1 \\ &= \langle P_{\mathbb{C}}\mathbb{S}(\cdot, w), \mathbb{S}(\cdot, z)\rangle_{H^2(\mathbb{D})},\end{aligned}$$

where $P_{\mathbb{C}}$ is the orthogonal projection of $H^2(\mathbb{D})$ onto the one-dimensional subspace of all constant functions on $\mathbb{D}$. Therefore,

$$I_{H^2(\mathbb{D})} - M_z M_z^* = P_{\mathbb{C}}.$$

To compute the kernel, $\ker(M_z - wI_{H^2(\mathbb{D})})^*$ for $w \in \mathbb{D}$, note that

$$\begin{aligned}M_z^*\mathbb{S}(\cdot, w) &= M_z^*(1 + \bar{w}z + \bar{w}^2 z^2 + \cdots) = \bar{w} + \bar{w}^2 z + \bar{w}^3 z^2 + \cdots = \bar{w}(1 + \bar{w}z + \bar{w}^2 z^2 + \cdots) \\ &= \bar{w}\mathbb{S}(\cdot, w).\end{aligned}$$

On the other hand, if $M_z^* f = \bar{w} f$ for some $f \in H^2(\mathbb{D})$ then

$$f(0) = P_{\mathbb{C}} f = (I_{H^2(\mathbb{D})} - M_z M_z^*) f = (1 - z\bar{w}) f,$$

that is, $f = f(0)\mathbb{S}(\cdot, w)$. Consequently, $M_z^* f = \bar{w} f$ if and only if $f = \lambda\mathbb{S}(\cdot, w)$ for some $\lambda \in \mathbb{C}$. That is,

$$\ker(M_z - wI_{H^2(\mathbb{D})})^* = \{\lambda\mathbb{S}(\cdot, w) : \lambda \in \mathbb{C}\}.$$

In particular,

$$\bigvee_{w \in \mathbb{D}} \ker(M_z - wI_{H^2(\mathbb{D})})^* = H^2(\mathbb{D}).$$

The following theorem summarizes the above observations.

THEOREM 2.1. *Let $H^2(\mathbb{D})$ denote the Hardy space over $\mathbb{D}$ and M_z denote the multiplication operator by the coordinate function z on $H^2(\mathbb{D})$. Then, the following properties hold:*

(i) The set $\{\mathbb{S}(\cdot, w) : w \in \mathbb{D}\}$ is total in $H^2(\mathbb{D})$.

(ii) The evaluation map $ev_w : H^2(\mathbb{D}) \to \mathbb{C}$ defined by $ev_w(f) = f(w)$ is continuous for each $w \in \mathbb{D}$.

(iii) $\sigma_p(M_z^*) = \mathbb{D}$ and $ker(M_z - wI_{H^2(\mathbb{D})})^* = \{\lambda \mathbb{S}(\cdot, w) : \lambda \in \mathbb{C}\}$.

(iv) $f(w) = \langle f, \mathbb{S}(\cdot, w)\rangle_{H^2(\mathbb{D})}$ for all $f \in H^2(\mathbb{D})$ and $w \in \mathbb{D}$.

(v)$I_{H^2(\mathbb{D})} - M_z M_z^* = P_{\mathbb{C}}$.

(vi) $\bigvee_{w\in\mathbb{D}} ker(M_z - wI_{H^2(\mathbb{D})})^* = H^2(\mathbb{D})$.

Let $\mathcal{E}$ be a Hilbert space. In what follows, $H^2_{\mathcal{E}}(\mathbb{D})$ stands for the Hardy space of $\mathcal{E}$-valued analytic functions on $\mathbb{D}$. Moreover, by virtue of the unitary $U : H^2_{\mathcal{E}}(\mathbb{D}) \to H^2(\mathbb{D}) \otimes \mathcal{E}$ defined by

$$z^m\eta \mapsto z^m \otimes \eta, \quad (\eta \in \mathcal{E}, m \in \mathbb{N})$$

the vector valued Hardy space $H^2_{\mathcal{E}}(\mathbb{D})$ will be identified with the Hilbert space tensor product $H^2(\mathbb{D}) \otimes \mathcal{E}$.

For a more extensive treatment of the Hardy space and related topics, the reader is referred to the books by Sz.-Nagy and Foias [NaFo70], Radjavi and Rosenthal [RaRo73], Rosenblum and Rovnyak [RRov97] and Halmos [Ha82].

2.1. **Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$.** Let $\{T_1, \ldots, T_n\}$ be a set of n commuting bounded linear operators on a Hilbert space $\mathcal{H}$. Then the n-tuple $(T_1, \ldots, T_n)$ turns $\mathcal{H}$ into a module over $\mathbb{C}[\boldsymbol{z}]$ in the following sense:

$$\mathbb{C}[\boldsymbol{z}] \times \mathcal{H} \to \mathcal{H}, \qquad (p, h) \mapsto p(T_1, \ldots, T_n)h,$$

where $p \in \mathbb{C}[\boldsymbol{z}]$ and $h \in \mathcal{H}$. The above module is usually called the *Hilbert module* over $\mathbb{C}[\boldsymbol{z}]$. Denote by $M_p : \mathcal{H} \to \mathcal{H}$ the bounded linear operator

$$M_p h = p \cdot h = p(T_1, \ldots, T_n)h, \qquad (h \in \mathcal{H})$$

for $p \in \mathbb{C}[\boldsymbol{z}]$. In particular, for $p = z_i \in \mathbb{C}[\boldsymbol{z}]$, this gives the *module multiplication* operators $\{M_j\}_{j=1}^n$ by the coordinate functions $\{z_j\}_{j=1}^n$ defined by

$$M_i h = z_i(T_1, \ldots, T_n)h = T_i h. \qquad (h \in \mathcal{H},\ 1 \leq i \leq n)$$

Here and in what follows, the notion of a Hilbert module $\mathcal{H}$ over $\mathbb{C}[\boldsymbol{z}]$ will be used in place of an n-tuple of commuting operators $\{T_1, \ldots, T_n\} \subseteq \mathcal{B}(\mathcal{H})$, where the operators are determined by module multiplication by the coordinate functions, and vice versa.

When necessary, the notation $\{M_{\mathcal{H},i}\}_{i=1}^n$ will be used to indicate the underlying Hilbert space $\mathcal{H}$ with respect to which the module maps are defined.

Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Then $\mathcal{S}$ is a *submodule* of $\mathcal{H}$ if $M_i\mathcal{S} \subseteq \mathcal{S}$ for all $i = 1, \ldots, n$. A closed subspace $\mathcal{Q}$ of $\mathcal{H}$ is said to be *quotient module* of $\mathcal{H}$ if $\mathcal{Q}^\perp \cong \mathcal{H}/\mathcal{Q}$ is a submodule of $\mathcal{H}$. Therefore, a closed subspace $\mathcal{Q}$ is a quotient module of $\mathcal{H}$ if and only if $M_i^*\mathcal{Q} \subseteq \mathcal{Q}$ for all $i = 1, \ldots, n$. In particular, if the module multiplication operators on a Hilbert module $\mathcal{H}$ are given by the commuting tuple of operators $(T_1, \ldots, T_n)$ then $\mathcal{S}$ is a submodule of $\mathcal{H}$ if and only if $\mathcal{S}$ is joint $(T_1, \ldots, T_n)$-invariant subspace of $\mathcal{H}$ and $\mathcal{Q}$ is a quotient module of $\mathcal{H}$ if and only if $\mathcal{Q}$ is joint $(T_1^*, \ldots, T_n^*)$-invariant subspace of $\mathcal{H}$.

Let $\mathcal{S}$ be a submodule and $\mathcal{Q}$ be a quotient module of a Hilbert module $\mathcal{H}$ over $\mathbb{C}[\boldsymbol{z}]$. Then $\mathcal{S}$ and $\mathcal{Q}$ are also Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$ where the module multiplication by the coordinate functions on $\mathcal{S}$ and $\mathcal{Q}$ are given by the restrictions $(R_1, \ldots, R_n)$ and the compressions $(C_1, \ldots, C_n)$ of the module multiplication operators on $\mathcal{H}$, respectively. That is,

$$R_i = M_i|_{\mathcal{S}} \quad \text{and} \quad C_i = P_{\mathcal{Q}} M_i|_{\mathcal{Q}}. \qquad (1 \leq i \leq n)$$

Evidently,

$$R_i^* = P_{\mathcal{S}} M_i^*|_{\mathcal{S}} \qquad \text{and} \qquad C_i^* = M_i^*|_{\mathcal{Q}}. \qquad (1 \leq i \leq n)$$

A bounded linear map $X : \mathcal{H} \to \mathcal{K}$ between two Hilbert modules $\mathcal{H}$ and $\mathcal{K}$ over $\mathbb{C}[\boldsymbol{z}]$ is said to be a *module map* if $XM_i = M_i X$ for $i = 1, \ldots, n$, or equivalently, if $XM_p = M_p X$ for $p \in \mathbb{C}[\boldsymbol{z}]$. A pair of Hilbert modules will be considered the 'same', that is, *isomorphic* provided there is a unitary module map between them, and *similar* if there is an invertible module map between them.

2.2. **Dilations.** The purpose of this subsection is to present a modified version of dilation theory for commuting tuples of operators.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$. Then

(1) A map $\Pi \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is called *co-module* map if $\Pi^* : \mathcal{K} \to \mathcal{H}$ is a module map, that is, $\Pi M_i^* = M_i^* \Pi$.

(2) $\mathcal{K}$ is said to be *dilation* of $\mathcal{H}$ if there exists a co-module isometry $\Pi : \mathcal{H} \to \mathcal{K}$. In this case, we also say that $\Pi \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is a dilation of $\mathcal{H}$.

(3) A dilation $\Pi \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ of $\mathcal{H}$ is *minimal* if $\mathcal{K} = \overline{\text{span}}\{M^{\boldsymbol{k}}(\Pi\mathcal{H}) : \boldsymbol{k} \in \mathbb{N}^n\}$.

Let $\Pi \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be a dilation of $\mathcal{H}$. Then $\Pi(\mathcal{H})$ is a quotient module of $\mathcal{K}$, that is, $\Pi(\mathcal{H})$ is a joint $(M_1^*, \ldots, M_n^*)$-invariant subspace of $\mathcal{K}$, and

$$M^{\boldsymbol{k}} = P_{\Pi(\mathcal{H})} M^{\boldsymbol{k}}|_{\Pi(\mathcal{H})},$$

for all $\boldsymbol{k} \in \mathbb{N}^n$. Moreover, one has the following short exact sequence of Hilbert modules

$$0 \longrightarrow \mathcal{S} \xrightarrow{i} \mathcal{K} \xrightarrow{\pi} \mathcal{H} \longrightarrow 0, \tag{2.1}$$

where $\mathcal{S} = (\Pi\mathcal{H})^{\perp} (\cong \mathcal{K}/\Pi\mathcal{H})$, a submodule of $\mathcal{K}$, i is the inclusion and $\pi := \Pi^*$ is the quotient map. In other words, if $\mathcal{K}$ is a dilation of $\mathcal{H}$ then there exists a quotient module $\mathcal{Q}$ and a submodule $\mathcal{S}$ of $\mathcal{K}$ such that $\mathcal{K} = \mathcal{S} \oplus \mathcal{Q}$, that is,

$$0 \longrightarrow \mathcal{S} \xrightarrow{i} \mathcal{K} \xrightarrow{\pi} \mathcal{Q} \longrightarrow 0,$$

and $\mathcal{Q} \cong \mathcal{H}$.
Conversely, let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$ and $\mathcal{H} \cong \mathcal{Q}$, a quotient module of $\mathcal{K}$. Therefore, $\mathcal{K}$ is a dilation of $\mathcal{H}$ and by defining $\mathcal{S} := \mathcal{Q}^{\perp}$, a submodule of $\mathcal{K}$, one arrives at the short exact sequence (2.1).

A Hilbert module $\mathcal{H}$ over $\mathbb{C}[z]$ is said to be *contractive Hilbert module over* $\mathbb{C}[z]$ if $I_{\mathcal{H}} - M^*M \geq 0$.

The famous isometric dilation theorem of Sz. Nagy (cf. [NaFo70]) states that:

THEOREM 2.2. *Every contractive Hilbert module $\mathcal{H}$ over $\mathbb{C}[z]$ has a minimal isometric dilation.*

Proof. Let $\mathcal{H}$ be a contractive Hilbert module over $\mathbb{C}[z]$. Let $D_{\mathcal{H}} = (I_{\mathcal{H}} - M^*M)^{\frac{1}{2}}$ and $\mathcal{N}_{\mathcal{H}} := \mathcal{H} \oplus H^2_{\mathcal{H}}(\mathbb{D})$. Define $N_{\mathcal{H}} \in \mathcal{B}(\mathcal{N}_{\mathcal{H}})$ by

$$N_{\mathcal{H}} := \begin{bmatrix} M & 0 \\ \boldsymbol{D}_{\mathcal{H}} & M_z \end{bmatrix},$$

where $\boldsymbol{D}_{\mathcal{H}} : \mathcal{H} \to H^2_{\mathcal{H}}(\mathbb{D})$ is the constant function defined by $(\boldsymbol{D}_{\mathcal{H}}h)(z) = D_{\mathcal{H}}h$ for all $h \in \mathcal{H}$ and $z \in \mathbb{D}$. Consequently,

$$N_{\mathcal{H}}^* N_{\mathcal{H}} = \begin{bmatrix} M^*M + D_{\mathcal{H}}^2 & 0 \\ 0 & I_{H^2_{\mathcal{H}}(\mathbb{D})} \end{bmatrix} = \begin{bmatrix} I_{\mathcal{H}} & 0 \\ 0 & I_{H^2_{\mathcal{H}}(\mathbb{D})} \end{bmatrix},$$

that is, $N_{\mathcal{H}}$ is an isometry. Moreover, one can check immediately that

$$N_{\mathcal{H}}^k := \begin{bmatrix} M^k & 0 \\ * & M_z^k \end{bmatrix}, \qquad (k \in \mathbb{N})$$

which along with the isometric embedding $\Pi_N \in \mathcal{B}(\mathcal{H}, \mathcal{N}_{\mathcal{H}})$ defined by $\Pi_N h = h \oplus 0$, for all $h \in \mathcal{H}$, implies that Π_N is an isometric dilation of $\mathcal{H}$ with the isometric map $N_{\mathcal{H}}$. Then $\tilde{\Pi}_N := P_{\tilde{\mathcal{N}}_{\mathcal{H}}} \Pi_N \in \mathcal{B}(\mathcal{H}, \tilde{\mathcal{N}}_{\mathcal{H}})$ is the minimal isometric dilation of $\mathcal{H}$, where $\tilde{\mathcal{N}}_{\mathcal{H}} = \overline{\text{span}}\{N_{\mathcal{H}}^k \mathcal{H} : k \in \mathbb{N}\}$ and $\tilde{N}_{\mathcal{H}} = N_{\mathcal{H}}|_{\tilde{\mathcal{N}}_{\mathcal{H}}}$. ■

Sz.-Nagy's minimal isometric dilation is unique in the following sense: if $\Pi \in \mathcal{B}(\mathcal{H}, \mathcal{M})$ is a minimal isometric dilation of $\mathcal{H}$ with isometry V, then there exists a (unique) unitary $\Phi : \tilde{\mathcal{N}}_{\mathcal{H}} \to \mathcal{M}$ such that $V\Phi = \Phi \tilde{N}_{\mathcal{H}}$.

It is also worth mentioning that the Schäffer isometric dilation of $\mathcal{H}$ is always minimal. The *Schäffer dilation space* is defined by $\mathcal{S}_{\mathcal{H}} := \mathcal{H} \oplus H^2_{\mathcal{D}_{\mathcal{H}}}(\mathbb{D})$ with

$$S_{\mathcal{H}} := \begin{bmatrix} M & 0 \\ \boldsymbol{D}_{\mathcal{H}} & M_z \end{bmatrix},$$

where $\mathcal{D}_{\mathcal{H}} = \overline{\text{ran}} D_{\mathcal{H}}$ (see [NaFo70]).

The von Neumann inequality [Jv51] follows from the isometric, and hence unitary (cf. [NaFo70]), dilation theorem for contractive Hilbert modules.

THEOREM 2.3. *Let $\mathcal{H}$ be a Hilbert module over $\mathbb{C}[z]$. Then $\mathcal{H}$ is contractive if and only if*

$$\|p(M)\| \leq \|p\|_\infty = \max_{|z| \leq 1} |p(z)|. \qquad (p \in \mathbb{C}[z])$$

As a consequence, the polynomial functional calculus of a contractive Hilbert module over $\mathbb{C}[z]$ extends to the disc algebra $A(\mathbb{D})$, where

$$A(\mathbb{D}) = \mathcal{O}(\mathbb{D}) \cap \mathbb{C}(\overline{\mathbb{D}}) = \overline{\mathbb{C}[z]}^{\|\cdot\|_\infty}.$$

This implies that

$$\|f(M)\| \leq \|f\|_\infty,$$

for all $f \in A(\mathbb{D})$.

2.3. **Hilbert modules over $A(\Omega)$.** Let Ω be a domain in $\mathbb{C}^n$ and $A(\Omega)$ be the unital Banach algebra obtained from the closure in the supremum norm on Ω of all functions holomorphic in some neighborhood of the closure of Ω. The most classical and familiar examples of $A(\Omega)$ are the ball algebra $A(\mathbb{B}^n)$ and the polydisc algebra $A(\mathbb{D}^n)$.

Now let $\mathcal{H}$ be a Hilbert space and π a norm continuous unital algebra homomorphism from the Banach algebra $A(\Omega)$ to the C^*-algebra $\mathcal{B}(\mathcal{H})$. Then the Hilbert space $\mathcal{H}$ is said to be a *Hilbert module over* $A(\Omega)$ if it is an $A(\Omega)$-module in the sense of algebra,

$$A(\Omega) \times \mathcal{H} \to \mathcal{H}, \qquad (\varphi, f) \mapsto \varphi \cdot f = \pi(\varphi)h,$$

with the additional property that the module multiplication $A(\Omega) \times \mathcal{H} \to \mathcal{H}$ is norm continuous. We say that a Hilbert module $\mathcal{H}$ over $A(\Omega)$ is contractive if π is a contraction, that is,

$$\|\varphi \cdot f\|_{\mathcal{H}} \leq \|\varphi\|_{A(\Omega)} \|f\|_{\mathcal{H}}. \qquad (\varphi \in A(\Omega),\ f \in \mathcal{H})$$

The following are some important and instructive examples of contractive Hilbert modules.
(1) the *Hardy module* $H^2(\mathbb{D}^n)$ ([FeSt72], [Ru69]), the closure of $\mathbb{C}[\boldsymbol{z}]$ in $L^2(\mathbb{T}^n)$, over $A(\mathbb{D}^n)$,
(2) the *Hardy module* $H^2(\mathbb{B}^n)$ [Ru80], the closure of $\mathbb{C}[\boldsymbol{z}]$ in $L^2(\partial\mathbb{B}^n)$, over $A(\mathbb{B}^n)$ and
(3) the *Bergman module over the ball* $L^2_a(\mathbb{B}^n)$, the closure of $A(\mathbb{B}^n)$ in $L^2(\mathbb{B}^n)$, over $A(\mathbb{B}^n)$.
(4) Quotient modules and submodules of (1), (2) and (3) over the corresponding algebras.

2.4. **Module tensor products and localizations.** Module tensor product and localizations are in the center of commutative algebra and algebraic geometry. The notion of module tensor products and localizations for Hilbert modules introduced by Douglas and Paulsen [DoPa89] was one of the inspiration points for Hilbert module method to operator theory.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert modules over $A(\Omega)$ and $\mathcal{H}_1 \otimes \mathcal{H}_2$ be the Hilbert space tensor product. Then $\mathcal{H}_1 \otimes \mathcal{H}_2$ turns into both a left and a right $A(\Omega)$ modules, $A(\Omega) \times \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$, by setting

$$(\varphi, h_1 \otimes h_2) \mapsto (\varphi \cdot h_1) \otimes h_2, \quad \text{and} \quad (\varphi, h_1 \otimes h_2) \mapsto h_1 \otimes (\varphi \cdot h_2),$$

respectively. Note that

$$\mathcal{N} = \overline{\text{span}}\{(\varphi \cdot h_1) \otimes h_2 - h_1 \otimes (\varphi \cdot h_2) : h_1 \in \mathcal{H}_1, h_2 \in \mathcal{H}_2, \varphi \in A(\Omega)\}$$

is both a left and a right $A(\Omega)$-submodule of $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then $\mathcal{N}^\perp (\cong (\mathcal{H}_1 \otimes \mathcal{H}_2)/\mathcal{N})$ is both a left and a right $A(\Omega)$-quotient module and

$$P_{\mathcal{N}^\perp}((\varphi \cdot h_1) \otimes h_2) = P_{\mathcal{N}^\perp}(h_1 \otimes (\varphi \cdot h_2)),$$

for all $h_1 \in \mathcal{H}_1, h_2 \in \mathcal{H}_2$ and $\varphi \in A(\Omega)$. In conclusion, these quotient modules are isomorphic Hilbert modules over $A(\Omega)$, which will denote by $\mathcal{H}_1 \otimes_{A(\Omega)} \mathcal{H}_2$ and referred as the *module tensor product* of the Hilbert modules $\mathcal{H}_1$ and $\mathcal{H}_2$ over $A(\Omega)$.

For each $\boldsymbol{w} \in \Omega$ denote by $\mathbb{C}_{\boldsymbol{w}}$ the one dimensional Hilbert module over $A(\Omega)$:

$$A(\Omega) \times \mathbb{C}_{\boldsymbol{w}} \to \mathbb{C}_{\boldsymbol{w}}, \qquad (\varphi, \lambda) \mapsto \varphi(\boldsymbol{w})\lambda.$$

Further, for each $\boldsymbol{w} \in \Omega$ denote by $A(\Omega)_{\boldsymbol{w}}$ the set of functions in $A(\Omega)$ vanishing at $\boldsymbol{w}$, that is,

$$A(\Omega)_{\boldsymbol{w}} = \{\varphi \in A(\Omega) : \varphi(\boldsymbol{w}) = 0\}.$$

Let $\mathcal{H}$ be a Hilbert module over $A(\Omega)$ and $\boldsymbol{w} \in \Omega$. Then the module tensor product $\mathcal{H} \otimes_{A(\Omega)} \mathbb{C}_{\boldsymbol{w}}$ is called the *localization* of the Hilbert module $\mathcal{H}$ at $\boldsymbol{w}$.

It is easy to see that

$$\mathcal{H}_{\boldsymbol{w}} := \overline{\text{span}}\{\varphi f : \varphi \in A(\Omega)_{\boldsymbol{w}}, f \in \mathcal{H}\}.$$

is a submodule of $\mathcal{H}$ for each $\boldsymbol{w} \in \Omega$. Moreover, the quotient module $\mathcal{H}/\mathcal{H}_{\boldsymbol{w}}$ is canonically isomorphic to $\mathcal{H} \otimes_{A(\Omega)} \mathbb{C}_{\boldsymbol{w}}$, the localization of $\mathcal{H}$ at $\boldsymbol{w} \in \Omega$, in the following sense

$$\mathcal{H} \otimes_{A(\Omega)} \mathbb{C}_{\boldsymbol{w}} \to \mathcal{H}/\mathcal{H}_{\boldsymbol{w}}, \qquad f \otimes_{A(\Omega)} 1 \mapsto P_{\mathcal{H}/\mathcal{H}_{\boldsymbol{w}}} f.$$

The following list of examples of localizations will be useful in a number of occasions later.

(1) For all $\boldsymbol{w} \in \mathbb{D}^n$,

$$H^2(\mathbb{D}^n) \otimes_{A(\mathbb{D}^n)} \mathbb{C}_{\boldsymbol{w}} \cong \mathbb{C}_{\boldsymbol{w}},$$

where $\cong$ stands for module isomorphism.

(2) Let $\mathcal{H} = H^2(\mathbb{B}^n)$ or $L^2_a(\mathbb{B}^n)$. Then for all $\boldsymbol{w} \in \mathbb{B}^n$,

$$\mathcal{H} \otimes_{A(\mathbb{B}^n)} \mathbb{C}_{\boldsymbol{w}} \cong \mathbb{C}_{\boldsymbol{w}}.$$

(3) Let $H^2(\mathbb{D}^2)_0 = \{f \in H^2(\mathbb{D}^2) : f(0) = 0\}$, the submodule of $H^2(\mathbb{D}^2)$ of functions vanishing at the origin. Then

$$H^2(\mathbb{D}^2)_0 \otimes_{A(\mathbb{D}^2)} \mathbb{C}_{\boldsymbol{w}} = \begin{cases} \mathbb{C}_{\boldsymbol{w}} & \text{if } \boldsymbol{w} \neq 0; \\ \mathbb{C}_0 \oplus \mathbb{C}_0 & \text{if } \boldsymbol{w} = 0. \end{cases}$$

Further results and comments:

(1) von Neumann inequality says that one can extend the functional calculus from $\mathbb{C}[z]$ to $A(\mathbb{D})$ for contractive Hilbert module over $\mathbb{C}[z]$. Another approach to extend the functional calculus is to consider the rational functions. More precisely, let K be a non-empty compact subset of $\mathbb{C}$ and $T \in \mathcal{B}(\mathcal{H})$. Denote $Rat(X)$ the set of rational functions with poles off K. Then K is a *spectral set* for T if $\sigma(T) \subseteq K$ and

$$\|f(T)\| \leq \|f\|_K := \sup \{|f(z)| : z \in K\}.$$

The notion of spectral set was introduced by J. von Neumann in [Jv51] where he proved that the closed unit disk is a spectral set of a bounded linear operator on a Hilbert space if and only if the operator is a contraction. Also recall that a bounded linear operator T on $\mathcal{H}$ has a normal ∂K-dilation if there exists a normal operator N on $\mathcal{K} \supseteq \mathcal{H}$ such that $\sigma(N) \subseteq \partial K$ and

$$P_{\mathcal{H}} f(N)|_{\mathcal{H}} = f(T). \qquad (f \in Rat(K))$$

The Sz. Nagy dilation theory shows that every contraction has a normal $\partial\overline{\mathbb{D}}$-dilation. It is known that the normal ∂K-dilation holds if K is the closure of an annulus [Ag85] and fails, in general, when K is a triply connected domain in $\mathbb{C}$ [DrMc05] (see also [AgHR08], [AgYo03] and [Sa14b]).

(2) Ando's theorem [An63] extends Sz.-Nagy's unitary dilation result to a pair of operators, that is, any pair of commuting contractions has a unitary dilation. However, the Ando dilation is not unique and it fails for three or more operators (see [Pa70] and [Va74]).

(3) The Ando dilation theorem is closely related to the commutant lifting theorem (see [Pa70]).
(4) A 2-variables analogue of von Neumanns inequality follows from Ando's dilation theorem [An63]. It is well known that for n-tuples of operators, $n \geq 3$, the von Neumann inequality fails in general. In [AVVW09], Anatolii, Kaliuzhnyi-Verbovetskyi, Vinnikov and Woerdeman proved a several variables analogue of von Neumanns inequality for a class of commuting n-tuples of strict contractions.
(5) The notion of module tensor product is due to Douglas (see [DoPa89]).
(6) The notion of localization of Hilbert modules, however, is by far not enough. The computation of higher order localizations is another important issue in the theory of Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$, which in general can be very difficult [XCDo92].
(7) In connection with this section see [DoPa89], [NaFo70], [Sa14b].

3. Hilbert modules of holomorphic functions

In various parts of operator theory and functional analysis, one is confronted with Hilbert spaces of functions, such that it is both simple and instructive to deal with a large class of operators (cf. [AraEn03]). The purpose of this section is to provide a brief introduction to the theory of Hilbert modules of holomorphic functions that will be used in subsequent sections.

3.1. **Reproducing kernel Hilbert modules.** A natural source of Hilbert module comes from the study of reproducing kernel Hilbert spaces (cf. [Aro50], [AgMc02], [CuSal84]) on domains in $\mathbb{C}^n$.

Let X be a non-empty set, and $\mathcal{E}$ a Hilbert space. An operator-valued function $K : X \times X \to \mathcal{B}(\mathcal{E})$ is said to be *positive definite kernel* if

$$\sum_{i,j=1}^{k} \langle K(z_i, z_j)\eta_j, \eta_i \rangle \geq 0,$$

for all $\eta_i \in \mathcal{E}$, $z_i \in X$, $i = 1, \cdots, k$, and $k \in \mathbb{N}$. Given such a positive definite kernel K on X, let $\mathcal{H}_K$ be the Hilbert space completion of the linear span of the set $\{K(\cdot, w)\eta : w \in X, \eta \in \mathcal{E}\}$ with respect to the inner product

$$\langle K(\cdot, w)\eta, K(\cdot, z)\zeta \rangle_{\mathcal{H}_K} = \langle K(z, w)\eta, \zeta \rangle_{\mathcal{E}},$$

for all $z, w \in X$ and $\eta, \zeta \in \mathcal{E}$. Therefore, $\mathcal{H}_K$ is a Hilbert space of $\mathcal{E}$-valued functions on X. The kernel function K has the reproducing property:

$$\langle f, K(\cdot, z)\eta \rangle_{\mathcal{H}_K} = \langle f(z), \eta \rangle_{\mathcal{E}},$$

for all $z \in X$, $f \in \mathcal{H}_K$ and $\eta \in \mathcal{E}$. In particular, for each $z \in X$, the evaluation operator $ev_z : \mathcal{H}_K \to \mathcal{E}$ defined by

$$\langle ev_z(f), \eta \rangle_{\mathcal{E}} = \langle f, K(\cdot, z)\eta \rangle_{\mathcal{H}_K}, \qquad (\eta \in \mathcal{E}, f \in \mathcal{H}_K)$$

is bounded. Conversely, let $\mathcal{H}$ be a Hilbert space of functions from X to $\mathcal{E}$ with bounded and non-zero evaluation operators ev_z for all $z \in X$. Therefore, $\mathcal{H}$ is a reproducing kernel Hilbert space with reproducing kernel

$$K(z,w) = ev_z \circ ev_w^* \in \mathcal{B}(\mathcal{E}). \qquad (z,w \in X)$$

Now let $X = \Omega$ a domain in $\mathbb{C}^n$ and $K : \Omega \times \Omega \to \mathcal{B}(\mathcal{E})$ be a kernel function, holomorphic in the first variable and anti-holomorphic in the second variable. Then $\mathcal{H}_K$ is a Hilbert space of holomorphic functions on Ω (cf. [CuSal84]).

A Hilbert module $\mathcal{H}_K$ is said to be *reproducing kernel Hilbert module* over Ω if $\mathcal{H}_K \subseteq \mathcal{O}(\Omega, \mathcal{E})$ and for each $1 \leq i \leq n$,

$$M_i f = z_i f,$$

where

$$(z_i f)(\boldsymbol{w}) = w_i f(\boldsymbol{w}). \qquad\qquad (f \in \mathcal{H}_K, \boldsymbol{w} \in \Omega)$$

It is easy to verify that

$$M_{z_i}^*(K(\cdot, \boldsymbol{w})\eta) = \bar{w}_i K(\cdot, \boldsymbol{w})\eta,$$

for all $\boldsymbol{w} \in \Omega, \eta \in \mathcal{E}$ and $i = 1, \ldots, n$.

In most of the following the module maps $\{M_i\}_{i=1}^n$ of a reproducing kernel Hilbert module will simply be denoted by the multiplication operators $\{M_{z_i}\}_{i=1}^n$ by the coordinate functions $\{z_i\}_{i=1}^n$.

Examples: (1) The Drury-Arveson module, denoted by H_n^2, is the reproducing kernel Hilbert module corresponding to the kernel $k_n : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$, where

$$k_n(\boldsymbol{z}, \boldsymbol{w}) = (1 - \sum_{i=1}^n z_i \bar{w}_i)^{-1}. \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n)$$

(2) Suppose $\alpha > n$. The weighted Bergman space $L^2_{a,\alpha}(\mathbb{B}^n)$ (see [Zh08]) is a reproducing kernel Hilbert space with kernel function

$$k_\alpha(\boldsymbol{z}, \boldsymbol{w}) = \frac{1}{(1 - \langle \boldsymbol{z}, \boldsymbol{w} \rangle_{\mathbb{C}^n})^\alpha}. \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n)$$

When $\alpha = n$, $L^2_{a,\alpha}(\mathbb{B}^n)$ is the usual Hardy module $H^2(\mathbb{B}^n)$.

(3) The kernel function for the Dirichlet module (see [Zh91]) $\mathcal{D}(\mathbb{B}^n)$ is given by

$$k_{\mathcal{D}(\mathbb{B}^n)}(\boldsymbol{z}, \boldsymbol{w}) = 1 + \log \frac{1}{1 - \langle \boldsymbol{z}, \boldsymbol{w} \rangle_{\mathbb{C}^n}}. \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n)$$

(4) $H^2(\mathbb{D}^n)$, the Hardy module over $\mathbb{D}^n$, is given by the reproducing kernel

$$\mathbb{S}_n(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^n (1 - z_i \bar{w}_i)^{-1}. \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n)$$

Finally, let I be a non-empty set and

$$l^2(I) = \{f : I \to \mathbb{C} : \sum_{i \in I} |f(i)|^2 < \infty\}.$$

Then $l^2(I)$ is a reproducing kernel Hilbert space with kernel $k(i,j) = \delta_{ij}$ for all $(i,j) \in I \times I$. Moreover, $\{k(\cdot, j) : j \in I\}$ is an orthonormal basis of $l^2(I)$. In general, $l^2(I)$ is not a reproducing kernel Hilbert module.

3.2. **Cowen-Douglas Hilbert modules.** Let m be a positive integer. A class of Hilbert modules over $\Omega \subseteq \mathbb{C}$, denoted by $B_m(\Omega)$, was introduced by Cowen and Douglas in [CoDo78]. This notion was extended to the multivariable setting, for $\Omega \subseteq \mathbb{C}^n$, by Curto and Salinas [CuSal84] and by Chen and Douglas [XCDo92]. See also [CoDo83].

Definition 3.1. *Let Ω be a domain in $\mathbb{C}^n$ and m be a positive integer. Then a Hilbert module $\mathcal{H}$ over $\mathbb{C}[\boldsymbol{z}]$ is said to be in $B_m^*(\Omega)$ if*

(i) the column operator $(M - \boldsymbol{w}I_{\mathcal{H}})^* : \mathcal{H} \to \mathcal{H}^n$ defined by

$$(M - \boldsymbol{w}I_{\mathcal{H}})^* h = (M_1 - w_1 I_{\mathcal{H}})^* h \oplus \cdots \oplus (M_n - w_n I_{\mathcal{H}})^* h, \qquad (h \in \mathcal{H})$$

has closed range for all $\boldsymbol{w} \in \Omega$, where $\mathcal{H}^n = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$.

(ii) dim ker $(M - \boldsymbol{w}I_{\mathcal{H}})^* = dim[\, \cap_{i=1}^n ker(M_i - w_i I_{\mathcal{H}})^*] = m$ for all $\boldsymbol{w} \in \Omega$, and

(iii) $\bigvee_{\boldsymbol{w} \in \Omega}$ ker $(M - \boldsymbol{w}I_{\mathcal{H}})^* = \mathcal{H}$.

Given a Hilbert module $\mathcal{H}$ in $B_m^*(\Omega)$, define

$$E_{\mathcal{H}}^* = \bigcup_{\boldsymbol{w} \in \Omega} \{\bar{\boldsymbol{w}}\} \times \ker(M - \boldsymbol{w}I_{\mathcal{H}})^*.$$

Then the mapping $\boldsymbol{w} \mapsto E_{\mathcal{H}}^*(\boldsymbol{w}) := \{\bar{\boldsymbol{w}}\} \times \ker\,(M - \boldsymbol{w}I_{\mathcal{H}})$ defines a rank m hermitian anti-holomorphic vector bundle over Ω. For a proof of this fact, the reader is referred to [CoDo78], [CoDo83], [CuSal84] and [EsSc14].

The fundamental relation between $\mathcal{H} \in B_m^*(\Omega)$ and the associated anti-holomorphic hermitian vector bundle [We80] over Ω defined by

$$\begin{array}{c} E_{\mathcal{H}}^* : \ker\,(M - \boldsymbol{w}I_{\mathcal{H}})^* \\ \downarrow \\ \boldsymbol{w} \end{array}$$

is the following identification:

Theorem 3.2. *Let $\Omega = \mathbb{B}^n$ or $\mathbb{D}^n$ and $\mathcal{H}, \tilde{\mathcal{H}} \in B_m^*(\Omega)$. Then $\mathcal{H} \cong \tilde{\mathcal{H}}$ if and only if the complex bundles $E_{\mathcal{H}}^*$ and $E_{\tilde{\mathcal{H}}}^*$ are equivalent as Hermitian anti-holomorphic vector bundles.*

Note that for U an open subset of Ω, the anti-holomorphic sections of $E_{\mathcal{H}}^*$ over U are given by $\gamma_f : U \to E_{\mathcal{H}}^*$, where $\gamma_f(\boldsymbol{w}) = (\bar{\boldsymbol{w}}, f(\boldsymbol{w}))$ and $f : U \to \mathcal{H}$ is an anti-holomorphic function with $f(\boldsymbol{w}) \in \ker(M - \boldsymbol{w}I_{\mathcal{H}})^*$ for all $\boldsymbol{w} \in U$.

The Grauert's theorem asserts that the anti-holomorphic vector bundle $E_{\mathcal{H}}^*$ over a domain in $\mathbb{C}$ or a contractible domain of holomorphy in $\mathbb{C}^n$ is holomorphically trivial, that is, $E_{\mathcal{H}}^*$ possesses a global anti-holomorphic frame. In particular, there exists anti-holomorphic functions $\{s_i\}_{i=1}^m \subseteq \mathcal{O}^*(\Omega, \mathcal{H})$ such that $\{s_i(\boldsymbol{w})\}_{i=1}^m$ is a basis of ker $(M - \boldsymbol{w}I_{\mathcal{H}})$ for all $\boldsymbol{w} \in \Omega$. Moreover, $\mathcal{H}$ is unitarily equivalent to a reproducing kernel Hilbert module with $\mathcal{B}(\mathbb{C}^m)$-valued kernel (see [Alp88], [CuSal84], [EsSc14]).

Theorem 3.3. *Let $\mathcal{H} \in B_m^*(\Omega)$ where Ω be a domain in $\mathbb{C}$ or a contractible domain of holomorphy in $\mathbb{C}^n$. Then there exits a reproducing kernel Hilbert module $\mathcal{H}_K \subseteq \mathcal{O}(\Omega, \mathbb{C}^m)$ such that $\mathcal{H} \cong \mathcal{H}_K$.*

Proof. Define $J_s : \mathcal{H} \to \mathcal{O}(\Omega, \mathbb{C}^m)$ by

$$(J_s(f))(\boldsymbol{w}) = (\langle f, s_1(\boldsymbol{w})\rangle_{\mathcal{H}}, \ldots, \langle f, s_m(\boldsymbol{w})\rangle_{\mathcal{H}}). \qquad (f \in \mathcal{H}, \boldsymbol{w} \in \Omega)$$

Note that J_s is an injective map. Consequently, the space $\mathcal{H}_{J_s} := \mathrm{ran} J_s \subseteq \mathcal{O}(\Omega, \mathbb{C}^m)$ equipped with the norm

$$\|J_s f\|_{\mathcal{H}_{J_s}} := \|f\|_{\mathcal{H}}, \qquad (f \in \mathcal{H})$$

is a $\mathbb{C}^m$-valued reproducing kernel Hilbert space with kernel $K_s : \Omega \times \Omega \to \mathcal{B}(\mathbb{C}^m)$ given by the "Gram matrix" of the frame $\{s_i(\boldsymbol{w}) : 1 \leq i \leq m\}$:

$$K_s(\boldsymbol{z}, \boldsymbol{w}) = \big(\langle s_j(\boldsymbol{w}), s_i(\boldsymbol{z})\rangle_{\mathcal{H}}\big)_{i,j=1}^m. \qquad (\boldsymbol{z}, \boldsymbol{w} \in \Omega)$$

Further, note that

$$\begin{aligned}(J_s M_i f)(\boldsymbol{w}) &= (\langle M_i f, s_1(\boldsymbol{w})\rangle_{\mathcal{H}}), \ldots, \langle M_i f, s_m(\boldsymbol{w})\rangle_{\mathcal{H}})) \\ &= (\langle f, M_i^* s_1(\boldsymbol{w})\rangle_{\mathcal{H}}), \ldots, \langle f, M_i^* s_m(\boldsymbol{w})\rangle_{\mathcal{H}})) \\ &= w_i(\langle f, s_1(\boldsymbol{w})\rangle_{\mathcal{H}}), \ldots, \langle f, s_m(\boldsymbol{w})\rangle_{\mathcal{H}})) \\ &= (M_{z_i} J_s f)(\boldsymbol{w}),\end{aligned}$$

for all $f \in \mathcal{H}$ and $\boldsymbol{w} \in \Omega$. This implies that $J_s M_i = M_{z_i} J_s$ for all $1 \leq i \leq n$ and hence the Hilbert module $\mathcal{H}$ is module isometric isomorphic with the reproducing kernel Hilbert module $\mathcal{H}_{J_s}$. ∎

If $E_{\mathcal{H}}^*$ is not trivial, then we can use an anti-holomorphic frame over an open subset $U \subseteq \Omega$ to define a kernel function K_U on U. Since a domain is connected, one can show that $\mathcal{H}_{K_U} \cong \mathcal{H}$. One way to obtain a local frame is to identify the fiber of the dual vector bundle $E_{\mathcal{H}}$ with $\mathcal{H}/I_{\boldsymbol{w}} \cdot \mathcal{H} \cong \mathbb{C}^m \cong \mathrm{span}\{s_i(\boldsymbol{w}) : 1 \leq i \leq m\}$, where $I_{\boldsymbol{w}} = \{p \in \mathbb{C}[\boldsymbol{z}] : p(\boldsymbol{w}) = 0\}$ is the maximal ideal of $\mathbb{C}[\boldsymbol{z}]$ at $\boldsymbol{w} \in \Omega$.

The curvature of the bundle $E_{\mathcal{H}}^*$ for the Chern connection determined by the metric defined by the Gram matrix or, if $E_{\mathcal{H}}^*$ is not trivial, then with the inner product on $E_{\mathcal{H}}^*(\boldsymbol{w}) = \ker(M_z - \boldsymbol{w} I_{\mathcal{H}})^* \subseteq \mathcal{H}$, is given by

$$\mathcal{K}_{E_{\mathcal{H}}^*}(\boldsymbol{w}) = (\bar{\partial}_j \{K(\boldsymbol{w}, \boldsymbol{w})^{-1} \partial_i K(\boldsymbol{w}, \boldsymbol{w})\})_{i,j=1}^n,$$

for all $\boldsymbol{w} \in \Omega$. Note that the representation of the curvature matrix defined above is with respect to the basis of two-forms $\{dw_i \wedge d\bar{w}_j : 1 \leq i, j \leq n\}$. In particular, for a line bundle, that is, when $m = 1$, the curvature form is given by

$$\begin{aligned}\mathcal{K}_{E_{\mathcal{H}}^*}(\boldsymbol{w}) &= \bar{\partial} K(\boldsymbol{w}, \boldsymbol{w})^{-1} \partial K(\boldsymbol{w}, \boldsymbol{w}) = -\partial\bar{\partial} \log \|K(\cdot, \boldsymbol{w})\|^2 \\ &= -\sum_{i,j=1}^n \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log K(\boldsymbol{w}, \boldsymbol{w}) dw_i \wedge d\bar{w}_j. \qquad (\boldsymbol{w} \in \Omega)\end{aligned}$$

The Hardy modules $H^2(\mathbb{B}^n)$ and $H^2(\mathbb{D}^n)$, the Bergman modules $L_a^2(\mathbb{B}^n)$ and $L_a^2(\mathbb{D}^n)$, the weighted Bergman modules $L_{a,\alpha}^2(\mathbb{B}^n)$ $(\alpha > n)$ and the Drury-Arveson module H_n^2 are the standard examples of Hilbert modules in $B_1^*(\Omega)$ with $\Omega = \mathbb{B}^n$ or $\mathbb{D}^n$. A further source of Hilbert modules in $B_m^*(\Omega)$ is a family of some quotient Hilbert modules, where the standard examples are used as building blocks (see Section 2 in [Sa14a]).

3.3. **Quasi-free Hilbert modules.** Besides reproducing kernel Hilbert modules, there is another class of function Hilbert spaces which will be frequently used throughout this article. These are the quasi-free Hilbert modules.

Recall that the Hardy module and the weighted Bergman modules over $\mathbb{D}^n$ (or $\mathbb{B}^n$) are singly-generated Hilbert module over $A(\mathbb{D}^n)$ (over $A(\mathbb{B}^n)$). In other words, these modules are the Hilbert space completion of $A(\Omega)$. More generally, every cyclic or singly-generated bounded Hilbert module over $\mathcal{A}(\Omega)$ is obtained as a Hilbert space completion of $\mathcal{A}(\Omega)$.

On the other hand, finitely generated free modules over $A(\Omega)$, in the sense of commutative algebra, have the form $A(\Omega) \otimes_{alg} l^2_m$ for some $m \in \mathbb{N}$ (see [Ei95]). However, the algebraic tensor product $A(\Omega) \otimes_{alg} l^2_m$ is not a Hilbert space. In order to construct "free Hilbert modules" we consider Hilbert space completions of free modules $A(\Omega) \otimes_{alg} l^2_m$:
Let $m \geq 1$. A Hilbert space $\mathcal{R}$ is said to be *quasi-free Hilbert module over* $A(\Omega)$ *and of rank* m if $\mathcal{R}$ is a Hilbert space completion of the algebraic tensor product $A(\Omega) \otimes_{alg} l^2_m$ and

(1) multiplication by functions in $\mathcal{A}(\Omega)$ define bounded operators on $\mathcal{R}$,
(2) the evaluation operators $ev_{\boldsymbol{w}} : \mathcal{R} \to l^2_m$ are locally uniformly bounded on Ω, and
(3) a sequence $\{f_k\} \subseteq \mathcal{A}(\Omega) \otimes l^2_m$ that is Cauchy in the norm of $\mathcal{R}$ converges to 0 in the norm of $\mathcal{R}$ if and only if $ev_{\boldsymbol{w}}(f_k)$ converges to 0 in l^2_m for $\boldsymbol{w} \in \Omega$.

Condition (1) implies that $\mathcal{R}$ is a bounded Hilbert module over $A(\Omega)$. Condition (2) ensures that $\mathcal{R}$ can be identified with a Hilbert space of l^2_m-valued holomorphic functions on Ω and condition (3) implies that the limit function of a Cauchy sequence in $A(\Omega) \otimes_{alg} l^2_m$ vanishes identically if and only if the limit in the $\mathcal{R}$-norm is the zero function. In other words, a quasi-free Hilbert module $\mathcal{R}$ over $A(\Omega)$ is a finitely generated reproducing kernel Hilbert module where the kernel function $K : \Omega \times \Omega \to \mathcal{B}(l^2_m)$ is holomorphic in the first variable and anti-holomorphic in the second variable.

In some instances, such as the Drury-Arveson module H^2_n, this definition does not apply. In such cases $\mathcal{R}$ is defined to be the completion of the polynomial algebra $\mathbb{C}[\boldsymbol{z}]$ relative to an inner product on it assuming that each $p(\boldsymbol{z})$ in $\mathbb{C}[\boldsymbol{z}]$ defines a bounded operator on $\mathcal{R}$ but there is no uniform bound. Hence, in this case $\mathcal{R}$ is a Hilbert module over $\mathbb{C}[\boldsymbol{z}]$.

3.4. **Multipliers.** Given $\mathcal{E}$- and $\mathcal{E}_*$-valued reproducing kernel Hilbert modules $\mathcal{H}$ and $\mathcal{H}_*$, respectively, over Ω, a function $\varphi : \Omega \to \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$ is said to be a *multiplier* if $\varphi f \in \mathcal{H}_*$, where $(\varphi f)(\boldsymbol{w}) = \varphi(\boldsymbol{w}) f(\boldsymbol{w})$ for $f \in \mathcal{H}$ and $\boldsymbol{w} \in \Omega$. The set of all such multipliers is denoted by $\mathcal{M}(\mathcal{H}, \mathcal{H}_*)$ or simply $\mathcal{M}$ if $\mathcal{H}$ and $\mathcal{H}_*$ are clear from the context (cf. [Bo03]). By the closed graph theorem, each $\varphi \in \mathcal{M}(\mathcal{H}, \mathcal{H}_*)$ induces a bounded linear map $M_\varphi : \mathcal{H} \to \mathcal{H}_*$ (cf. [Ha82]) defined by

$$M_\varphi f = \varphi f,$$

for all $f \in \mathcal{H}_K$. Consequently, $\mathcal{M}(\mathcal{H}, \mathcal{H}_*)$ is a Banach space with

$$\|\varphi\|_{\mathcal{M}(\mathcal{H}, \mathcal{H}_*)} = \|M_\varphi\|_{\mathcal{B}(\mathcal{H}, \mathcal{H}_*)}.$$

For $\mathcal{H} = \mathcal{H}_*$, $\mathcal{M}(\mathcal{H}) = \mathcal{M}(\mathcal{H}, \mathcal{H})$ is a Banach algebra with this norm.

Let $\mathcal{R} \subseteq \mathcal{O}(\Omega, \mathbb{C})$ be a reproducing kernel Hilbert module with kernel $k_{\mathcal{R}}$ and $\mathcal{E}$ be a Hilbert space. Then $\mathcal{R} \otimes \mathcal{E}$ is a reproducing kernel Hilbert module with kernel function $(\boldsymbol{z}, \boldsymbol{w}) \mapsto k_{\mathcal{R}}(\boldsymbol{z}, \boldsymbol{w}) I_{\mathcal{E}}$. By $\mathcal{M}_{\mathcal{B}(\mathcal{E}, \mathcal{E}_*)}(\mathcal{R})$ we denote the set of all multipliers $\mathcal{M}(\mathcal{R} \otimes \mathcal{E}, \mathcal{R} \otimes \mathcal{E}_*)$.

The following characterization result is well known and easy to prove.

THEOREM 3.4. *Let X be a non-empty set and for $i = 1, 2$, $K_i : X \times X \to \mathcal{B}(\mathcal{E}_i)$ be positive definite kernel functions with reproducing kernel Hilbert spaces $\mathcal{H}_{K_i}$. Suppose also that $\Theta : X \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ is a function. Then the following are equivalent:*
(1) $\Theta \in \mathcal{M}(\mathcal{H}_{K_1}, \mathcal{H}_{K_2})$.
(2) There exists a constant $c > 0$ such that

$$(x, y) \to c^2 K_2(x, y) - \Theta(x) K_1(x, y) \Theta(y)^*$$

is positive definite. In this case, the multiplier norm of Θ is the infimum of all such constants $c > 0$. Moreover, the infimum is achieved.

Examples:
(1) For the Drury-Aveson space H^2_n, the multiplier space is given by

$$\mathcal{M}_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(H^2_n) = \{\Theta \in \mathcal{O}(\mathbb{B}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*)) : \sup\|\Theta(rT)\| < \infty\},$$

where the supremum ranges over $0 < r < 1$ and commuting n-tuples $(T_, \dots, T_n)$ on Hilbert spaces $\mathcal{H}$ such that $\sum_{i=1}^n T_i T_i^* \leq I_{\mathcal{H}}$ (see [EsP02], [BaTV01] for more details).
(2) Let $\mathcal{H} = H^2(\mathbb{B}^n)$ or $L^2_a(\mathbb{B}^n)$. Then

$$\mathcal{M}_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathcal{H}) = H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{B}^n).$$

(3) Let $\mathcal{H} = H^2(\mathbb{D}^n)$ or $L^2_a(\mathbb{D}^n)$. Then

$$\mathcal{M}_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathcal{H}) = H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D}^n).$$

One striking fact about the Dirichlet space is that the multiplier space $\mathcal{M}(\mathcal{D}(\mathbb{D}))$ is a proper subset of $H^\infty(\mathbb{D})$ (see [St80]). Also, it is bounded but not a contractive Hilbert module over $\mathbb{C}[z]$. Note that also the multiplier space $\mathcal{M}(H^2_n)$ is a proper subspace of $H^\infty(\mathbb{B}^n)$. Moreover, $\mathcal{M}(H^2_n)$ does not contain the ball algebra $A(\mathbb{B}^n)$ (see [Dr78], [Arv98]).

This subsection concludes with a definition. Let $\Theta_i \in \mathcal{M}_{\mathcal{B}(\mathcal{E}_i,\mathcal{E}_{*i})}(\mathcal{R})$ and $i = 1, 2$. Then Θ_1 and Θ_2 are said to *coincide*, denoted by $\Theta_1 \cong \Theta_2$, if there exists unitary operators $\tau : \mathcal{E}_1 \to \mathcal{E}_2$ and $\tau_* : \mathcal{E}_{*1} \to \mathcal{E}_{*2}$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{R} \otimes \mathcal{E}_1 & \xrightarrow{M_{\Theta_1}} & \mathcal{R} \otimes \mathcal{E}_{*1} \\
{\scriptstyle I_{\mathcal{R}} \otimes \tau} \Big\downarrow & & {\scriptstyle I_{\mathcal{R}} \otimes \tau_*} \Big\downarrow \\
\mathcal{R} \otimes \mathcal{E}_2 & \xrightarrow{M_{\Theta_2}} & \mathcal{R} \otimes \mathcal{E}_{*2}
\end{array}$$

Further results and comments:

(1) Let $\Omega \subseteq \mathbb{C}$ and $V^*(\mathcal{H})$ be the von Neumann algebra of operators commuting with both M_z and M_z^*. Note that projections in $V^*(\mathcal{H})$, or reducing submodules of $\mathcal{H}$, are in one-to-one correspondence with reducing subbundles of $E^*_{\mathcal{H}}$. A subbundle F of an anti-holomorphic Hermitian vector bundle E is said to be a *reducing subbundle* if both F and its orthogonal complement $F^\perp$ in E are anti-holomorphic subbundles.
Also note that if S is an operator commuting with M_z^*, then $SE^*_{\mathcal{H}}(w) \subseteq E^*_{\mathcal{H}}(w)$ for each $w \in \Omega$ and hence S induces a holomorphic bundle map, denoted by $\Gamma(S)$, on $E^*_{\mathcal{H}}$.

In [ChDoGu11], Chen, Douglas and Guo proved that if S lies in $V^*(\mathcal{H})$, then $\Gamma(S)$ is not only anti-holomorphic, but also connection-preserving.

THEOREM 3.5. *Let $\mathcal{H} \in B_m^*(\Omega)$ and Φ be a bundle map on $E_{\mathcal{H}}^*$. There exists an operator T_Φ in $V^*(\mathcal{H})$ such that $\Phi = \Gamma(T_\Phi)$ if and only if Φ is connection preserving. Consequently, the map Γ is a $*$-isomorphism from $V^*(\mathcal{H})$ to connection-preserving bundle maps on $E_{\mathcal{H}}^*$.*

(2) Let $\mathcal{H}_1 \in B_{m_1}^*(\Omega)$ and $\mathcal{H}_2 \in B_{m_2}^*(\Omega)$ and $\Omega \subseteq \mathbb{C}$. It is natural to ask the following question: Determine the Hilbert module $\mathcal{H}$, if such exists, in $B_{m_1 m_2}(\Omega)$ corresponding to the anti-holomorphic vector bundle $E_{\mathcal{H}_1}^* \otimes E_{\mathcal{H}_2}^*$. That is, find $\mathcal{H} \in B_{m_1 m_2}(\Omega)$ such that $E_{\mathcal{H}}^* \cong E_{\mathcal{H}_1}^* \otimes E_{\mathcal{H}_2}^*$, where the equivalence is in terms of the anti-holomorphic vector bundle isomorphism. In [Li88], Q. Lin proved the following remarkable result.

THEOREM 3.6. *Let $\mathcal{H}_1 \in B_{m_1}^*(\Omega)$ and $\mathcal{H}_2 \in B_{m_2}^*(\Omega)$ and $\Omega \subseteq \mathbb{C}$. Define*

$$\mathcal{H} = \bigvee_{z \in \Omega} [ker(M - zI_{\mathcal{H}_1})^* \otimes ker(M - zI_{\mathcal{H}_2})^*].$$

Then $\mathcal{H}$ is a submodule of $\mathcal{H}_1 \otimes \mathcal{H}_2$, and the module multiplications on $\mathcal{H}$ coincides: $(M \otimes I_{\mathcal{H}_2})|_{\mathcal{H}} = (I_{\mathcal{H}_1} \otimes M)|_{\mathcal{H}}$. Moreover, $\mathcal{H} \in B_{m_1 m_2}^*(\Omega)$ and $E_{\mathcal{H}}^* \cong E_{\mathcal{H}_1}^* \otimes E_{\mathcal{H}_2}^*$.

(3) In [Zh00], Zhu suggested an alternative approach to the Cowen-Douglas theory based on the notion of spanning holomorphic cross-sections. More precisely, let $\Omega \subseteq \mathbb{C}$ and $\mathcal{H} \in B_m^*(\Omega)$. Then $E_{\mathcal{H}}^*$ possesses a spanning anti-holomorphic cross-section, that is, there is an anti-holomorphic function $\gamma : \Omega \to \mathcal{H}$ such that $\gamma(w) \in \ker\, (M - wI_{\mathcal{H}})^*$ for all $w \in \Omega$ and $\mathcal{H}$ is the closed linear span of the range of γ.
More recently, Eschmeier and Schmitt [EsSc14] extended Zhu's results to general domains in $\mathbb{C}^n$.
(4) The concept of quasi-free Hilbert module is due to Douglas and Misra [DoMi03], [DoMi05]. The notion is closely related to the generalized Bergman kernel introduced by Curto and Salinas [CuSal84].
(5) For a systematic exposition of the theory of quasi-free Hilbert modules, see the work by Chen [Ch09].
(6) In connection with Cowen-Douglas theory see Apostol and Martin [ApMa81], McCarthy [Mc96] and Martin [Ma85].
(7) In [Ba11], Barbian proved that an operator T between reproducing kernel Hilbert spaces is a multiplier if and only if $(Tf)(x) = 0$ holds for all f and x satisfying $f(x) = 0$.
(8) The reader is referred to [Aro50], [AgMc02], [CuSal84], [DoMiVa00] and [BuMa84] for some introduction to the general theory of reproducing kernel Hilbert spaces. For recent results on reproducing kernel Hilbert spaces see [Ba11], [Ba08] and the reference therein.

4. Contractive Hilbert modules over $A(\mathbb{D})$

This section gives a brief review of contractive Hilbert modules over $A(\mathbb{D})$ and begins with the definition of free resolutions from commutative algebra. The following subsection recast the canonical model of Sz.-Nagy and Foias in terms of Hilbert modules. It is proved that for a contractive Hilbert module over $A(\mathbb{D})$ there exists a unique free resolution. The final subsection is devoted to prove that the free resolutions of contractive Hilbert modules over $A(\mathbb{D})$ are uniquely determined by a nice class of bounded holomorphic functions on $\mathbb{D}$.

4.1. **Free resolutions.** The purpose of this subsection is to recall the notion of *free modules* from commutative algebra. Let M be a module over a commutative ring R. Then M is free if and only if M is a direct sum of isomorphic copies of the underlying ring R.

It is well known and easy to see that every module has a free resolution with respect to the underlying ring. More precisely, given a module M over a ring R, there exists a sequence of free R-modules $\{F_i\}_{i=0}^{\infty}$ and module maps $\varphi_i : F_i \to F_{i-1}$, for all $i \geq 1$, such that the sequence of modules

$$\cdots \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0,$$

is exact where $F_0/\text{ran}\varphi_1 = M$ and hence that φ_0 is a projection. The above resolution is said to be a *finite resolution of length* l, for some $l \geq 0$, if $F_{l+1} = \{0\}$ and $F_i \neq \{0\}$ for $0 \leq i \leq l$.

A celebrated result in commutative algebra, namely, the Hilbert Syzygy theorem, states that: Every finitely generated graded $\mathbb{C}[\boldsymbol{z}]$-module has a finite graded free resolution of length l for some $l \leq n$ by finitely generated free modules.

It is also a question of general interest: given a free resolution of a module over $\mathbb{C}[\boldsymbol{z}]$ when does the resolution stop.

4.2. **Dilations and free resolutions.** A contractive Hilbert module $\mathcal{H}$ over $A(\mathbb{D})$ is said to be *completely non-unitary* (or c.n.u.) if there is no non-zero reducing submodule $\mathcal{S} \subseteq \mathcal{H}$ such that $M|_{\mathcal{S}}$ is unitary.

Let $\mathcal{H}$ be a contractive Hilbert module over $A(\mathbb{D})$. Then the defect operators of $\mathcal{H}$ are defined by $D_{\mathcal{H}} = (I_{\mathcal{H}} - M^*M)^{\frac{1}{2}} \in \mathcal{B}(\mathcal{H})$ and $D_{*\mathcal{H}} = (I_{\mathcal{H}} - MM^*)^{\frac{1}{2}} \in \mathcal{B}(\mathcal{H})$, and the defect spaces by $\mathcal{D}_{\mathcal{H}} = \overline{\text{ran}} D_{\mathcal{H}}$ and $\mathcal{D}_{*\mathcal{H}} = \overline{\text{ran}} D_{*\mathcal{H}}$. The *characteristic function* $\Theta_{\mathcal{H}} \in H^{\infty}_{\mathcal{B}(\mathcal{D}_{\mathcal{H}},\mathcal{D}_{*\mathcal{H}})}(\mathbb{D})$ is defined by

$$\Theta_{\mathcal{H}}(z) = [-M + zD_{*\mathcal{H}}(I_{\mathcal{H}} - zM^*)^{-1}D_{\mathcal{H}}]|_{\mathcal{D}_{\mathcal{H}}}. \quad (z \in \mathbb{D})$$

Define $\Delta_{\mathcal{H}}(t) = [I_{\mathcal{D}_{\mathcal{H}}} - \Theta_{\mathcal{H}}(e^{it})^*\Theta_{\mathcal{H}}(e^{it})]^{\frac{1}{2}} \in \mathcal{B}(L^2_{\mathcal{D}_{\mathcal{H}}}(\mathbb{T}))$ for $t \in [0,1]$. Then

$$\mathcal{M}_{\mathcal{H}} = H^2_{\mathcal{D}_{*\mathcal{H}}}(\mathbb{D}) \oplus \overline{\Delta_{\mathcal{H}} L^2_{\mathcal{D}_{\mathcal{H}}}(\mathbb{T})},$$

is a contractive Hilbert module over $A(\mathbb{D})$. Then

$$\mathcal{S}_{\mathcal{H}} = \{M_{\Theta_{\mathcal{H}}} f \oplus \Delta_{\mathcal{H}} f : f \in H^2_{\mathcal{D}_{\mathcal{H}}}(\mathbb{D})\} \subseteq \mathcal{M}_{\mathcal{H}},$$

defines a submodule of $\mathcal{M}_{\mathcal{H}}$. Also consider the quotient module

$$\mathcal{Q}_{\mathcal{H}} = \mathcal{M}_{\mathcal{H}} \ominus \mathcal{S}_{\mathcal{H}}.$$

Here the module map $M_z \oplus M_{e^{it}}|_{\overline{\Delta_{\mathcal{H}} L^2_{\mathcal{D}_{\mathcal{H}}}(\mathbb{T})}}$ on $\mathcal{M}_{\mathcal{H}}$ is an isometry where M_z on $H^2_{\mathcal{D}_{*\mathcal{H}}}(\mathbb{D})$ is the pure part and $M_{e^{it}}|_{\overline{\Delta_{\mathcal{H}} L^2_{\mathcal{D}_{\mathcal{H}}}(\mathbb{T})}}$ on $\overline{\Delta_{\mathcal{H}} L^2_{\mathcal{D}_{\mathcal{H}}}(\mathbb{T})}$ is the unitary part in the sense of the Wold decomposition of isometries, Theorem 5.1. Consequently,

$$0 \longrightarrow H^2_{\mathcal{D}_{\mathcal{H}}}(\mathbb{D}) \xrightarrow{\begin{bmatrix} \Theta_{\mathcal{H}} \\ \Delta_{\mathcal{H}} \end{bmatrix}} \mathcal{M}_{\mathcal{H}} \xrightarrow{\Pi^*_{NF}} \mathcal{Q}_{\mathcal{H}} \longrightarrow 0,$$

where Π^*_{NF} is the quotient (module) map.

Theorem 4.1. *(Sz.-Nagy and Foias) Let $\mathcal{H}$ be a c.n.u. contractive Hilbert module over $A(\mathbb{D})$. Then*
(i) $\mathcal{H} \cong \mathcal{Q}_{\mathcal{H}}$.
(ii) $\mathcal{M}_{\mathcal{H}}$ is the minimal isometric dilation of $\mathcal{H}$.

Minimality of Sz.-Nagy-Foias isometric dilation, conclusion (ii) in Theorem 4.1, can be interpreted as a factorization of dilation maps in the following sense:
Let $\mathcal{H}$ be a c.n.u. contractive Hilbert module over $A(\mathbb{D})$ and $\Pi : \mathcal{H} \to \mathcal{K}$ be an isometric dilation of $\mathcal{H}$ with isometry V on $\mathcal{K}$. Then there exists a unique co-module isometry $\Phi \in \mathcal{B}(\mathcal{M}_{\mathcal{H}}, \mathcal{K})$ such that

$$\Pi = \Phi \Pi_{NF},$$

that is, the following diagram commutes:

$$\begin{tikzcd} & \mathcal{M}_{\mathcal{H}} \arrow[d, "\Phi"] \\ \mathcal{H} \arrow[ur, "\Pi_{NF}"] \arrow[r, "\Pi"'] & \mathcal{K} \end{tikzcd}$$

As will be shown below, specializing to the case of $C_{\cdot 0}$ class and using localization technique one can recover the characteristic function of a given $C_{\cdot 0}$-contractive Hilbert module. Recall that a contractive Hilbert module $\mathcal{H}$ over $A(\mathbb{D})$ is said to be in $C_{\cdot 0}$ class if $M^{*k} \to 0$ in SOT as $k \to \infty$. Submodules and quotient modules of vector-valued Hardy modules are examples of Hilbert modules in $C_{\cdot 0}$ class.

Let $\mathcal{H}$ be a $C_{\cdot 0}$ contractive Hilbert module over $A(\mathbb{D})$. Then there exists a Hilbert space $\mathcal{E}_*$ such that $\mathcal{H} \cong \mathcal{Q}$ for some quotient module $\mathcal{Q}$ of $H^2_{\mathcal{E}_*}(\mathbb{D})$ (cf. Corollary 8.2). Now by Beurling-Lax-Halmos theorem, Theorem 5.3, there exists a Hilbert space $\mathcal{E}$ such that the submodule $\mathcal{Q}^\perp \cong H^2_{\mathcal{E}}(\mathbb{D})$. This yields the following short exact sequence of modules:

$$0 \longrightarrow H^2_{\mathcal{E}}(\mathbb{D}) \xrightarrow{X} H^2_{\mathcal{E}_*}(\mathbb{D}) \xrightarrow{\pi} \mathcal{H} \longrightarrow 0,$$

where X is isometric module map, and π is co-isometric module map. Localizing the isometric part of the short exact sequence, $H^2_{\mathcal{E}}(\mathbb{D}) \xrightarrow{X} H^2_{\mathcal{E}_*}(\mathbb{D})$, at $z \in \mathbb{D}$ one gets

$$H^2_{\mathcal{E}}(\mathbb{D})/(A(\mathbb{D})_z \cdot H^2_{\mathcal{E}}(\mathbb{D})) \xrightarrow{X_z} H^2_{\mathcal{E}_*}(\mathbb{D})/(A(\mathbb{D})_z \cdot H^2_{\mathcal{E}_*}(\mathbb{D})).$$

Identifying $H^2_{\mathcal{E}}(\mathbb{D})/(A(\mathbb{D})_z \cdot H^2_{\mathcal{E}}(\mathbb{D}))$ with $\mathcal{E}$ and $H^2_{\mathcal{E}_*}(\mathbb{D})/(A(\mathbb{D})_z H^2_{\mathcal{E}_*}(\mathbb{D}))$ with $\mathcal{E}_*$ one can recover the characteristic function of $\mathcal{H}$ as the map $z \mapsto X_z \in \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$.

4.3. **Invariants.** This subsection begins by proving a theorem, due to Sz.-Nagy and Foias ([NaFo70]), on a complete unitary invariant of c.n.u. contractions.

THEOREM 4.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be c.n.u. contractive Hilbert modules over $A(\mathbb{D})$. Then $\mathcal{H}_1 \cong \mathcal{H}_2$ if and only if $\Theta_{\mathcal{H}_1} \cong \Theta_{\mathcal{H}_2}$.*

Proof. Denote the module multiplication operator on $\mathcal{H}_1$ and $\mathcal{H}_2$ by M_1 and M_2, respectively. Now let $uM_1 = M_2 u$, for some unitary $u : \mathcal{H}_1 \to \mathcal{H}_2$. Since $uD_{*\mathcal{H}_1} = D_{*\mathcal{H}_2} u$ and $uD_{\mathcal{H}_1} = D_{\mathcal{H}_2} u$

$$u|_{\mathcal{D}_{\mathcal{H}_1}} : \mathcal{D}_{\mathcal{H}_1} \to \mathcal{D}_{\mathcal{H}_2} \quad \text{and} \quad u|_{\mathcal{D}_{*\mathcal{H}_1}} : \mathcal{D}_{*\mathcal{H}_1} \to \mathcal{D}_{*\mathcal{H}_2},$$

are unitary operators. A simple computation now reveals that

$$u|_{\mathcal{D}_{*\mathcal{H}_1}} \Theta_{\mathcal{H}_1}(z) = \Theta_{\mathcal{H}_2}(z) u|_{\mathcal{D}_{\mathcal{H}_1}},$$

for all $z \in \mathbb{D}$, that is, $\Theta_{\mathcal{H}_1} \cong \Theta_{\mathcal{H}_2}$.
Conversely, given unitary operators $u \in \mathcal{B}(\mathcal{D}_{\mathcal{H}_1}, \mathcal{D}_{\mathcal{H}_2})$ and $u_* \in \mathcal{B}(\mathcal{D}_{*\mathcal{H}_1}, \mathcal{D}_{*\mathcal{H}_2})$ with the intertwining property $u_* \Theta_{\mathcal{H}_1}(z) = \Theta_{\mathcal{H}_2}(z) u$ for all $z \in \mathbb{D}$,

$$\boldsymbol{u} = I_{H^2(\mathbb{D})} \otimes u|_{\mathcal{D}_{\mathcal{H}_1}} : H^2_{\mathcal{D}_{\mathcal{H}_1}}(\mathbb{D}) \to H^2_{\mathcal{D}_{\mathcal{H}_2}}(\mathbb{D}),$$

and

$$\boldsymbol{u_*} = I_{H^2(\mathbb{D})} \otimes u|_{\mathcal{D}_{*\mathcal{H}_1}} : H^2_{\mathcal{D}_{*\mathcal{H}_1}}(\mathbb{D}) \to H^2_{\mathcal{D}_{*\mathcal{H}_2}}(\mathbb{D}),$$

and

$$\boldsymbol{\tau} = (I_{L^2(\mathbb{T})} \otimes u)|_{\overline{\Delta_{\mathcal{H}_1} L^2_{\mathcal{D}_{\mathcal{H}_1}}(\mathbb{T})}} : \overline{\Delta_{\mathcal{H}_1} L^2_{\mathcal{D}_{\mathcal{H}_1}}(\mathbb{T})} \to \overline{\Delta_{\mathcal{H}_2} L^2_{\mathcal{D}_{\mathcal{H}_2}}(\mathbb{T})},$$

are module maps. Moreover,

$$\boldsymbol{u_*} M_{\Theta_{\mathcal{H}_1}} = M_{\Theta_{\mathcal{H}_2}} \boldsymbol{u}.$$

Consequently, one arrives at the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^2_{\mathcal{D}_{\mathcal{H}_1}}(\mathbb{D}) \arrow[r, "\begin{bmatrix} M_{\Theta_{\mathcal{H}_1}} \\ \Delta_{\mathcal{H}_1} \end{bmatrix}"] \arrow[d, "\boldsymbol{u}"'] & H^2_{\mathcal{D}_{*\mathcal{H}_1}}(\mathbb{D}) \arrow[r, "\Pi^*_{NF,1}"] \arrow[d, "\boldsymbol{u_*} \oplus \boldsymbol{\tau}"'] & \mathcal{Q}_{\mathcal{H}_1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^2_{\mathcal{D}_{\mathcal{H}_2}}(\mathbb{D}) \arrow[r, "\begin{bmatrix} M_{\Theta_{\mathcal{H}_2}} \\ \Delta_{\mathcal{H}_2} \end{bmatrix}"'] & H^2_{\mathcal{D}_{*\mathcal{H}_2}}(\mathbb{D}) \arrow[r, "\Pi^*_{NF,2}"'] & \mathcal{Q}_{\mathcal{H}_2} \arrow[r] & 0
\end{tikzcd}$$

where the third vertical arrow is given by the unitary operator

$$\Pi^*_{NF,2}(\boldsymbol{u_*} \oplus \boldsymbol{\tau}) \Pi_{NF,1} : \mathcal{Q}_{\mathcal{H}_1} \to \mathcal{Q}_{\mathcal{H}_2}.$$

To see this, first note that

$$
\begin{aligned}
(\boldsymbol{u}_* \oplus \boldsymbol{\tau})(\mathrm{ran}\Pi_{NF,1}) &= (\boldsymbol{u}_* \oplus \boldsymbol{\tau})((\mathrm{ker}\Pi^*_{NF,1}))^{\perp} = (\boldsymbol{u}_* \oplus \boldsymbol{\tau})((\mathrm{ran} \begin{bmatrix} M_{\Theta_{\mathcal{H}_1}} \\ \Delta_{\mathcal{H}_1} \end{bmatrix})^{\perp}) \\
&= [(\boldsymbol{u}_* \oplus \boldsymbol{\tau})(\mathrm{ran} \begin{bmatrix} M_{\Theta_{\mathcal{H}_1}} \\ \Delta_{\mathcal{H}_1} \end{bmatrix})]^{\perp} = [\mathrm{ran} \begin{bmatrix} M_{\Theta_{\mathcal{H}_2}} \\ \Delta_{\mathcal{H}_2} \end{bmatrix}]^{\perp} \\
&= \mathrm{ran}\Pi_{\mathcal{H}_2}.
\end{aligned}
$$

Moreover, the unitary operator

$$(\boldsymbol{u}_* \oplus \boldsymbol{\tau})|_{\mathrm{ran}\Pi_{NF,1}} : \mathrm{ran}\Pi_{NF,1} \to \mathrm{ran}\Pi_{NF,2},$$

is a module map. This completes the proof. ∎

Further results and comments:

(1) All results presented in this section can be found in the book by Sz.-Nagy and Foias [NaFo70]. Here the Hilbert module point of view is slightly different from the classical one.
(2) Theorem 4.2 is due to Sz.-Nagy and Foias [NaFo70].
(3) For non-commutative tuples of operators, Theorems 4.1 and 4.2 were generalized by Popescu [Po89] and Ball and Vinnikov [BaV05] (see also [Po99], [BEsSa05], [BenTi07b], [BenTi07a], [Po06], [Po10], [Po11], [Va92], [Ba78], and the references therein).
(4) The notion of isometric dilation of contractions is closely related to the invariant subspace problem (see [ChPa11], [Bea88], [RaRo73]). The reader is referred to [Sa13c], [Sa13b] for further recent developments in this area.
(5) There are many other directions to the model theory (both in single and several variables) that are not presented in this survey. For instance, coordinate free approach by Douglas, Vasyunin and Nikolski, and the de Branges-Rovnyak model by de Branges, Rovnyak, Ball and Dritschel. We recommend the monographs by Nikolski [Ni02] which is a comprehensive source of these developments.
(6) The paper by Ball and Kriete [BaKr87] contains a remarkable connection between the Sz.-Nagy and Foias functional model and the de Branges-Rovnyak model on the unit disc.

5. Submodules

This section contains classical theory of isometries on Hilbert spaces, invariant subspaces of M_z on $H^2(\mathbb{D})$ and some more advanced material on this subject.

Let S be an isometry on a Hilbert space $\mathcal{H}$, that is, $S^*S = I_{\mathcal{H}}$. A closed subspace $\mathcal{W} \subseteq \mathcal{H}$ is said to be *wandering subspace* for S if $S^k\mathcal{W} \perp S^l\mathcal{W}$ for all $k, l \in \mathbb{N}$ with $k \neq l$, or equivalently, if $S^k\mathcal{W} \perp \mathcal{W}$ for all $k \geq 1$. An isometry S on $\mathcal{H}$ is said to be *shift* if

$$\mathcal{H} = \bigoplus_{k \geq 0} S^k\mathcal{W},$$

for some wandering subspace $\mathcal{W}$ for S. Equivalently, an isometry S on $\mathcal{H}$ is shift if and only if (see Theorem 5.1 below)

$$\bigcap_{k=0}^{\infty} S^k \mathcal{H} = \{0\}.$$

For a shift S on $\mathcal{H}$ with a wandering subspace $\mathcal{W}$ one sees that

$$\mathcal{H} \ominus S\mathcal{H} = \bigoplus_{k \geq 0} S^k \mathcal{W} \ominus S(\bigoplus_{k \geq 0} S^k \mathcal{W}) = \bigoplus_{k \geq 0} S^k \mathcal{W} \ominus \bigoplus_{m \geq 1} S^k \mathcal{W} = \mathcal{W}.$$

In other words, wandering subspace of a shift is uniquely determined by $\mathcal{W} = \mathcal{H} \ominus S\mathcal{H}$. The dimension of the wandering subspace of a shift is called the *multiplicity* of the shift.

As for the examples, the only invariant of a shift operator is its multiplicity, that is, the wandering subspace, up to unitary equivalence.

5.1. **von Neumann and Wold decomposition.** One of the most important results in operator algebras, operator theory and stochastic processes is the Wold decomposition theorem [Wo38] (see also page 3 in [NaFo70]), which states that every isometry on a Hilbert space is either a shift, or a unitary, or a direct sum of shift and unitary.

THEOREM 5.1. *Let S be an isometry on $\mathcal{H}$. Then $\mathcal{H}$ admits a unique decomposition $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_u$, where $\mathcal{H}_s$ and $\mathcal{H}_u$ are S-reducing subspaces of $\mathcal{H}$ and $S|_{\mathcal{H}_s}$ is a shift and $S|_{\mathcal{H}_u}$ is unitary. Moreover,*

$$\mathcal{H}_s = \bigoplus_{k=0}^{\infty} S^k \mathcal{W} \qquad \text{and} \qquad \mathcal{H}_u = \bigcap_{k=0}^{\infty} S^k \mathcal{H},$$

where $\mathcal{W} = ran(I - SS^*)$ is the wandering subspace for S.

Proof. Let $\mathcal{W} = \text{ran}(I - SS^*)$ be the wandering subspace for S and $\mathcal{H}_s = \bigoplus_{k=0}^{\infty} V^k \mathcal{W}$. Consequently, $\mathcal{H}_s$ is a S-reducing subspace of $\mathcal{H}$ and that $S|_{\mathcal{H}_s}$ is an isometry. On the other hand, for all $k \geq 0$,

$$\begin{aligned}
(S^k \mathcal{W})^{\perp} &= (S^k \text{ran}(I - SS^*))^{\perp} = \text{ran}(I - S^k(I - SS^*)S^{*k}) \\
&= \text{ran}[(I - S^k S^{*k}) + S^{k+1} S^{*\,k+1}] = \text{ran}(I - S^k S^{*k}) \oplus \text{ran} S^{k+1} \\
&= (S^k \mathcal{H})^{\perp} \oplus S^{k+1} \mathcal{H}.
\end{aligned}$$

Therefore

$$\mathcal{H}_u := \mathcal{H}_s^{\perp} = \bigcap_{k=0}^{\infty} S^k \mathcal{H}.$$

Uniqueness of the decomposition readily follows from the uniqueness of the wandering subspace $\mathcal{W}$ for S. This completes the proof. ∎

COROLLARY 5.2. *Let $\mathcal{H}$ be a Hilbert module over $\mathbb{C}[z]$. If the module multiplication M on $\mathcal{H}$ is a shift then there exists a Hilbert space $\mathcal{W}$ and a module isometry U from $H^2_{\mathcal{W}}(\mathbb{D})$ onto $\mathcal{H}$.*

Proof. Let $\mathcal{W}$ be the wandering subspace for M. Define

$$U : H^2_{\mathcal{W}}(\mathbb{D}) \to \mathcal{H} = \bigoplus_{k=0}^{\infty} M^k \mathcal{W},$$

by $U(z^k f) = M^k f$ for all $f \in \mathcal{W}$ and $k \in \mathbb{N}$. One can check that this is indeed the isometric module map from $H^2_{\mathcal{W}}(\mathbb{D})$ onto $\mathcal{H}$. ■

5.2. **Submodules of $H^2_{\mathcal{E}}(\mathbb{D})$.** The purpose of this subsection is to show that a submodule of $H^2_{\mathcal{E}}(\mathbb{D})$ is uniquely determined (up to unitary multipliers) by inner multipliers. The present methodology applies the von Neumann-Wold decomposition theorem, to the submodules of the Hardy module $H^2_{\mathcal{E}}(\mathbb{D})$ (see page 239, Theorem 2.1 in [FoFr90] and [Do11]).

Theorem 5.3. (Beurling-Lax-Halmos Theorem) *Let $\mathcal{S}$ be a submodule of the Hardy module $H^2_{\mathcal{E}}(\mathbb{D})$. Then there exists a closed subspace $\mathcal{F} \subseteq \mathcal{E}$ such that*

$$\mathcal{S} \cong H^2_{\mathcal{F}}(\mathbb{D}).$$

In particular, there exists an inner function $\Theta \in H^\infty_{\mathcal{L}(\mathcal{F},\mathcal{E})}(\mathbb{D})$ such that $M_\Theta : H^2_{\mathcal{F}}(\mathbb{D}) \to H^2_{\mathcal{E}}(\mathbb{D})$ is a module isometry and $\mathcal{S} = \Theta H^2_{\mathcal{F}}(\mathbb{D})$. Moreover, Θ is unique up to a unitary constant right factor, that is, if $\mathcal{S} = \tilde{\Theta} H^2_{\tilde{\mathcal{F}}}(\mathbb{D})$ for some Hilbert space $\tilde{\mathcal{F}}$ and inner function $\tilde{\Theta} \in H^\infty_{\mathcal{B}(\tilde{\mathcal{F}},\mathcal{E})}(\mathbb{D})$, then $\Theta = \tilde{\Theta} W$ where W is a unitary operator in $\mathcal{B}(\mathcal{F}, \tilde{\mathcal{F}})$.

Proof. Let $\mathcal{S}$ be a submodule of $H^2_{\mathcal{E}}(\mathbb{D})$. Then

$$\bigcap_{l=0}^{\infty} (M_z|_{\mathcal{S}})^l \mathcal{S} \subseteq \bigcap_{l=0}^{\infty} M_z^l H^2_{\mathcal{E}}(\mathbb{D}) = \{0\}.$$

By Corollary 5.2 there exists an isometric module map U from $H^2_{\mathcal{F}}(\mathbb{D})$ onto $\mathcal{S} \subseteq H^2_{\mathcal{E}}(\mathbb{D})$. Consequently, $U = M_\Theta$ for some inner function $\Theta \in H^\infty_{\mathcal{L}(\mathcal{F},\mathcal{E})}(\mathbb{D})$. ■

In the particular case of the space $\mathcal{E} = \mathbb{C}$, the above result recovers Beurling's characterization of submodules of $H^2(\mathbb{D})$.

Corollary 5.4. (Beurling) *Let $\mathcal{S}$ be a non-zero submodule of $H^2(\mathbb{D})$. Then $\mathcal{S} = \theta H^2(\mathbb{D})$ for some inner function $\theta \in H^\infty(\mathbb{D})$.*

Moreover, one also has the following corollary:

Corollary 5.5. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be submodules of $H^2(\mathbb{D})$. Then $\mathcal{S}_1 \cong \mathcal{S}_2$.*

The conclusion of Beurling's theorem, Corollary 5.4, fails if $H^2(\mathbb{D})$ is replaced by the Bergman module $L^2_a(\mathbb{D})$. However, a module theoretic interpretation of Beurling-Lax-Halmos theorem states that: Let $\mathcal{S}$ be a closed subspace of the "free module" $H^2(\mathbb{D}) \otimes \mathcal{E} (\cong H^2_{\mathcal{E}}(\mathbb{D}))$. Then $\mathcal{S}$ is a submodule of $H^2_{\mathcal{E}}(\mathbb{D})$ if and only if $\mathcal{S}$ is also "free" with $\mathcal{S} \ominus z\mathcal{S}$ as a generating set. Moreover, in this case $\dim[\mathcal{S} \ominus \mathcal{S}] \leq \dim \mathcal{E}$. In particular, the wandering subspace $\mathcal{S} \ominus z\mathcal{S}$ is a generating set of $\mathcal{S}$.

Recall that a bounded linear operator T on a Hilbert space $\mathcal{H}$ is said to have the *wandering subspace property* if $\mathcal{H}$ is generated by the subspace $\mathcal{W}_T := \mathcal{H} \ominus T\mathcal{H}$, that is,

$$\mathcal{H} = [\mathcal{W}_T] = \overline{\text{span}}\{T^m \mathcal{W}_T : m \in \mathbb{N}\}.$$

In that case $\mathcal{W}_T$ is said to be a wandering subspace for T.

The following statements, due to Aleman, Richter and Sundberg [AlRiSu96], assert that the same conclusion hold also in the Bergman module $L^2_a(\mathbb{D})$.

THEOREM 5.6. *Let $\mathcal{S}$ be a submodule of $L^2_a(\mathbb{D})$. Then*

$$\mathcal{S} = \bigvee_{k=0}^{\infty} z^k(\mathcal{S} \ominus z\mathcal{S}).$$

The same conclusion holds for the weighted Bergman space $L^2_{a,\alpha}(\mathbb{D})$ with weight $\alpha = 3$ [Sh01] but for $\alpha > 3$, the issue is more subtle (see [HePe04], [SMR02]).

Another important consequence of the Beurling-Lax-Halmos theorem is the characterization of cyclic submodules of $H^2_{\mathcal{E}}(\mathbb{D})$: Let f be a non-zero vector in $H^2_{\mathcal{E}}(\mathbb{D})$. Then the cyclic submodule of $H^2_{\mathcal{E}}(\mathbb{D})$ generated by f (and denoted by $[f]$) is isomorphic to $H^2(\mathbb{D})$.
There is no analog of the preceding result for the Bergman module:

THEOREM 5.7. *There does not exists any submodule $\mathcal{S}$ of $L^2_a(\mathbb{D})$ such that $\mathcal{S} \cong [1 \oplus z]$, the cyclic submodule of $L^2_a(\mathbb{D}) \oplus L^2_a(\mathbb{D}) (\cong L^2_a(\mathbb{D}) \otimes \mathbb{C}^2)$ generated by $1 \oplus z$.*

Proof. Let $\mathcal{S}$ be a submodule of $L^2_a(\mathbb{D})$ and U be a module isometric isomorphism from $[1 \oplus z]$ onto $\mathcal{S}$. Let

$$U(1 \oplus z) = f,$$

for some $f \in L^2_a(\mathbb{D})$. Then the fact that the closed support of Lebesgue measure on $\mathbb{D}$ is $\mathbb{D}$ implies that

$$|f(z)|^2 = 1 + |z|^2. \qquad (z \in \mathbb{D})$$

By Taylor series expansion of $f(z)$ one can show this is impossible for any holomorphic function f on $\mathbb{D}$. ■

In the language of Hilbert modules, Beurling-Lax-Halmos theorem says that the set of all non-zero submodules of $H^2_{\mathcal{E}_*}(\mathbb{D})$ are uniquely determined by the set of all module isometric maps from $H^2_{\mathcal{E}}(\mathbb{D})$ to $H^2_{\mathcal{E}_*}(\mathbb{D})$ where $\mathcal{E}$ is a Hilbert space so that $\dim \mathcal{E} \leq \dim \mathcal{E}_*$. On the other hand, a module map $U : H^2_{\mathcal{E}}(\mathbb{D}) \to H^2_{\mathcal{E}_*}(\mathbb{D})$ is uniquely determined by a multiplier $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$ and that Θ is inner if and only if U is isometry (cf. [NaFo70]). Consequently, there exists a bijective correspondence, modulo the unitary group, between the set of all non-zero submodules of $H^2_{\mathcal{E}_*}(\mathbb{D})$ and the set of all isometric module maps from $H^2_{\mathcal{E}}(\mathbb{D})$ to $H^2_{\mathcal{E}_*}(\mathbb{D})$, where $\mathcal{E} \subseteq \mathcal{E}_*$ and the set of all inner multipliers $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$, where $\mathcal{E} \subseteq \mathcal{E}_*$.

5.3. **Submodules of H^2_n.** This subsection will show how to extend the classification result of submodules of $H^2_{\mathcal{E}}(\mathbb{D})$, the Beurling-Lax-Halmos theorem, to $H^2_n \otimes \mathcal{E}$. This important generalization was given by McCullough and Trent [SMT00].

Recall that the Drury-Arveson module $H^2_n \otimes \mathcal{E}$ is a reproducing kernel Hilbert module corresponding to the kernel

$$(\boldsymbol{z}, \boldsymbol{w}) \mapsto (1 - \sum_{i=1}^{n} z_i \bar{w}_i)^{-1} I_{\mathcal{E}},$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n$ (see Section 3). A multiplier $\Theta \in \mathcal{M}_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(H^2_n)$ is said to be *inner* if M_Θ is a partial isometry in $\mathcal{L}(H^2_n \otimes \mathcal{E}, H^2_n \otimes \mathcal{E}_*)$.

THEOREM 5.8. *Let $\mathcal{S}(\neq \{0\})$ be a closed subspace of $H^2_n \otimes \mathcal{E}_*$. Then $\mathcal{S}$ is a submodule of $H^2_n \otimes \mathcal{E}_*$ if and only if*

$$\mathcal{S} = \Theta(H^2_n \otimes \mathcal{E}),$$

for some inner multiplier $\Theta \in \mathcal{M}_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(H^2_n)$.

Proof. Let $\mathcal{S}$ be a submodule of $H^2_n \otimes \mathcal{E}_*$ and $R_i = M_{z_i}|_{\mathcal{S}}$, $i = 1, \ldots, n$. Then

$$\sum_{i=1}^n R_i R_i^* = \sum_{i=1}^n P_{\mathcal{S}} M_{z_i} P_{\mathcal{S}} M_{z_i}^* P_{\mathcal{S}} \leq \sum_{i=1}^n P_{\mathcal{S}} M_{z_i} M_{z_i}^* P_{\mathcal{S}},$$

and consequently,

$$\begin{aligned} P_{\mathcal{S}} - \sum_{i=1}^n R_i R_i^* = P_{\mathcal{S}} - \sum_{i=1}^n P_{\mathcal{S}} M_{z_i} P_{\mathcal{S}} M_{z_i}^* P_{\mathcal{S}} &\geq P_{\mathcal{S}} - \sum_{i=1}^n P_{\mathcal{S}} M_{z_i} M_{z_i}^* P_{\mathcal{S}} \\ &= P_{\mathcal{S}} (I_{H^2_n \otimes \mathcal{E}_*} - \sum_{i=1}^n M_{z_i} M_{z_i}^*) P_{\mathcal{S}}. \end{aligned}$$

Define $K : \mathbb{B}^n \otimes \mathbb{B}^n \to \mathcal{L}(\mathcal{E}_*)$, a positive definite kernel, by

$$\langle K(\boldsymbol{z}, \boldsymbol{w}) x_l, x_m \rangle = \langle (P_{\mathcal{S}} - \sum_{i=1}^n P_{\mathcal{S}} M_{z_i} P_{\mathcal{S}} M_{z_i}^* P_{\mathcal{S}})(k_n(\cdot, \boldsymbol{w}) \otimes x_l), k_n(\cdot, \boldsymbol{z}) \otimes x_m \rangle$$

where $\{x_l\}$ is a basis of $\mathcal{E}_*$. By Kolmogorov theorem, there exists a Hilbert space $\mathcal{E}$, a function $\Theta \in \mathcal{O}(\mathbb{B}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ such that

$$K(\boldsymbol{z}, \boldsymbol{w}) = \Theta(\boldsymbol{z}) \Theta(\boldsymbol{w})^*. \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n)$$

On the other hand, since

$$P_{\mathcal{S}} M_{z_i} P_{\mathcal{S}} M_{z_i}^* P_{\mathcal{S}} = M_{z_i} P_{\mathcal{S}} M_{z_i}^*,$$

for $i = 1, \ldots, n$, we have

$$\begin{aligned} &\langle (P_{\mathcal{S}} - \sum_{i=1}^n P_{\mathcal{S}} M_{z_i} P_{\mathcal{S}} M_{z_i}^* P_{\mathcal{S}})(k_n(\cdot, \boldsymbol{w}) \otimes x_l), k_n(\cdot, \boldsymbol{z}) \otimes x_m \rangle \\ &\qquad = \langle (P_{\mathcal{S}} - \sum_{i=1}^n M_{z_i} P_{\mathcal{S}} M_{z_i}^*)(k_n(\cdot, \boldsymbol{w}) \otimes x_l), k_n(\cdot, \boldsymbol{z}) \otimes x_m \rangle \\ &\qquad = k_n^{-1}(\boldsymbol{z}, \boldsymbol{w}) \langle P_{\mathcal{S}}(k_n(\cdot, \boldsymbol{w}) \otimes x_l), k_n(\cdot, \boldsymbol{z}) \otimes x_m \rangle. \end{aligned}$$

Thus

$$\langle k_n(\boldsymbol{z}, \boldsymbol{w}) K(\boldsymbol{z}, \boldsymbol{w}) x_l, x_m \rangle = \langle P_{\mathcal{S}}(k_n(\cdot, \boldsymbol{w}) \otimes x_l), k_n(\cdot, \boldsymbol{z}) \otimes x_m \rangle.$$

This implies that

$$(\boldsymbol{z}, \boldsymbol{w}) \mapsto \big(I_{\mathcal{E}_*} - K(\boldsymbol{z}, \boldsymbol{w})\big) k_n(\boldsymbol{z}, \boldsymbol{w}) = \big(I_{\mathcal{E}_*} - \Theta(\boldsymbol{z}) \Theta(\boldsymbol{w})^*\big) k_n(\boldsymbol{z}, \boldsymbol{w})$$

is a $\mathcal{B}(\mathcal{E}_*)$-valued positive definite kernel, from which it follows that Θ is a multiplier, that is, $\Theta \in \mathcal{M}_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(H^2_n)$. Finally,

$$\begin{aligned}\langle M_\Theta M_\Theta^*(k_n(\cdot,\boldsymbol{w})\otimes x_l), k_n(\cdot,\boldsymbol{z})\otimes x_m\rangle &= \langle k_n(\boldsymbol{z},\boldsymbol{w})\Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^* x_l, x_m\rangle \\ &= \langle k_n(\boldsymbol{z},\boldsymbol{w})K(\boldsymbol{z},\boldsymbol{w})x_l, x_m\rangle \\ &= \langle P_{\mathcal{S}}(k_n(\cdot,\boldsymbol{w})\otimes x_l), k_n(\cdot,\boldsymbol{z})\otimes x_m\rangle,\end{aligned}$$

and hence $P_{\mathcal{S}} = M_\Theta M_\Theta^*$ and that M_Θ a partial isometry. This completes the proof. ∎

In [GRSu02], Green, Richter and Sundberg prove that for almost every $\boldsymbol{\zeta} \in \partial\mathbb{B}^n$ the nontangential limit $\Theta(\boldsymbol{\zeta})$ of the inner multiplier Θ is a partial isometry. Moreover, the rank of $\Theta(\boldsymbol{\zeta})$ is equal to a constant almost everywhere.

5.4. **Solution to a Toeplitz operator equation.** This subsection contains an application of Hilbert module approach to a problem concerning the classical analytic Toeplitz operators. This Toeplitz operator equation problem can be formulated in a more general framework.

Let $\mathcal{S} = M_\Theta H^2_{\mathcal{E}}(\mathbb{D})$ be a M_z-invariant subspace of $H^2_{\mathcal{E}_*}(\mathbb{D})$ for some inner multiplier $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$. Moreover, let $\mathcal{S}$ be invariant under M_Φ for some $\Phi \in \mathcal{B}(\mathcal{E}_*)$. Then

$$\Phi\,\Theta = \Theta\Psi,$$

for some unique $\Psi \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$.

Problem: Determine Ψ, that is, find a representation of the unique multiplier Ψ. If Φ is a polynomial, then under what conditions will Ψ be a polynomial, or a polynomial of the same degree as Φ?

More precisely, given Θ and Φ as above, one seeks a (unique) solution $X \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ to the Toeplitz equation $\Theta X = \Phi\,\Theta$.

This problem appears to be difficult because there are infinitely many obstructions (rather, equations, if one expands Θ and Φ in power series). Thus a priori the answer is not expected to be tractable in general. However, it turns out that if $\Phi(z) = A + A^*z$, then $\Psi = B + B^*z$ for some unique B. The proof is a straightforward application of methods introduced by Agler and Young in [AgYo03]. However, the intuitive idea behind this "guess" is that, Φ turns $H^2_{\mathcal{E}_*}(\mathbb{D})$ into a natural Hilbert module over $\mathbb{C}[z_1, z_2]$ (see Corollary 5.12).

It is now time to proceed to the particular framework for the Toeplitz operator equation problem. Let

$$\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1|, |z_2| \leq 1\} \subseteq \mathbb{C}^2,$$

be the symmetrized bidisc. A Hilbert module $\mathcal{H}$ over $\mathbb{C}[z_1, z_2]$ is said to be Γ-*normal Hilbert module* if M_1 and M_2 are normal operators and $\sigma_{Tay}(M_1, M_2)$, the Taylor spectrum of (M_1, M_2) (see Section 7), is contained in the distinguished boundary of Γ. A Hilbert module $\mathcal{H}$ over $\mathbb{C}[z_1, z_2]$ is said to be Γ-*isometric Hilbert module* if $\mathcal{H}$ is a submodule of a Γ-normal Hilbert module. A Γ-isometric Hilbert module $\mathcal{H}$ is *pure* if M_2 is a shift operator.

Let $\mathcal{E}_*$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{E}_*)$ with $w(A)$, the numerical radius of A, not greater than one. By $[H^2_{\mathcal{E}_*}(\mathbb{D})]_A$ we denote the Hilbert module $H^2_{\mathcal{E}_*}(\mathbb{D})$ with

$$\mathbb{C}[z_1, z_2] \times H^2_{\mathcal{E}_*}(\mathbb{D}) \to H^2_{\mathcal{E}_*}(\mathbb{D}), \qquad (p(z_1, z_2), h) \mapsto p(A + A^* M_z, M_z)h.$$

The following theorem is due to Agler and Young (see [AgYo03]).

THEOREM 5.9. *Let $\mathcal{H}$ be a Hilbert module over $\mathbb{C}[z_1, z_2]$. Then $\mathcal{H}$ is a pure Γ-isometric Hilbert module if and only if $\mathcal{H} \cong [H^2_{\mathcal{E}_*}(\mathbb{D})]_A$ for some Hilbert space $\mathcal{E}_*$, $A \in \mathcal{B}(\mathcal{E}_*)$ and $w(A) \leq 1$.*

Given a Hilbert space $\mathcal{E}_*$ and $A \in \mathcal{B}(\mathcal{E}_*)$ with $w(A) \leq 1$, the Hilbert module $[H^2_{\mathcal{E}_*}(\mathbb{D})]_A$ is called a Γ-*isometric Hardy module* with symbol A.

Now let $\mathcal{S}$ be a non-zero submodule of $[H^2_{\mathcal{E}}(\mathbb{D})]_A$. Then in particular, by the Beurling-Lax-Halmos theorem, Theorem 5.3, we have

$$\mathcal{S} = \Theta H^2_{\mathcal{E}}(\mathbb{D}),$$

for some Hilbert space $\mathcal{E}$ and inner multiplier $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$.

Now everything is in place to state and prove the main result of this subsection.

THEOREM 5.10. *Let $\mathcal{S} \neq \{0\}$ be a closed subspace of $H^2_{\mathcal{E}_*}(\mathbb{D})$ and $A \in \mathcal{B}(\mathcal{E}_*)$ with $w(A) \leq 1$. Then $\mathcal{S}$ is a submodule of $[H^2_{\mathcal{E}_*}(\mathbb{D})]_A$ if and only*

$$(A + A^* M_z) M_\Theta = M_\Theta (B + B^* M_z),$$

for some unique $B \in \mathcal{B}(\mathcal{E})$ (up to unitary equivalence) with $w(B) \leq 1$ where $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$ is the Beurling-Lax-Halmos representation of $\mathcal{S}$.

Proof. Assume that $\mathcal{S}$ be a non-zero submodule of $[H^2_{\mathcal{E}_*}(\mathbb{D})]_A$ and $\mathcal{S} = M_\Theta H^2_{\mathcal{E}}(\mathbb{D})$ be the Beurling-Lax-Halmos representation of $\mathcal{S}$ where $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$ is an inner multiplier and $\mathcal{E}$ is an auxiliary Hilbert space. Also

$$(A + A^* M_z)(M_\Theta H^2_{\mathcal{E}}(\mathbb{D})) \subseteq M_\Theta H^2_{\mathcal{E}}(\mathbb{D}),$$

implies that $(A + A^* M_z) M_\Theta = M_\Theta M_\Psi$ for some unique $\Psi \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$. Therefore,

$$M_\Theta^* (A + A^* M_z) M_\Theta = M_\Psi.$$

Multiplying both sides by M_z^*, one arrives at

$$M_z^* M_\Theta^* (A + A^* M_z) M_\Theta = M_z^* M_\Psi.$$

Then $M_\Theta^* (A M_z^* + A^*) M_\Theta = M_z^* M_\Psi$ and hence, $M_z^* M_\Psi = M_\Psi^*$, or equivalently, $M_\Psi = M_\Psi^* M_z$. Since $\|M_\Psi\| \leq 2$, it follows that (M_Ψ, M_z) is a Γ-isometry. By Theorem 5.9, it follows that

$$M_\Psi = B + B^* M_z,$$

for some $B \in \mathcal{B}(\mathcal{E})$ and $w(B) \leq 1$, and uniqueness of B follows from that of Ψ.
The converse part is trivial, and the proof is complete. ∎

One of the important applications of the above theorem is the following result concerning Toeplitz operators with analytic polynomial symbols of the form $A + A^* z$.

THEOREM 5.11. *Let $\mathcal{S} = M_\Theta H^2_{\mathcal{E}}(\mathbb{D}) \subseteq H^2_{\mathcal{E}_*}(\mathbb{D})$ be a non-zero M_z-invariant subspace of $H^2_{\mathcal{E}_*}(\mathbb{D})$ and $A \in \mathcal{B}(\mathcal{E}_*)$. Then $\mathcal{S}$ is invariant under the Toeplitz operator with analytic polynomial symbol $A + A^* z$ if and only if there exists a unique operator $B \in \mathcal{B}(\mathcal{E})$ such that*

$$(A + A^* z)\Theta = \Theta(B + B^* z).$$

The following result relates Theorem 5.10 to module maps of Γ-isometric Hardy modules.

COROLLARY 5.12. *Let $\mathcal{S} \neq \{0\}$ be a closed subspace of $H^2_{\mathcal{E}_*}(\mathbb{D})$. Then $\mathcal{S}$ is a submodule of the Γ-isometric Hardy module $[H^2_{\mathcal{E}_*}(\mathbb{D})]_A$ with symbol A if and only if there exists a Γ-isometric Hardy module $[H^2_{\mathcal{E}}(\mathbb{D})]_B$ with a unique symbol $B \in \mathcal{B}(\mathcal{E})$ and an isometric module map*

$$U : [H^2_{\mathcal{E}}(\mathbb{D})]_B \longrightarrow [H^2_{\mathcal{E}_*}(\mathbb{D})]_A,$$

such that $\mathcal{S} = UH^2_{\mathcal{E}}(\mathbb{D})$.

Another application of Theorem 5.10 concerns unitary equivalence of Γ-isometric Hardy module submodules.

COROLLARY 5.13. *A non-zero submodule of a Γ-isometric Hardy module is isometrically isomorphic with a Γ-isometric Hardy module.*

Further results and comments:

(1) The classification result of invariant subspaces, Corollary 5.4, is due to Beurling [Beu49]. The Beurling-Lax-Halmos theorem was obtained by Lax [La59] and Halmos [Ha61] as a generalization of Beurling's theorem (see [NaFo70]). See also the generalization by Ball and Helton in [BaHe83].
The simple proof of the Beurling-Lax-Halmos theorem presented here requires the von Neumann-Wold decomposition theorem which appeared about two decades earlier than Beurling's classification result on invariant subspaces of $H^2(\mathbb{D})$.
(2) Let $\mathcal{S} \neq \{0\}$ be a submodule of $H^2(\mathbb{D})$. Then the wandering subspace of $\mathcal{S}$, $\mathcal{S} \ominus z\mathcal{S}$, has dimension one. However, in contrast with the Hardy module $H^2(\mathbb{D})$, the dimension of the generating subspace $\mathcal{S} \ominus z\mathcal{S}$ of a submodule $\mathcal{S}$ of the Bergman module $L^2_a(\mathbb{D})$ could be any number in the range $1, 2, \ldots$ including ∞. This follows from the dilation theory developed by Apostol, Bercovici, Foias and Pearcy (see [ApBerFP85]).
(3) Beurling type theorem for the Bergman space, Theorem 5.6, is due to Aleman, Richter and Sundberg. This result was further generalized by Shimorin [Sh01] in the context of operators close to isometries. His results include the Dirichlet space on the unit disc. A several variables analogue of the wandering subspace problem for the Bergman space over $\mathbb{D}^n$ is proposed in [CDSaS14].
(4) See [III10] for a simple and ingenious proof of the Aleman-Richter-Sundberg theorem concerning invariant subspaces of the Bergman space.
(5) The proof of Theorem 5.1 is from [Sa14c]. It is slightly simpler than the one in [NaFo70] and [FoFr90]. Theorem 5.10 and Corollary 5.12 are due to the author. Theorem 5.7 is due to Douglas ([Do11]).
(6) Theorem 5.8 is due to McCollough and Trent [SMT00]. For more related results in one variable, see the article by Jury [Ju05]. See [Sa13c] for a new approach to Theorem 5.8.
(7) One possible approach to solve the problem mentioned in the last subsection is to consider first the finite dimension case, that is, $\mathcal{E}_* = \mathbb{C}^k$ for $k > 1$.
(8) Let $\mathcal{S} \neq \{0\}$ be a closed subspace of $H^2_{H^2(\mathbb{D}^{n-1})}(\mathbb{D})$. By Beurling-Lax-Halmos theorem, that $\mathcal{S}$ is a submodule of $H^2_{H^2(\mathbb{D}^{n-1})}(\mathbb{D})$ if and only if $\mathcal{S} = \Theta H^2_{\mathcal{E}_*}(\mathbb{D})$, for some closed

subspace $\mathcal{E}_* \subseteq H^2(\mathbb{D}^{n-1})$ and inner function $\Theta \in H^\infty_{\mathcal{L}(\mathcal{E}_*, H^2(\mathbb{D}^{n-1}))}(\mathbb{D})$. Here one is naturally led to formulate the following problem.
Problem: For which closed subspace $\mathcal{E}_* \subseteq H^2(\mathbb{D}^{n-1})$ and inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*, H^2(\mathbb{D}^{n-1}))}(\mathbb{D})$ the submodule $\Theta H^2_{\mathcal{E}_*}(\mathbb{D})$ of $H^2_{H^2(\mathbb{D}^{n-1})}(\mathbb{D})$, realized as a subspace of $H^2(\mathbb{D}^n)$, is a submodule of $H^2(\mathbb{D}^n)$?
This problem is hard to tackle in general. However, see [Sa13a] for some partial results.

(9) The Beurling-Lax-Halmos theorem for submodules of vector-valued Hardy modules can be restated by saying that the non-trivial submodules of $H^2_{\mathcal{E}_*}(\mathbb{D})$ are the images of vector-valued Hardy modules under partially isometric module maps (see [RaRo73]). This classification result for $C_{\cdot 0}$-contractive Hilbert modules over $A(\mathbb{D})$ have also been studied (see [Sa13b]).

THEOREM 5.14. *Let $\mathcal{H}$ be a $C_{\cdot 0}$-contractive Hilbert module over $A(\mathbb{D})$ and $\mathcal{S}$ be a non-trivial closed subspace of $\mathcal{H}$. Then $\mathcal{S}$ is a submodule of $\mathcal{H}$ if and only if there exists a Hilbert space $\mathcal{E}$ and a partially isometric module map $\Pi : H^2_{\mathcal{E}}(\mathbb{D}) \to \mathcal{H}$ such that*

$$\mathcal{S} = ran\ \Pi,$$

or equivalently,

$$P_{\mathcal{S}} = \Pi\Pi^*.$$

An analogous assertion is true also for Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$ (see [Sa13c]).

(10) Let $\mathcal{E}$ and $\mathcal{E}_*$ be Hilbert spaces and $m \in \mathbb{N}$. Let $\Theta \in \mathcal{M}(H^2_{\mathcal{E}}(\mathbb{D}), L^2_{a,m}(\mathbb{D}) \otimes \mathcal{E}_*)$ be a partially isometric multiplier. It follows easily from the definition of multipliers that $\Theta H^2_{\mathcal{E}}(\mathbb{D})$ is a submodule of $L^2_{a,m}(\mathbb{D}) \otimes \mathcal{E}_*$. The following converse was proved by Ball and Bolotnikov in [BaBo13b] (see also [BaBo13a] and Olofsson [Ol06]).

THEOREM 5.15. *Let $\mathcal{S}$ be a non-trivial submodule of the vector-valued weighted Bergman module $L^2_{a,m}(\mathbb{D}) \otimes \mathcal{E}_*$. Then there exists a Hilbert space $\mathcal{E}$ and partially isometric multiplier $\Theta \in \mathcal{M}(H^2_{\mathcal{E}}(\mathbb{D}), L^2_{a,m}(\mathbb{D}) \otimes \mathcal{E}_*)$ such that*

$$\mathcal{S} = \Theta H^2_{\mathcal{E}}(\mathbb{D}).$$

Another representation for $\mathcal{S}$, a submodule of $L^2_{a,m}(\mathbb{D}) \otimes \mathcal{E}_*$, is based on the observation that for any such $\mathcal{S}$, the subspace $z^k\mathcal{S} \ominus z^{k+1}\mathcal{S}$ can be always represented as $z^k\Theta_k\mathcal{U}_k$ for an appropriate subspace $\mathcal{U}_k$ and an $L^2_{a,m}(\mathbb{D}) \otimes \mathcal{E}_*$-inner function $z^k\Theta_k$, $k \geq 0$. This observation leads to the orthogonal representation:

$$\mathcal{S} = \oplus_{k\geq 0}(z^k\mathcal{S} \ominus z^{k+1}\mathcal{S}) = \oplus_{k\geq 0} z^k\Theta_k\mathcal{U}_k,$$

of $\mathcal{S}$ in terms of a Bergman-inner family $\{\Theta_k\}_{k\geq 0}$ (see [BaBo13b] and [BaBo13a] for more details).
More recently, Theorem 5.15 has been extended by the author [Sa13c], [Sa13b] to the case of reproducing kernel Hilbert modules.

6. Unitarily equivalent submodules

Let $\mathcal{H} \subseteq \mathcal{O}(\mathbb{D}, \mathbb{C})$ be a reproducing kernel Hilbert module and $\mathcal{S}_1$ and $\mathcal{S}_2$ be two non-zero submodules of $\mathcal{H}$.

(1) If $\mathcal{H} = H^2(\mathbb{D})$, then $\mathcal{S}_1 \cong \mathcal{S}_2$ (see Corollary 5.5).

(2) If $\mathcal{H} = L^2_a(\mathbb{D})$ and $\mathcal{S}_1 \cong \mathcal{S}_2$, then $\mathcal{S}_1 = \mathcal{S}_2$ (see [R88] or Corollary 8.5 in [Sa14a]).

Therefore, on the one hand every non-zero submodule is isometrically isomorphic to the module itself while on the other hand, no proper submodule is.
Now let $n > 1$. For submodules of $H^2(\mathbb{D}^n)$ over $A(\mathbb{D}^n)$, some are unitarily equivalent to $H^2(\mathbb{D}^n)$ and some are not (cf. [Ru69], [Ma85], [SaASW13]). For the Hardy module $H^2(\partial\mathbb{B}^n)$, the existence of inner functions on $\mathbb{B}^n$ [Ale82] established the existence of proper submodules of $H^2(\partial\mathbb{B}^n)$ that are unitarily equivalent to $H^2(\mathbb{B}^n)$.

These observations raise a number of interesting questions concerning Hilbert modules with unitarily equivalent submodules. The purpose of this section is to investigate and classify a class of Hilbert modules with proper submodules unitarily equivalent to the original.

6.1. **Isometric module maps.** This subsection begins with a simple observation concerning unitarily equivalent submodules of Hilbert modules. Let $\mathcal{H}$ be a Hilbert module over $A(\Omega)$ and $\mathcal{S}$ be a non-trivial submodule of $\mathcal{H}$. Then $\mathcal{S}$ is unitarily equivalent to $\mathcal{H}$ if and only if $\mathcal{S} = U\mathcal{H}$ for some isometric module map U on $\mathcal{H}$.

Now let $U\mathcal{H}$ be a submodule of $\mathcal{H}$ for some isometric module map U. Then $U\mathcal{H}$ is said to be *pure* unitarily equivalent submodule of $\mathcal{H}$ if

$$\bigcap_{k\geq 0}^{\infty} U^k\mathcal{H} = \{0\}.$$

Proposition 6.1. *Let $\mathcal{H}$ be a Hilbert module over $A(\Omega)$ for which there exists an isometric module map U satisfying $\bigcap_{k=0}^{\infty} U^k\mathcal{H} = (0)$. Then there exists an isomorphism $\Psi\colon\ H^2_{\mathcal{W}}(\mathbb{D}) \to \mathcal{H}$ with $\mathcal{W} = \mathcal{H} \ominus U\mathcal{H}$ and a commuting n-tuple of functions $\{\varphi_i\}$ in $H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ so that $U = \Psi M_z \Psi^*$ and $M_i = \Psi M_{\varphi_i} \Psi^*$ for $i = 1, 2, \ldots, n$.*

Proof. By Corollary 5.2, there is a canonical isomorphism $\Psi\colon\ H^2_{\mathcal{W}}(\mathbb{D}) \to \mathcal{H}$ such that $\Psi T_z = U\Psi$ where $\mathcal{W} = \mathcal{H} \ominus U\mathcal{H}$. Further, $X_i = \Psi^* M_i \Psi$ is an operator on $H^2_{\mathcal{E}}(\mathbb{D})$ which commutes with T_z. Hence, there exists a function φ_i in $H^\infty_{\mathcal{L}(\mathcal{W})}(\mathbb{D})$ such that $X_i = M_{\varphi_i}$. Moreover, since the $\{M_i\}$ commute, so do the $\{X_i\}$ and hence the functions $\{\varphi_i\}$ commute pointwise a.e. on $\mathbb{T}$. ■

6.2. **Hilbert-Samuel Polynomial.** A Hilbert module $\mathcal{H}$ over $\mathbb{C}[z]$ is said to be semi-Fredholm at $\boldsymbol{w} \in \mathbb{C}^n$ if

$$\dim[\mathcal{H}/I_{\boldsymbol{w}} \cdot \mathcal{H}] < \infty.$$

In particular, note that $\mathcal{H}$ semi-Fredholm at $\boldsymbol{w}$ implies that $I_{\boldsymbol{w}} \cdot \mathcal{H}$ is a closed submodule of $\mathcal{H}$ and

$$\dim[I^k_{\boldsymbol{w}} \cdot \mathcal{H}/I^{k+1}_{\boldsymbol{w}} \cdot \mathcal{H}] < \infty,$$

for all $k \in \mathbb{N}$. In this case the direct sum

$$\mathrm{gr}(\mathcal{H}) := \bigoplus_{k\geq 0} I_{\boldsymbol{w}}^k \cdot \mathcal{H}/I_{\boldsymbol{w}}^{k+1} \cdot \mathcal{H},$$

can be turned into a graded finitely generated $\mathbb{C}[\boldsymbol{z}]$-module. It is a fundamental result of commutative algebra that to any such module there is a polynomial $h_{\mathcal{H}}^{\boldsymbol{w}} \in \mathbb{Q}[x]$ of degree not greater than n, the *Hilbert-Samuel polynomial*, with

$$h_{\mathcal{H}}^{\boldsymbol{w}}(k) = \dim[I_{\boldsymbol{w}}^k \cdot \mathcal{H}/I_{\boldsymbol{w}}^{k+1} \cdot \mathcal{H}],$$

for all $k \geq N_{\mathcal{H}}$ for some positive integer $N_{\mathcal{H}}$ (see [DoYa93]).

In some cases it is possible to calculate the Hilbert-Samuel polynomial for a Hilbert module directly. For example (see [Fa03]), let Ω be a Reinhardt domain in $\mathbb{C}^n$ and $\mathcal{H} \subseteq \mathcal{O}(\Omega, \mathbb{C})$ be a reproducing kernel Hilbert module. Let $\mathcal{S}$ be a singly generated submodule of $\mathcal{H}$ and $\boldsymbol{w} \in \Omega$. Then

$$h_{\mathcal{S}}^{\boldsymbol{w}}(k) = \binom{n+k-1}{n}.$$

In general, it is difficult to compute the Hilbert-Samuel polynomial directly.

The following result demonstrates that the Hilbert-Samuel polynomial does not depend on the choice of a quasi-free Hilbert module.

THEOREM 6.2. *If $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ be a pair of rank m quasi-free Hilbert modules over $A(\Omega)$. If both $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are semi-Fredholm at $\boldsymbol{w} \in \Omega$ then $h_{\mathcal{R}}^{\boldsymbol{w}} \equiv h_{\widetilde{\mathcal{R}}}^{\boldsymbol{w}}$.*

Proof. Consider rank m quasi-free Hilbert modules $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ over $A(\Omega)$ with $1 \leq m < \infty$. Following Lemma 1 in [DoMi05], construct the rank m quasi-free Hilbert module Δ, which is the graph of a closed densely defined module map from $\mathcal{R}$ to $\widetilde{\mathcal{R}}$ obtained as the closure of the set $\{\varphi f_i \oplus \varphi g_i \colon\ \varphi \in A(\Omega)\}$, where $\{f_i\}_{i=1}^m$ and $\{g_i\}_{i=1}^m$ are generators for $\mathcal{R}$ and $\widetilde{\mathcal{R}}$, respectively. Then the module map $X\colon\ \Delta \to \mathcal{R}$ defined by $f_i \oplus g_i \to f_i$ is bounded, one-to-one and has dense range. Note that for fixed $\boldsymbol{w}_0$ in Ω,

$$X^*(I_{\boldsymbol{w}_0} \cdot \mathcal{R})^{\perp} \subset (I_{\boldsymbol{w}_0} \cdot \Delta)^{\perp}.$$

Since the rank of Δ is also k, this map is an isomorphism. Let $\{\gamma_i(\boldsymbol{w}_0)\}$ be anti-holomorphic functions from a neighborhood Ω_0 of $\boldsymbol{w}_0$ to $\mathcal{R}$ such that $\{\gamma_i(\boldsymbol{w})\}$ spans $(I_{\boldsymbol{w}} \cdot \mathcal{R})^{\perp}$ for $\boldsymbol{w} \in \Omega_0$. Then

$$\{\frac{\partial^{\boldsymbol{k}}}{\partial z^{\boldsymbol{k}}} \gamma_i(\boldsymbol{w})\}_{|\boldsymbol{k}|<k},$$

forms a basis for $(I_{\boldsymbol{w}}^k \cdot \mathcal{R})^{\perp}$ for $k = 0, 1, 2, \ldots$, using the same argument as in Section 4 in [CuSal84] and Section 4 in [DoMiVa00]. Similarly, since $\{X^*\gamma_i(\boldsymbol{w})\}$ is a basis for $(I_{\boldsymbol{w}} \cdot \Delta)^{\perp}$, it follows that X^* takes $(I_{\boldsymbol{w}}^k \cdot \mathcal{R})^{\perp}$ onto $(I_{\boldsymbol{w}}^k \cdot \Delta)^{\perp}$ for $k = 0, 1, 2, \ldots$. Therefore

$$\dim(I_{\boldsymbol{w}}^k \cdot \mathcal{R})^{\perp} = \dim(I_{\boldsymbol{w}}^k \cdot \Delta)^{\perp},$$

for all k. Hence

$$h_{\mathcal{R}}^{\boldsymbol{w}} = h_{\Delta}^{\boldsymbol{w}}. \qquad (\boldsymbol{w} \in \Omega)$$

The result now follows by interchanging the roles of $\mathcal{R}$ and $\widetilde{\mathcal{R}}$. ∎

In particular, one can calculate the Hilbert-Samuel polynomial by considering only the Bergman module over $A(\Omega)$ since

$$h^{\boldsymbol{w}}_{\mathcal{R}\otimes\mathbb{C}^k} \equiv k h^{\boldsymbol{w}}_{\mathcal{R}},$$

for all finite integers k. To accomplish that one can reduce to the case of a ball as follows.

Theorem 6.3. *If $\mathcal{R}$ is a quasi-free Hilbert module over $A(\Omega)$ for $\Omega \subset \mathbb{C}^n$ which is semi-Fredholm for $\boldsymbol{w}$ in a neighborhood of $\boldsymbol{w}_0$ in Ω with constant codimension, then $h^{\boldsymbol{w}_0}_{\mathcal{R}}$ has degree n.*

Proof. Let $B_\varepsilon(\boldsymbol{w}_0)$ be a ball with radius ε centered at $\boldsymbol{w}_0$, whose closure is contained in Ω. An easy argument shows that the map $X\colon\ L^2_a(\Omega) \to L^2_a(B_\varepsilon(\boldsymbol{w}_0))$ defined by

$$Xf \equiv f|_{B_\varepsilon(\boldsymbol{w}_0)}, \qquad (f \in L^2_a(\Omega))$$

is bounded, one-to-one and has dense range. Moreover, by a similar argument to the one used in Theorem 6.2 for $\boldsymbol{w} \in B_\varepsilon(\boldsymbol{w}_0)$, it follows that

$$h^{\boldsymbol{w}}_{L^2_a(\Omega)} \equiv h^{\boldsymbol{w}}_{B_\varepsilon(\boldsymbol{w})}.$$

The proof is completed by considering the Hilbert–Samuel polynomials at $\boldsymbol{w}_0$ of the Bergman module for the ball $B_\varepsilon(\boldsymbol{w}_0)$ for some $\varepsilon > 0$ which is centered at $\boldsymbol{w}_0$. This calculation reduces to that of the module $\mathbb{C}[\boldsymbol{z}]$ over the algebra $\mathbb{C}[\boldsymbol{z}]$ since the monomials in $L^2_a(B_\varepsilon(\boldsymbol{w}_0))$ are orthogonal. Hence

$$h^{\boldsymbol{w}_0}_{L^2_a(B_\varepsilon(\boldsymbol{w}_0))}(k) = \binom{n+k-1}{n}.$$

This completes the proof. ∎

6.3. **On complex dimension.** The purpose of this subsection is to show that the complex dimension of the domain Ω is one, that is $n = 1$, whenever $\mathcal{H}$ is quasi-free, semi-Fredholm and $\dim \mathcal{H}/U\mathcal{H} < \infty$.

The following result relates pure isometrically isomorphic submodules of finite codimension and linear Hilbert-Samuel polynomials.

Theorem 6.4. *If $\mathcal{H}$ is semi-Fredholm at $\boldsymbol{w}_0$ in Ω and $\mathcal{S}$ is a pure isometrically isomorphic submodule of $\mathcal{H}$ having finite codimension in $\mathcal{H}$, then $h^{\boldsymbol{w}_0}_{\mathcal{H}}$ has degree at most one.*

Proof. As in the proof of Proposition 6.1, the existence of $\mathcal{S}$ in $\mathcal{H}$ yields a module isomorphism Ψ of $\mathcal{H}$ with $H^2_{\mathcal{W}}(\mathbb{D})$ for $\mathcal{W} = \mathcal{H} \ominus \mathcal{S}$. Assume that $\boldsymbol{w}_0 = 0$ for simplicity and note that the assumption that $\mathcal{H}$ is semi-Fredholm at $\boldsymbol{w}_0 = 0$ implies that

$$M_{z_1} \cdot \mathcal{H} + \cdots + M_{z_n} \cdot \mathcal{H},$$

has finite codimension in $\mathcal{H}$. Hence

$$\widetilde{\mathcal{S}} = M_{\varphi_1} \cdot H^2_{\mathcal{W}}(\mathbb{D}) + \cdots + M_{\varphi_n} \cdot H^2_{\mathcal{W}}(\mathbb{D}),$$

has finite codimension in $H^2_{\mathcal{W}}(\mathbb{D})$, where $M_{z_i} = \Psi M_{\varphi_i} \Psi^*$. Moreover, $\widetilde{\mathcal{S}}$ is invariant under the action of M_z. Therefore, by the Beurling–Lax–Halmos Theorem, Theorem 5.3, there is an inner function Θ in $H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ for which $\widetilde{\mathcal{S}} = \Theta H^2_{\mathcal{W}}(\mathbb{D})$. Further, since $\widetilde{\mathcal{S}}$ has finite codimension in $H^2_{\mathcal{W}}(\mathbb{D})$ and the dimension of $\mathcal{W}$ is finite, it follows that the matrix entries of

Θ are rational functions with poles outside the closed unit disk and $\Theta(e^{it})$ is unitary for e^{it} in $\mathbb{T}$ (cf. [NaFo70], Chapter VI, Section 4).

Now the determinant, $\det\Theta$, is a scalar-valued rational inner function in $H^\infty(\mathbb{D})$ and hence is a finite Blaschke product. Using Cramer's Rule one can show that (cf. [He64], Theorem 11)

$$(\det\Theta)H^2_{\mathcal{W}}(\mathbb{D}) \subseteq \Theta H^2_{\mathcal{W}}(\mathbb{D}),$$

which implies that

$$\dim_{\mathbb{C}} H^2_{\mathcal{W}}(\mathbb{D})/\Theta H^2_{\mathcal{W}}(\mathbb{D}) \leq \dim_{\mathbb{C}} H^2_{\mathcal{W}}(\mathbb{D})/(\det\Theta)H^2_{\mathcal{W}}(\mathbb{D}).$$

Continuing, we have

$$\begin{aligned}\Psi(I^2_{\boldsymbol{w}_0}\cdot\mathcal{H}) &= \Psi\left(\bigvee_{i,j=1}^n M_{z_i}M_{z_j}\mathcal{H}\right) = \bigvee_{i,j=1}^n M_{\varphi_i}M_{\varphi_j}H^2_{\mathcal{W}}(\mathbb{D}) = \bigvee_{i=1}^n M_{\varphi_i}(\Theta H^2_{\mathcal{W}}(\mathbb{D}))\\ &\supseteq \bigvee_{i=1}^n M_{\varphi_i}(\det\Theta)H^2_{\mathcal{W}}(\mathbb{D}) \supseteq \bigvee_{i=1}^n \det\Theta(M_{\varphi_i}H^2_{\mathcal{W}}(\mathbb{D})) = (\det\Theta)\Theta H^2_{\mathcal{W}}(\mathbb{D})\\ &\supseteq (\det\Theta)^2H^2_{\mathcal{W}}(\mathbb{D}).\end{aligned}$$

Therefore

$$\dim(\mathcal{H}/I^2_{\boldsymbol{w}_0}\cdot\mathcal{H}) \leq \dim H^2_{\mathcal{W}}(\mathbb{D})/(\det\Theta)^2H^2_{\mathcal{W}}(\mathbb{D}).$$

Proceeding by induction, one arrives at

$$\dim(\mathcal{H}/I^k_{\boldsymbol{w}_0}\cdot\mathcal{H}) \leq \dim H^2_{\mathcal{W}}(\mathbb{D})/(\det\Theta)^kH^2_{\mathcal{W}}(\mathbb{D}),$$

for each positive integer k. Also

$$h^{\boldsymbol{w}_0}_{\mathcal{H}}(k) \leq \dim H^2_{\mathcal{W}}(\mathbb{D})/(\det\Theta)^kH^2_{\mathcal{W}}(\mathbb{D}) = kd\dim\mathcal{W} \quad \text{for} \quad k \geq N_{\mathcal{H}},$$

where d is the dimension of $H^2/(\det\Theta)H^2$. Hence, the degree of $h^{\boldsymbol{w}_0}_{\mathcal{H}}$ is at most one. ■

Combining this theorem with Theorem 6.3 yields the following result.

THEOREM 6.5. *If $\mathcal{R}$ is a semi-Fredholm, quasi-free Hilbert module over $A(\Omega)$ with $\Omega \subset \mathbb{C}^n$ having a pure isometrically isomorphic submodule of finite codimension, then $n = 1$.*

6.4. **Hilbert modules over $A(\mathbb{D})$.** By virtue of Theorem 6.5, one can immediately reduce to the case of domains Ω in $\mathbb{C}$ if there exists a pure isometrically isomorphic submodule of finite codimension.

The purpose of this subsection is to prove that for a quasi-free Hilbert module $\mathcal{R}$ over $A(\mathbb{D})$, the existence of a pure unitarily equivalent submodule of finite codimension implies that $\mathcal{R}$ is unitarily equivalent to $H^2_{\mathcal{E}}(\mathbb{D})$ with $\dim\mathcal{E} = \text{rank}\mathcal{R}$.

THEOREM 6.6. *Let $\mathcal{R}$ be a finite rank, quasi-free Hilbert module over $A(\mathbb{D})$ which is semi-Fredholm for ω in $\mathbb{D}$. Assume there exists a pure module isometry U such that $\dim\mathcal{R}/U\mathcal{R} < \infty$. Then $\mathcal{R}$ and $H^2_{\mathcal{E}}(\mathbb{D})$ are $A(\mathbb{D})$-module isomorphic where $\mathcal{E}$ is a Hilbert space with dim $\mathcal{E}$ equal to the multiplicity of $\mathcal{R}$.*

Proof. As in Proposition 6.1, without loss of generality one can assume that $\mathcal{R} \cong H^2_{\mathcal{W}}(\mathbb{D})$, where $\mathcal{W} = \mathcal{R} \ominus U\mathcal{R}$ with $\dim \mathcal{W} < \infty$ and U corresponds to M_z. Let M_φ denote the operator on $H^2_{\mathcal{W}}(\mathbb{D})$ unitarily equivalent to module multiplication by z on $\mathcal{R}$, where φ is in $H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ with $\|\varphi(z)\| \leq 1$ for all z in $\mathbb{D}$.
Since the operator M_φ is defined by module multiplication on $H^2_{\mathcal{W}}(\mathbb{D})$ and the corresponding $A(\mathbb{D})$-module has finite rank, it is enough to show that φ is inner. Hence M_φ would be a pure isometry so that $H^2_{\mathcal{W}}(\mathbb{D})$ and $H^2(\mathbb{D})$ would be $A(\mathbb{D})$-module isomorphic.
Since the range of $M_\varphi - wI$ has finite codimension in $H^2_{\mathcal{W}}(\mathbb{D})$, it follows that the operator $M_\varphi - wI$ has closed range for each $w \in \mathbb{D}$. Now $\ker(M_\varphi - wI) = \{0\}$, by Lemma 1.1 in [DoSa08], implies that $M_\varphi - wI$ is bounded below. Then by Lemma 2.1 in [DoSa08], $(L_\varphi - wI)$ is bounded below on $L^2_{\mathcal{E}}(\mathbb{D})$, where L_φ is the Laurent operators with symbol φ.
For each w in $\mathbb{D}$ and k in $\mathbb{N}$, define

$$E^w_k = \{e^{it} :\ \operatorname{dist}(\sigma(\varphi(e^{it})), w) < \frac{1}{k}\},$$

where $\sigma(\varphi(e^{it}))$ denotes the spectrum of the matrix $\varphi(e^{it})$.
Then either $\sigma(\varphi(e^{it})) \subset \mathbb{T}$ a.e or there exists a w_0 in $\mathbb{D}$ such that $m(E^{w_0}_k) > 0$ for all $k \in \mathbb{N}$. In the latter case, one can find a sequence of functions $\{f_k\}$ in $L^2_{\mathcal{E}}(\mathbb{T})$ such that f_k is supported on $E^{w_0}_k$, $\|f_k(e^{it})\| = 1$ for e^{it} in $E^{\omega_0}_k$ and

$$\|\varphi(e^{it}) f_k(e^{it}) - w_0 f_k(e^{it})\| \leq \frac{1}{k}.$$

It then follows that

$$\|(L_\varphi - w_0) f_k\| \leq \frac{1}{k}\|f_k\|$$

for all k in $\mathbb{N}$, which contradicts the fact that $L_\varphi - w_0 I$ is bounded below. Hence, $\sigma(\varphi(e^{it})) \subset \mathbb{T}$, a.e. and hence $\varphi(e^{it})$ is unitary a.e. Therefore, T_φ is a pure isometry and the Hilbert module $H^2_{\mathcal{E}}(\mathbb{D})$ determined by T_φ is $A(\mathbb{D})$-module isomorphic with $H^2_{\mathcal{E}}(\mathbb{D})$. ■

This result can not be extended to the case in which U is not pure. For example, for $\mathcal{R} = H^2(\mathbb{D}) \oplus L^2_a(\mathbb{D})$, one could take $U = M_z \oplus I$.

Further results and comments:

(1) All of the material in this section is taken from [DoSa08].
(2) For the Bergman modules over the unit ball, one can show (cf. [XCGu03, P94, R88]) that no proper submodule is unitarily equivalent to the Bergman module itself. These issues are thoroughly discussed in [Sa14a].
(3) In a sense, the existence of a Hilbert module with unitarily equivalent submodules is a rare phenomenon. The following example shows that the problem is more complicated even in the sense of quasi-similarity.
Example: The Hardy module $H^2(\mathbb{D}^2)$ is not quasi-similar to the submodule $H^2(\mathbb{D}^2)_0 = \{f \in H^2(\mathbb{D}^2) : f(0) = 0\}$ of $H^2(\mathbb{D}^2)$. Suppose X and Y define a quasi-affinity between $H^2(\mathbb{D}^2)$ and $H^2(\mathbb{D}^2)_0$. Then the localized maps X_0 and Y_0 are isomorphisms between $\mathbb{C}_0$ and $\mathbb{C}_0 \oplus \mathbb{C}_0$ (see Section 4) which is impossible.
(4) Theorem 6.6 can be extended to the case of a finitely-connected domain Ω with a nice boundary, that is, Ω for which $\partial\Omega$ is the finite union of simple closed curves. Here

it is convenient to recall the notion of the *bundle shift* $H^2_\alpha(\Omega)$ for Ω determined by the unitary representation α of the fundamental group $\pi_1(\Omega)$ of Ω. The bundle shift $H^2_\alpha(\Omega)$ is the Hardy space of holomorphic sections of the flat unitary bundle over Ω determined by α (see [AbDo76], [Ba79]). The reader is referred to ([DoSa08], Theorem 2.8) for a proof of the following theorem.

THEOREM 6.7. *Let $\mathcal{R}$ be a finite rank, quasi-free Hilbert module over $A(\Omega)$, where Ω is a finitely-connected domain in $\mathbb{C}$ with nice boundary, which is semi-Fredholm for ω in Ω. Let U be a pure module isometry such that $\dim \mathcal{R}/U\mathcal{R} < \infty$. Then there is a unitary representation α of $\pi^1(\Omega)$ on some finite dimensional Hilbert space such that $\mathcal{R}$ and the bundle shift $H^2_\alpha(\Omega)$, are $A(\Omega)$-module isomorphic.*

(5) In [Do11], Douglas proved the following result on rank one quasi-free Hilbert modules.

THEOREM 6.8. *Let $\mathcal{R}$ be a rank one quasi-free Hilbert module over $A(\Omega)$, where $\Omega = \mathbb{B}^n$ or $\mathbb{D}^n$. Suppose each submodule $\mathcal{S}$ of $\mathcal{R}$ is isometrically isomorphic to $\mathcal{R}$. Then $n = 1$ and $\mathcal{R} \cong H^2(\mathbb{D})$ and the module map M on $\mathcal{R}$ is the Toeplitz operator M_φ, where φ is a conformal self map of $\mathbb{D}$ onto itself.*

(6) The notion of Hilbert-Samuel polynomials for Hilbert modules is a relatively new concept and was introduced by Douglas and Yan in 1993 [DoYa93]. Because of its strong interaction with commutative algebra and complex analytic geometry, Hilbert module approach to Hilbert-Samuel polynomial and Samuel multiplicity has had a spectacular development since its origin. The reader is referred to the recent work by Eschmeier [Es08a], [Es08b], [Es07b], [Es07a] and Fang [Fa06], [Fa08], [Fa09].

7. Corona condition and Fredholm Hilbert modules

The purpose of this section is to apply techniques from Taylor's theory, in terms of Koszul complex, Berezin transforms and reproducing kernel method to quasi-free Hilbert modules and obtain a connection between Fredholm theory and corona condition.

7.1. **Koszul complex and Taylor invertibility.** In this subsection, the notion of Taylor's invertibility (see [Ta70a], [Ta70b]) for commuting tuples of operators on Hilbert spaces will be discussed.

Let $\mathcal{E}^n$ be the exterior algebra generated by n symbols $\{e_1, \ldots, e_n\}$ along with identity e_0, that is, $\mathcal{E}^n$ is the algebra of forms in $\{e_1, \ldots, e_n\}$ with complex coefficients and $e_i \wedge e_j = -e_j \wedge e_i$ for all $1 \leq i, j \leq n$. Let $\mathcal{E}^n_k$ be the vector subspace of $\mathcal{E}^n$ generated by the basis

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < \ldots < i_k \leq n\}.$$

In particular,

$$\mathcal{E}^n_i \wedge \mathcal{E}^n_j \subseteq \mathcal{E}^n_{i+j},$$

and

$$\mathcal{E}^n = \mathbb{C} e_1 \wedge \cdots \wedge e_n.$$

Moreover

$$\dim \mathcal{E}^n_k = \binom{n}{k},$$

that is, $\mathcal{E}^n_k$ is isomorphic to $\mathbb{C}^{\binom{n}{k}}$ as a vector space over $\mathbb{C}$. Also note that $\mathcal{E}^n$ is graded:

$$\mathcal{E}^n = \sum_{k=0}^{\infty} \mathcal{E}^n_k.$$

Define the *creation operator* $E_i : \mathcal{E}^n \to \mathcal{E}^n$, for each $1 \leq i \leq n$, by $E_i \eta = e_i \wedge \eta$ and $E_0 \eta = \eta$ for all $\eta \in \mathcal{E}^n$. In particular, note that $\mathcal{E}^n$ is a finite dimensional vector space. Then the anticommutation relation follows easily:

$$E_i E_j = -E_j E_i \qquad \text{and} \qquad E_i^* E_j + E_j E_i^* = \delta_{ij} E_0.$$

Now let $T = (T_1, \ldots, T_n)$ be a commuting tuple of operators on $\mathcal{H}$. Let $\mathcal{E}^n(T) = \mathcal{H} \otimes_{\mathbb{C}} \mathcal{E}^n$ and $\mathcal{E}^n_k(T) = \mathcal{H} \otimes_{\mathbb{C}} \mathcal{E}^n_k \subset \mathcal{E}^n(T)$ and define $\partial_T \in \mathcal{B}(\mathcal{E}^n(T))$ by

$$\partial_T = \sum_{i=1}^{n} T_i \otimes E_i.$$

It follows easily from the anticommutation relationship that $\partial_T^2 = 0$. The *Koszul complex* $K(T)$ is now defined to be the (chain) complex

$$K(T) : 0 \longrightarrow \mathcal{E}^n_0(T) \xrightarrow{\partial_{1,\mathcal{H}}} \mathcal{E}^n_1(T) \xrightarrow{\partial_{2,\mathcal{H}}} \cdots \xrightarrow{\partial_{n-1,\mathcal{H}}} \mathcal{E}^n_{n-1}(T) \xrightarrow{\partial_{n,\mathcal{H}}} \mathcal{E}^n_n(T) \longrightarrow 0,$$

where $\mathcal{E}^n_k(T)$ is the collection of all k-forms in $\mathcal{E}^n(T)$ and $\partial_{k,T}$, *the differential*, is defined by

$$\partial_{k,T} = \partial_T|_{\mathcal{E}^n_{k-1}(T)} : \mathcal{E}^n_{k-1}(T) \to \mathcal{E}^n_k(T). \qquad (k = 1, \ldots, n)$$

For each $k = 0, \ldots, n$ the cohomology vector space associated to the Koszul complex $K(T)$ at k-th stage is the vector space

$$H^k(T) = \ker\, \partial_{k+1,T} / \mathrm{ran} \partial_{k,T}.$$

Here $\partial_{0,T}$ and $\partial_{n+1,T}$ are the zero map. A commuting tuple of operators T on $\mathcal{H}$ is said to be *invertible* if $K(T)$ is exact. The *Taylor spectrum* of T is defined as

$$\sigma_{Tay}(T) = \{\boldsymbol{w} \in \mathbb{C}^n : K(T - \boldsymbol{w} I_{\mathcal{H}}) \text{ is not exact } \}.$$

The tuple T is said to be a *Fredholm tuple* if

$$\dim\, \Big[H^k(T)\Big] < \infty, \qquad (k = 0, 1, \ldots, n)$$

and *semi-Fredholm tuple* if the last cohomology group,

$$H^n(T) = \mathcal{H} / \sum_{i=1}^{n} T_i \mathcal{H},$$

of its Koszul complex in finite dimensional.
If T is a Fredholm tuple, then the index of T is

$$\mathrm{ind} T := \sum_{k=0}^{n} (-1)^k \dim\, \Big[H^k(T)\Big].$$

The tuple T is said to be Fredholm (or semi-Fredholm) at $\boldsymbol{w} \in \mathbb{C}^n$ if the tuple $T - \boldsymbol{w} I_{\mathcal{H}}$ is Fredholm (or semi-Fredholm).

Viewing the tuple T as a Hilbert module over $\mathbb{C}[\boldsymbol{z}]$, it follows that T is semi-Fredholm at $\boldsymbol{w}$ if and only if

$$\dim[\mathcal{H}/I_{\boldsymbol{w}} \cdot \mathcal{H}] < \infty.$$

In particular, note that $\mathcal{H}$ semi-Fredholm at $\boldsymbol{w}$ implies that $I_{\boldsymbol{w}}\mathcal{H}$ is a closed submodule of $\mathcal{H}$.

7.2. **Weak corona property.** Let $\{\varphi_1, \ldots, \varphi_k\} \subseteq H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{B}^n)$ be a k-tuple of commuting $\mathcal{B}(\mathcal{E})$-valued functions where $\mathcal{E}$ is a Hilbert space. Then the tuple is said to have the *weak corona property* if there exists an $\epsilon > 0$ and $1 > \delta > 0$ such that

$$\sum_{i=1}^{k} \varphi_i(\boldsymbol{z})\varphi_i(\boldsymbol{z})^* \geq \epsilon I_{\mathcal{E}},$$

for all $\boldsymbol{z}$ satisfying $1 > \|\boldsymbol{z}\| \geq 1 - \delta$.
The tuple $\{\varphi_1, \ldots, \varphi_k\}$ is said to have the *corona property* if

$$\sum_{i=1}^{k} \varphi_i(\boldsymbol{z})\varphi_i(\boldsymbol{z})^* \geq \epsilon I_{\mathcal{E}},$$

for all $\boldsymbol{z} \in \mathbb{B}^n$.

For n=1 and $\mathcal{E} = \mathbb{C}$, the Carleson's corona theorem (see [Car62]) asserts that:

THEOREM 7.1. *(Carleson) A set $\{\varphi_1, \ldots, \varphi_k\}$ in $H^\infty(\mathbb{D})$ satisfies $\sum_{i=1}^k |\varphi_i(z)| \geq \epsilon$ for all z in $\mathbb{D}$ for some $\epsilon > 0$ if and only if there exist $\{\psi_1, \ldots, \psi_k\} \subset H^\infty(\mathbb{D})$ such that*

$$\sum_{i=1}^{k} \varphi_i \psi_i = 1.$$

Also one has the following fundamental result of Taylor (see [Ta70b], Lemma 1):

LEMMA 7.2. *Let $(T_1, \ldots, T_k)$ be in the center of an algebra $\mathcal{A}$ contained in $\mathcal{L}(\mathcal{H})$ such that there exists $(S_1, \ldots, S_k)$ in $\mathcal{A}$ satisfying $\sum_{i=1}^k T_i S_i = I_{\mathcal{H}}$. Then the Koszul complex for $(T_1, \ldots, T_k)$ is exact.*

Now consider a contractive quasi-free Hilbert module $\mathcal{R}$ over $A(\mathbb{D})$ of multiplicity one, which therefore has $H^\infty(\mathbb{D})$ as the multiplier algebra.

PROPOSITION 7.3. *Let $\mathcal{R}$ be a contractive quasi-free Hilbert module over $A(\mathbb{D})$ of multiplicity one and $\{\varphi_1, \ldots, \varphi_k\}$ be a subset of $H^\infty(\mathbb{D})$. Then the Koszul complex for the k-tuple $(M_{\varphi_1}, \ldots, M_{\varphi_k})$ on $\mathcal{R}$ is exact if and only if $\{\varphi_1, \ldots, \varphi_k\}$ satisfies the corona property.*

Proof. If $\sum_{i=1}^k \varphi_i \psi_i = 1$ for some $\{\psi_1, \ldots, \psi_k\} \subset H^\infty(\mathbb{D})$, then the fact that M_Φ is Taylor invertible follows from Lemma 7.2. On the other hand, the last group of the Koszul complex is $\{0\}$ if and only if the row operator M_φ in $\mathcal{B}(\mathcal{R}^k, \mathcal{R})$ is bounded below which, as before, shows that $\sum_{i=1}^k |\varphi_i(z)|$ is bounded below on $\mathbb{D}$. This completes the proof. ∎

The missing step to extend the result from $\mathbb{D}$ to the open unit ball $\mathbb{B}^n$ is the fact that it is unknown if the corona condition for $\{\varphi_1, \ldots, \varphi_k\}$ in $H^\infty(\mathbb{B}^n)$ is equivalent to the Corona property.

7.3. **Semi-Fredholm implies weak corona.** Let $\mathcal{H}_K$ be a scalar-valued reproducing kernel Hilbert space over Ω and $F \in \mathcal{B}(\mathcal{H}_K)$. Then the *Berezin transform* (see [DaDo05]) of F is denoted by $\hat{F}$ and defined by

$$\hat{F}(\boldsymbol{z}) = \langle F \frac{K(\cdot,\boldsymbol{z})}{\|K(\cdot,\boldsymbol{z})\|}, \frac{K(\cdot,\boldsymbol{z})}{\|K(\cdot,\boldsymbol{z})\|}\rangle. \qquad (\boldsymbol{z} \in \Omega)$$

Note that the multiplier space of a rank one quasi-free Hilbert module $\mathcal{R}$ over $A(\mathbb{B}^n)$ is precisely $H^\infty(\mathbb{B}^n)$, since $\mathcal{R}$ is the completion of $A(\mathbb{B}^n)$, by definition (see Proposition 5.2 in [DaDo05]).

THEOREM 7.4. *Let $\mathcal{R}$ be a contractive quasi-free Hilbert module over $A(\mathbb{B}^n)$ of multiplicity one and $\{\varphi_1, \ldots, \varphi_k\}$ be a subset of $H^\infty(\mathbb{B}^n)$. If $(M_{\varphi_1}, \ldots, M_{\varphi_k})$ is a semi-Fredholm tuple, then $\{\varphi_1, \ldots, \varphi_k\}$ satisfies the weak corona condition.*

Proof. Let $K : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$ be the kernel function for the quasi-free Hilbert module $\mathcal{R}$. By the assumption, the range of the row operator $M_\Phi = (M_{\varphi_1}, \ldots, M_{\varphi_k}) : \mathcal{R}^k \to \mathcal{R}$ in $\mathcal{R}$ has finite co-dimension, that is,

$$\dim[\mathcal{R}/(M_{\varphi_1}\mathcal{R} + \ldots + M_{\varphi_k}\mathcal{R})] < \infty,$$

and, in particular, M_Φ has closed range. Consequently, there is a finite rank projection F in $\mathcal{B}(\mathcal{R})$ such that

$$M_\Phi M_\Phi^* + F = \sum_{i=1}^k M_{\varphi_i} M_{\varphi_i}^* + F : \mathcal{R} \to \mathcal{R}$$

is bounded below. Therefore, there exists a $c > 0$ such that

$$\langle FK(\cdot,\boldsymbol{z}), K(\cdot,\boldsymbol{z})\rangle + \langle \sum_{i=1}^k M_{\varphi_i} M_{\varphi_i}^* K(\cdot,\boldsymbol{z}), K(\cdot,\boldsymbol{z})\rangle \ \geq c\|K(\cdot,\boldsymbol{z})\|^2,$$

for all $\boldsymbol{z} \in \mathbb{B}^n$. Therefore,

$$\|K(\cdot,\boldsymbol{z})\|^2 \hat{F}(\boldsymbol{z}) + \|K(\cdot,\boldsymbol{z})\|^2 \Big(\sum_{i=1}^k \varphi_i(\boldsymbol{z})\varphi_i^*(\boldsymbol{z})\Big) \geq c\|K(\cdot,\boldsymbol{z})\|^2,$$

and so

$$\hat{F}(\boldsymbol{z}) + \sum_{i=1}^k \varphi_i(\boldsymbol{z})\varphi_i(\boldsymbol{z})^* \geq c,$$

for all $\boldsymbol{z}$ in $\mathbb{B}^n$. Using the known boundary behavior of the Berezin transform (see Theorem 3.2 in [DaDo05]), since F is finite rank we have that $|\hat{F}(\boldsymbol{z})| \leq \frac{c}{2}$ for all $\boldsymbol{z}$ such that $1 > \|\boldsymbol{z}\| > 1-\delta$ for some $1 > \delta > 0$ depending on c. Hence

$$\sum_{i=1}^k \varphi_i(\boldsymbol{z})\varphi_i(\boldsymbol{z})^* \geq \frac{c}{2},$$

for all $\boldsymbol{z}$ such that $1 > \|\boldsymbol{z}\| > 1 - \delta > 0$, which completes the proof. ■

The key step in this proof is the vanishing of the Berezin transform at the boundary of $\mathbb{B}^n$ for a compact operator. The proof of this statement depends on the fact that $\frac{K(\cdot,\boldsymbol{z})}{\|K(\cdot,\boldsymbol{z})\|}$ converges weakly to zero as $\boldsymbol{z}$ approaches the boundary which rests on the fact that $\mathcal{R}$ is contractive.

7.4. **A Sufficient condition.**

THEOREM 7.5. *Let $\mathcal{R}$ be a contractive quasi-free Hilbert module over $A(\mathbb{D})$ of multiplicity one, which is semi-Fredholm at each point z in $\mathbb{D}$. If $\{\varphi_1, \ldots, \varphi_k\}$ is a subset of $H^\infty(\mathbb{D})$, then the k-tuple $M_\Phi = (M_{\varphi_1}, \ldots, M_{\varphi_k})$ is semi-Fredholm if and only if it is Fredholm if and only if $(\varphi_1, \ldots, \varphi_k)$ satisfies the weak corona condition.*

Proof. If M_Φ is semi-Fredholm, then by Proposition 7.4 there exist $\epsilon > 0$ and $1 > \delta > 0$ such that

$$\sum_{i=1}^{k} |\varphi_i(z)|^2 \geq \epsilon,$$

for all z such that $1 > |z| > 1 - \delta > 0$. Let $\mathcal{Z}$ be the set

$$\mathcal{Z} = \{z \in \mathbb{D} : \varphi_i(z) = 0 \text{ for all } i = 1, \ldots, k\}.$$

Since the functions $\{\varphi_i\}_{i=1}^k$ can not all vanish for z satisfying $1 > |z| > 1 - \delta$, it follows that the cardinality of the set $\mathcal{Z} := N$ is finite. Let

$$\mathcal{Z} = \{z_1, z_2, \ldots, z_N\}$$

and l_j be the smallest order of the zero at z_j for all φ_j and $1 \leq j \leq k$. Let $B(z)$ be the finite Blaschke product with zero set precisely $\mathcal{Z}$ counting the multiplicities. Note that $\xi_i := \frac{\varphi_i}{B} \in H^\infty(\mathbb{D})$ for all $i = 1, \ldots, k$. Since $\{\varphi_1, \ldots, \varphi_k\}$ satisfies the weak corona property, it follows that $\sum_{i=1}^k |\xi_i(z)|^2 \geq \epsilon$ for all z such that $1 > |z| > 1 - \delta$. Note that $\{\xi_1, \ldots, \xi_n\}$ does not have any common zero and so $\sum_{i=1}^k |\xi_i(z)|^2 \geq \epsilon$, for all z in $\mathbb{D}$. Therefore, $\{\xi_1, \ldots, \xi_k\}$ satisfies the corona property and hence there exists $\{\eta_1, \ldots, \eta_k\}$, a subset of $H^\infty(\mathbb{D})$, such that

$$\sum_{i=1}^{k} \xi_i(z)\eta_i(z) = 1,$$

for all z in $\mathbb{D}$. Thus,

$$\sum_{i=1}^{k} \varphi_i(z)\eta_i(z) = B,$$

for all z in $\mathbb{D}$. This implies $\sum_{i=1}^k M_{\varphi_i} M_{\eta_i} = M_B$, and consequently, $\sum_{i=1}^k \overline{M}_{\varphi_i} \overline{M}_{\eta_i} = \overline{M}_B$, where $\overline{M}_{\varphi_i}$ is the image of M_{φ_i} in the Calkin algebra, $\mathcal{Q}(\mathcal{R}) = \mathcal{B}(\mathcal{R})/\mathcal{K}(\mathcal{R})$. But the assumption that M_{z-w} is Fredholm for all w in $\mathbb{D}$ yields that M_B is Fredholm. Therefore, $X = \sum_{i=1}^k \overline{M}_{\varphi_i} \overline{M}_{\eta_i}$ is invertible. Moreover, since X commutes with the set

$$\{\overline{M}_{\varphi_1}, \ldots, \overline{M}_{\varphi_k}, \overline{M}_{\eta_1}, \ldots, \overline{M}_{\eta_k}\},$$

it follows that $(M_{\varphi_1}, \ldots, M_{\varphi_k})$ is a Fredholm tuple, which completes the proof. ∎

Further results and comments:

(1) In Theorem 8.2.6 in [EsP96], a version of Theorem 7.4 is established in case $\mathcal{R}$ is the Bergman module on $\mathbb{B}^n$.
(2) The converse of Theorem 7.4 is known for the Bergman space for certain domains in $\mathbb{C}^n$ (see Theorem 8.2.4 in [EsP96] and pages 241-242). A necessary condition for the converse to hold for the situation in Theorem 7.4 is for the n-tuple of co-ordinate multiplication operators to have essential spectrum equal to $\partial\mathbb{B}^n$, which is not automatic, but is true for the classical spaces.
(3) One prime reason to establish a converse, in Theorem 7.5, is that one can represent the zero variety of the ideal generated by the functions in terms of a single function, the finite Blaschke product (or polynomial). This is not surprising since $\mathbb{C}[z]$ is a principal ideal domain.
(4) As pointed out in the monograph by Eschmeier and Putinar, the relation between corona problem and the Taylor spectrum is not new (cf. [Ho67], [Wol97]).
(5) This section is mainly based on [DoSa10b] and closely related to [DoEs12] and [DoSa10a].
(6) In [Ve72], Venugopalkrishna developed a Fredholm theory and index theory for the Hardy module over strongly pseudoconvex domains in $\mathbb{C}^n$.
(7) An excellent source of information concerning Taylor spectrum is the monograph by Muller [Mu07]. See also the paper [Cu81] and the survey [Cu88] by Curto and the book by Eschmeier and Putinar [EsP96].

8. Co-spherically contractive Hilbert modules

A Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ is said to be *co-spherically contractive*, or define a *row contraction*, if

$$\| \sum_{i=1}^{n} M_i h_i \|^2 \leq \sum_{i=1}^{n} \|h_i\|^2, \quad (h_1, \ldots, h_n \in \mathcal{H}),$$

or, equivalently, if $\sum_{i=1}^{n} M_i M_i^* \leq I_{\mathcal{H}}$. Define the defect operator and the defect space of $\mathcal{H}$ as

$$D_{*\mathcal{H}} = (I_{\mathcal{H}} - \sum_{i=1}^{n} M_i M_i^*)^{\frac{1}{2}} \in \mathcal{L}(\mathcal{H}),$$

and

$$\mathcal{D}_{*\mathcal{H}} = \overline{\operatorname{ran}} D_{*\mathcal{H}},$$

respectively. We denote $D_{*\mathcal{H}}$ and $\mathcal{D}_{*\mathcal{H}}$ by D_* and $\mathcal{D}_*$ respectively, if $\mathcal{H}$ is clear from the context.

If $n = 1$ then $\mathcal{H}$ is a contractive Hilbert module over $A(\mathbb{D})$ (see Section 4).

8.1. **Drury-Arveson Module.** Natural examples of co-spherically contractive Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$ are the Drury-Arveson module, the Hardy module and the Bergman module, all defined on $\mathbb{B}^n$.

One can identify the Hilbert tensor product $H_n^2 \otimes \mathcal{E}$ with the $\mathcal{E}$-valued H_n^2 space $H_n^2(\mathcal{E})$ or the $\mathcal{B}(\mathcal{E})$-valued reproducing kernel Hilbert space with kernel function $(\boldsymbol{z}, \boldsymbol{w}) \mapsto (1 - \sum_{i=1}^{n} z_i \bar{w}_i)^{-1} I_{\mathcal{E}}$.

Then

$$H^2_n(\mathcal{E}) = \{f \in \mathcal{O}(\mathbb{B}^n, \mathcal{E}) : f(z) = \sum_{\boldsymbol{k} \in \mathbb{N}^n} a_{\boldsymbol{k}} z^{\boldsymbol{k}}, a_{\boldsymbol{k}} \in \mathcal{E}, \|f\|^2 := \sum_{\boldsymbol{k} \in \mathbb{N}^n} \frac{\|a_{\boldsymbol{k}}\|^2}{\gamma_{\boldsymbol{k}}} < \infty\},$$

where $\gamma_{\boldsymbol{k}} = \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!}$ are the multinomial coefficients and $\boldsymbol{k} \in \mathbb{N}^n$.

Given a co-spherically contractive Hilbert module $\mathcal{H}$, define the completely positive map $P_{\mathcal{H}} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ by

$$P_{\mathcal{H}}(X) = \sum_{i=1}^n M_i X M_i^*,$$

for all $X \in \mathcal{L}(\mathcal{H})$. Note that

$$I_{\mathcal{H}} \geq P_{\mathcal{H}}(I_{\mathcal{H}}) \geq P^2_{\mathcal{H}}(I_{\mathcal{H}}) \geq \cdots \geq P^l_{\mathcal{H}}(I_{\mathcal{H}}) \geq \cdots \geq 0.$$

In particular,

$$P_\infty(\mathcal{H}) := \text{SOT} - \lim_{l \to \infty} P^l_{\mathcal{H}}(I_{\mathcal{H}})$$

exists and $0 \leq P_\infty(\mathcal{H}) \leq I_{\mathcal{H}}$. The Hilbert module $\mathcal{H}$ is said to be *pure* if

$$P_\infty(\mathcal{H}) = 0.$$

Examples of pure co-spherically contractive Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$ includes the submodules and quotient modules of vector-valued Drury-Arveson module.

8.2. **Quotient modules of $H^2_n(\mathcal{E})$.** First recall a standard result from algebra: Any module is isomorphic to a quotient of a free module. The purpose of this subsection is to prove an analogous result for co-spherically contractive Hilbert modules: any pure co-spherically contractive Hilbert module is isomorphic to a quotient module of the Drury-Arveson module with some multiplicity.

THEOREM 8.1. *Let $\mathcal{H}$ be a co-spherically contractive Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. Then there exists a unique co-module map $\Pi_{\mathcal{H}} : \mathcal{H} \to H^2_n(\mathcal{D}_*)$ such that*

$$(\Pi_{\mathcal{H}} h)(\boldsymbol{w}) = D_*(I_{\mathcal{H}} - \sum_{i=1}^n w_i M_i^*)^{-1} h, \qquad (\boldsymbol{w} \in \mathbb{B}^n,\ h \in \mathcal{H})$$

and $\Pi^*_{\mathcal{H}} \Pi_{\mathcal{H}} = I_{\mathcal{H}} - P_\infty(\mathcal{H})$. Moreover, $\Pi^*_{\mathcal{H}}(k_n(\cdot, \boldsymbol{w})\eta) = (I_{\mathcal{H}} - \sum_{i=1}^n \bar{w}_i M_i)^{-1} D_* \eta$ for $\boldsymbol{w} \in \mathbb{B}^n$ and $\eta \in \mathcal{D}_*$.

Proof. First, note that for each $\boldsymbol{w} \in \mathbb{B}^n$ that

$$\begin{aligned}\|\sum_{i=1}^n w_i M_i^*\| &= \|(w_1 I_{\mathcal{H}}, \dots, w_n I_{\mathcal{H}})^*(M_1, \dots, M_n)\| \leq \|(w_1 I_{\mathcal{H}}, \dots, w_n I_{\mathcal{H}})^*\| \|(M_1, \dots, M_n)\| \\ &= (\sum_{i=1}^n |w_i|^2)^{\frac{1}{2}} \|\sum_{i=1}^n M_i M_i^*\|^{\frac{1}{2}} = \|\boldsymbol{w}\|_{\mathbb{C}^n} \|\sum_{i=1}^n M_i M_i^*\|^{\frac{1}{2}} < 1.\end{aligned}$$

Therefore, $\Pi_{\mathcal{H}} : \mathcal{H} \to H^2_n(\mathcal{D}_*)$ defined by

$$(\Pi_{\mathcal{H}} h)(\boldsymbol{z}) := D_*(I_{\mathcal{H}} - \sum_{i=1}^n z_i M_i^*)^{-1} h = \sum_{\boldsymbol{k}\in\mathbb{N}^n} (\gamma_{\boldsymbol{k}} D_* M^{*\boldsymbol{k}} h) z^{\boldsymbol{k}},$$

for all $h \in \mathcal{H}$ and $\boldsymbol{z} \in \mathbb{B}^n$, is a bounded linear map. Also the equalities

$$\begin{aligned}
\|\Pi_{\mathcal{H}} h\|^2 &= \|\sum_{\boldsymbol{k}\in\mathbb{N}^n} (\gamma_{\boldsymbol{k}} D_* M^{*\boldsymbol{k}} h) z^{\boldsymbol{k}}\|^2 = \sum_{\boldsymbol{k}\in\mathbb{N}^n} \gamma_{\boldsymbol{k}}^2 \|D_* M^{*\boldsymbol{k}} h\|^2 \|z^{\boldsymbol{k}}\|^2 = \sum_{\boldsymbol{k}\in\mathbb{N}^n} \gamma_{\boldsymbol{k}}^2 \|D_* M^{*\boldsymbol{k}} h\|^2 \frac{1}{\gamma_{\boldsymbol{k}}} \\
&= \sum_{\boldsymbol{k}\in\mathbb{N}^n} \gamma_{\boldsymbol{k}} \|D_* M^{*\boldsymbol{k}} h\|^2 = \sum_{l=0}^\infty \sum_{|\boldsymbol{k}|=l} \gamma_{\boldsymbol{k}} \|D_* M^{*\boldsymbol{k}} h\|^2 = \sum_{l=0}^\infty \sum_{|\boldsymbol{k}|=l} \gamma_{\boldsymbol{k}} \langle M^{\boldsymbol{k}} D_*^2 M^{*\boldsymbol{k}} h, h\rangle \\
&= \sum_{l=0}^\infty \langle \sum_{|\boldsymbol{k}|=l} \gamma_{\boldsymbol{k}} M^{\boldsymbol{k}} D_*^2 M^{*\boldsymbol{k}} h, h\rangle = \sum_{l=0}^\infty \langle P^l_{\mathcal{H}}(D_*^2) h, h\rangle = \sum_{l=0}^\infty \langle P^l_{\mathcal{H}}(I_{\mathcal{H}} - P_{\mathcal{H}}(I_{\mathcal{H}})) h, h\rangle \\
&= \sum_{l=0}^\infty \langle (P^l_{\mathcal{H}}(I_{\mathcal{H}}) - P^{l+1}_{\mathcal{H}}(I_{\mathcal{H}})) h, h\rangle = \sum_{l=0}^\infty (\langle P^l_{\mathcal{H}}(I_{\mathcal{H}}) h, h\rangle - \langle P^{l+1}_{\mathcal{H}}(I_{\mathcal{H}}) h, h\rangle) \\
&= \|h\|^2 - \langle P_\infty(\mathcal{H}) h, h\rangle,
\end{aligned}$$

holds for all $h \in \mathcal{H}$, where the last equality follows from the fact that $\{P^l_{\mathcal{H}}(I_{\mathcal{H}})\}_{l=0}^\infty$ is a decreasing sequence of positive operators and that $P^0_{\mathcal{H}}(I_{\mathcal{H}}) = I_{\mathcal{H}}$ and $P_\infty(\mathcal{H}) = \lim_{l\to\infty} P^l_{\mathcal{H}}(I_{\mathcal{H}})$. Therefore, $\Pi_{\mathcal{H}}$ is a bounded linear operator and

$$\Pi^*_{\mathcal{H}} \Pi_{\mathcal{H}} = I_{\mathcal{H}} - P_\infty(\mathcal{H}).$$

On the other hand, for all $h \in \mathcal{H}$ and $\boldsymbol{w} \in \mathbb{B}^n$ and $\eta \in \mathcal{D}_*$, it follows that

$$\begin{aligned}
\langle \Pi^*_{\mathcal{H}}(k_n(\cdot, \boldsymbol{w})\eta), h\rangle_{\mathcal{H}} &= \langle k_n(\cdot, \boldsymbol{w})\eta, D_*(I_{\mathcal{H}} - \sum_{i=1}^n w_i M_i^*)^{-1} h\rangle_{H^2_n(\mathcal{D}_*)} \\
&= \langle \sum_{\boldsymbol{k}\in\mathbb{N}^n} (\gamma_{\boldsymbol{k}} \bar{w}^{\boldsymbol{k}} \eta) z^{\boldsymbol{k}}, \sum_{\boldsymbol{k}\in\mathbb{N}^n} (\gamma_{\boldsymbol{k}} D_* M^{*\boldsymbol{k}} h) z^{\boldsymbol{k}}\rangle_{H^2_n(\mathcal{D}_*)} = \sum_{\boldsymbol{k}\in\mathbb{N}^n} \gamma_{\boldsymbol{k}} \bar{w}^{\boldsymbol{k}} \langle M^{\boldsymbol{k}} D_* \eta, h\rangle_{\mathcal{H}} \\
&= \langle (I_{\mathcal{H}} - \sum_{i=1}^n \bar{w}_i M_i)^{-1} D_* \eta, h\rangle_{\mathcal{H}},
\end{aligned}$$

that is,

$$\Pi^*_{\mathcal{H}}(k_n(\cdot, \boldsymbol{w})\eta) = (I_{\mathcal{H}} - \sum_{i=1}^n \bar{w}_i M_i)^{-1} D_* \eta.$$

Also for all $\eta \in \mathcal{D}_*$ and $\boldsymbol{l} \in \mathbb{N}^n$,

$$\langle \Pi^*_{\mathcal{H}}(z^{\boldsymbol{l}}\eta), h\rangle = \langle z^{\boldsymbol{l}}\eta, \sum_{\boldsymbol{k}\in\mathbb{N}^n} (\gamma_{\boldsymbol{k}} D_* M^{*\boldsymbol{k}} h) z^{\boldsymbol{k}}\rangle = \gamma_{\boldsymbol{l}} \|z^{\boldsymbol{l}}\|^2 \langle \eta, D_* M^{*\boldsymbol{l}} h\rangle = \langle M^{\boldsymbol{l}} D_* \eta, h\rangle,$$

and hence $\Pi_{\mathcal{H}}$ is a co-module map. Finally, uniqueness of $\Pi_{\mathcal{H}}$ follows from the fact that $\{z^{\boldsymbol{k}}\eta : \boldsymbol{k} \in \mathbb{N}^n, \eta \in \mathcal{D}_*\}$ is a total set of $H^2_n(\mathcal{D}_*)$. This completes the proof. ■

It is an immediate consequence of this result that if $\mathcal{H}$ is a pure co-spherical contractive Hilbert module over $\mathbb{C}[\boldsymbol{z}]$, then $P_\infty(\mathcal{H}) = 0$. Equivalently, that $\Pi_{\mathcal{H}}$ is an isometry. This yields the dilation result for pure co-spherical contractive Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$.

COROLLARY 8.2. *Let $\mathcal{H}$ be a pure co-spherical contractive Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. Then*

$$\mathcal{H} \cong \mathcal{Q},$$

for some quotient module $\mathcal{Q}$ of $H^2_n(\mathcal{D}_*)$.

Proof. By Theorem 8.1, the co-module map $\Pi_{\mathcal{H}} : \mathcal{H} \to H^2_n(\mathcal{D}_*)$ is an isometry. In particular, $\mathcal{Q} = \Pi_{\mathcal{H}}\mathcal{H}$ is a quotient module of $H^2_n(\mathcal{D}_*)$. This completes the proof. ∎

A Hilbert module $\mathcal{H}$ over $\mathbb{C}[\boldsymbol{z}]$ is said to be *spherical Hilbert module* if M_i is normal operator for each $1 \leq i \leq n$ and

$$\sum_{i=1}^{n} M_i M_i^* = I_{\mathcal{H}}.$$

Given a spherical Hilbert module $\mathcal{H}$ over $\mathbb{C}[\boldsymbol{z}]$, there exists a unique unital $*$-representation $\pi : C^*(\partial\mathbb{B}^n) \to \mathcal{B}(\mathcal{H})$ such that $\pi(z_i) = M_i$ and vice versa (see [At90], [At92], [Arv98]).

The following dilation theorem is due to Arveson [Arv98].

THEOREM 8.3. *Let $\mathcal{H}$ be a co-spherical contractive Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. Then there exists a spherical Hilbert module $\mathcal{S}$ over $\mathbb{C}[\boldsymbol{z}]$ such that $H^2_n(\mathcal{D}_*) \oplus \mathcal{S}$ is a dilation of $\mathcal{H}$. Equivalently, there exists a spherical Hilbert module $\mathcal{S}$ over $\mathbb{C}[\boldsymbol{z}]$ and a co-module isometry $U : \mathcal{H} \to H^2_n(\mathcal{D}_*) \oplus \mathcal{S}$. In particular,*

$$\mathcal{H} \cong \mathcal{Q},$$

for some quotient module $\mathcal{Q}$ of $H^2_n(\mathcal{D}_*) \oplus \mathcal{S}$. Moreover, the minimal dilation is unique.

8.3. **Curvature Inequality.** The purpose of this subsection is to compare the curvatures of the bundles $E^*_{\mathcal{Q}}$ associated with a quotient module $\mathcal{Q} = \mathcal{H} \otimes \mathcal{E}/\mathcal{S} \in B^*_m(\Omega)$ and $E^*_{\mathcal{H}}$, where $\mathcal{H} \in B^*_1(\Omega)$ and $\mathcal{E}$, a coefficient Hilbert space. First, we need to recall some results from complex geometry concerning curvatures of sub-bundles and quotient bundles (cf. [GH94], pp. 78-79).

Let E be a Hermitian anti-holomorphic bundle over Ω (possibly infinite rank) and F be an anti-holomorphic sub-bundle of E such that the quotient $Q = E/F$ is also anti-holomorphic. Let ∇_E denote the Chern connection on E and $\mathcal{K}_E$ the corresponding curvature form. There are two canonical connections that we can define on F and the quotient bundle Q. The first ones are the Chern connections ∇_F and ∇_Q on F and Q, respectively. To obtain the second connections, let P denote the projection-valued bundle map of E so that $P(\boldsymbol{z})$ is the orthogonal projection of $E(\boldsymbol{z})$ onto $F(\boldsymbol{z})$. Then

$$\nabla_{PE} = P \nabla_E P \qquad \text{and} \qquad \nabla_{P^\perp E} = P^\perp \nabla_E P^\perp,$$

define connections on F and Q, respectively, where $P^\perp = I - P$ and Q is identified fiber wise with $P^\perp E$. The following result from complex geometry relates the curvatures for these pairs of connections.

THEOREM 8.4. *If F is an anti-holomorphic sub-bundle of the anti-holomorphic bundle E over Ω such that E/F is anti-holomorphic, then the curvature functions for the connections ∇_F, ∇_{PE}, ∇_Q and $\nabla_{P^\perp E}$ satisfy*

$$\mathcal{K}_F(\boldsymbol{w}) \geq \mathcal{K}_{PE}(\boldsymbol{w}) \qquad \text{and} \qquad \mathcal{K}_Q(\boldsymbol{w}) \leq \mathcal{K}_{P^\perp E}(\boldsymbol{w}). \qquad (\boldsymbol{w} \in \Omega).$$

The proof is essentially a matrix calculation involving the off-diagonal entries of ∇_E, one of which is the second fundamental form and the other its dual (cf. [GH94]). (Note in [GH94], E is finite rank but the proof extends to the more general case.)

An application of this result to Hilbert modules yields the following:

THEOREM 8.5. *Let $\mathcal{H} \in B_1^*(\Omega)$ be a Hilbert module over $A(\Omega)$ (or over $\mathbb{C}[\boldsymbol{z}]$) and $\mathcal{S}$ be a submodule of $\mathcal{H} \otimes \mathcal{E}$ for a Hilbert space $\mathcal{E}$ such that the quotient module $\mathcal{Q} = (\mathcal{H} \otimes \mathcal{E})/\mathcal{S}$ is in $B_m^*(\Omega)$. If $E_{\mathcal{H}}^*$ and $E_{\mathcal{Q}}^*$ are the corresponding Hermitian anti-holomorphic bundles over Ω, then*

$$P^\perp(\boldsymbol{w})(\mathcal{K}_{E_{\mathcal{H}}^*}(\boldsymbol{w}) \otimes I_{\mathcal{E}})P^\perp(\boldsymbol{w}) \geq \mathcal{K}_{E_{\mathcal{Q}}^*}(\boldsymbol{w}). \quad (\boldsymbol{w} \in \Omega)$$

Proof. The result follows from the previous theorem by setting $E = E_{\mathcal{H}}^* \otimes \mathcal{E}$, $F = E_{\mathcal{S}}^*$ and $Q = E_{\mathcal{Q}}^*$. ■

In particular, one has the following extremal property of the curvature functions.

THEOREM 8.6. *Let $\mathcal{H} \in B_m^*(\Omega)$ be a Hilbert module over $A(\Omega)$. If $\mathcal{H}$ is dilatable to $\mathcal{R} \otimes \mathcal{E}$ for some Hilbert space $\mathcal{E}$, then*

$$\mathcal{K}_{E_{\mathcal{R}}^*}(\boldsymbol{w}) \otimes I_{\mathcal{E}} \geq \mathcal{K}_{E_{\mathcal{H}}^*}(w). \qquad (\boldsymbol{w} \in \Omega)$$

The following factorization result is a special case of Arveson's dilation result (see Corollary 2 in [DoMiSa12] for a proof).

THEOREM 8.7. *Let $\mathcal{H}_k$ be a reproducing kernel Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ with kernel function k over $\mathbb{B}^n$. Then $\mathcal{H}_k$ is co-spherically contractive if and only if the function $(1 - \sum_{i=1}^n z_i \bar{w}_i)k(\boldsymbol{z}, \boldsymbol{w})$ is positive definite.*

The following statement is now an easy consequence of Theorem 8.7.

COROLLARY 8.8. *Let $\mathcal{H}_k$ be a co-spherically contractive reproducing kernel Hilbert module over $\mathbb{B}^n$. Then*

$$\mathcal{K}_{E_{H_n^2}^*} - \mathcal{K}_{E_{\mathcal{H}_k}^*} \geq 0.$$

Further results and comments:

(1) The Drury-Arveson space has been used, first in connection with the models for commuting contractions by Lubin in 1976 [Lu76] (see also [Lu77]), and then by Drury in 1978 in connection with the von Neumann inequality for commuting contractive tuples. However, the Drury-Arveson space has been popularized by Arveson in 1998 [Arv98].

(2) The proof of Theorem 8.1 is a classic example of technique introduced by Rota [Ro60] in the context of similarity problem for strict contractions. In [Ba77], J. Ball obtained a several-variables analogue of Rota's model. In connection with Rota's model, see also the work by Curto and Herrero [CuHe85].

(3) The converse of Theorem 8.8 is false in general. A converse of Theorem 8.8 is related to the notion of infinite divisibility (see [BKMi13]).
(4) Theorem 8.8 is from [DoMiSa12]. For $n = 1$, this result was obtained by Misra in [Mi84] and was further generalized by Uchiyama in [Uc90].
(5) Theorem 8.3 was proved independently by many authors (see [MuVa93], [Po99]). Most probably, the existence of dilation was proved for the first time by Jewell and Lubin in [JeLu79] and [Lu76]. However, the uniqueness part of the minimal dilation is due to Arveson.
(6) The inequality in Theorem 8.8 shows in view of Theorem 8.6 that the module H^2_n is an extremal element in the set of co-spherically contractive Hilbert modules over the algebra $\mathbb{C}[\boldsymbol{z}]$. Similarly, for the polydisk $\mathbb{D}^n$, the Hardy module is an extremal element in the set of those modules over the algebra $A(\mathbb{D}^n)$ which admit a dilation to the Hardy space $H^2(\mathbb{D}^n) \otimes \mathcal{E}$.
(7) We refer the reader to Athavale [At92], [At90] for an analytic approach and Attele and Lubin [AtLu96] for a geometric approach to the (regular unitary) dilation theory. In particular, Athavale proved that a spherical isometry must be subnormal. Other related work concerning dilation of commuting tuples of operators appears in [RSu10], [CuVa93], [CuVa95].
(8) Motivated by the Gauss-Bonnet theorem and the curvature of a Riemannian manifold, in [Arv00] Arveson introduced a notion of curvature which is a numerical invariant. His notion of curvature is related to the Samuel multiplicity [Fa03], Euler characteristic [Arv00] and Fredholm index [GRSu02].

References

[AbDo76] M. Abrahamse and R. Douglas, *A class of subnormal operators related to multiply connected domains*, Adv. Math. 19 (1976), 106–148.

[Ag85] J. Agler, *Rational dilation on an annulus*, Ann. of Math. (2) 121 (1985), 537-563. 8 (160), 182, 147–163.

[AgHR08] J. Agler, J. Harland and B. Raphael, *Classical function theory, operator dilation theory, and machine computation on multiply-connected domains*, Mem. Amer. Math. Soc. 191 (2008), no. 892.

[AgMc02] J. Agler and J. McCarthy, *Pick interpolation and Hilbert function spaces*, Graduate Studies in Mathematics, 44. American Mathematical Society, Providence, RI, 2002.

[AgYo03] J. Agler and N. Young, *A model theory for Γ-contractions*, J. Operator Theory, 49 (2003), 45-60.

[Ale82] A. Aleksandrov, *The existence of inner functions in a ball*, Mat. Sb., 118 (1982) 147-163.

[AlRiSu96] A. Aleman, S. Richter and C. Sundberg, *Beurling's theorem for the Bergman space*, Acta Math. 177 (1996), 275-310.

[Alp88] D. Alpay, *A remark on the Cowen Douglas classes $B_n(\Omega)$*, Arch. Math., 51 (1988), 539–546.

[An63] T. Ando, *On a pair of commuting contractions*, Acta Sci. Math. (Szeged), 24 (1963), 88–90.

[AVVW09] A. Anatolii, D. Kaliuzhnyi-Verbovetskyi, V. Vinnikov and H. Woerdeman, *Classes of tuples of commuting contractions satisfying the multivariable von Neumann inequality*, J. Funct. Anal. 256 (2009), 3035-3054.

[ApBerFP85] C. Apostol, H. Bercovici, C. Foias, and C. Pearcy, *Invariant subspaces, dilation theory, and the structure of the predual of a dual algebra. I*, J. Funct. Anal. 63 (1985) 369-404.

[ApMa81] C. Apostol and M. Martin, *A C^*–algebra approach to the Cowen-Douglas theory*, Topics in modern operator theory (Timisoara/Herculane, 1980), 45-51, Operator Theory: Adv. Appl. 2, Birkhsuser, Basel-Boston, Mass. (1981).

[AraEn03] J. Arazy and M. Englis, *Analytic models for commuting operator tuples on bounded symmetric domains*, Trans. Amer. Math. Soc. 355 (2003), 837-864.

[Aro50] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. 68 (1950), 337-404.

[Arv98] W. Arveson, *Subalgebras of C^*-algebras III: Multivariable operator theory*, Acta Math. 181 (1998), 159-228.

[Arv00] W. Arveson, *The curvature invariant of a Hilbert module over* $\mathbb{C}[z_1, \ldots, z_n]$, J. Reine Angew. Math. 522 (2000), 173-236.

[At90] A. Athavale, *On the intertwining of joint isometries*, J. Operator Theory 23 (1990), 339-350.

[At92] A. Athavale, *Model theory on the unit ball in* $\mathbb{C}^m$, J. Operator Theory 27 (1992), 347-358.

[AtLu96] K. Attele and A. Lubin, *Dilations and commutant lifting for jointly isometric operators– a geometric approach*, J. Funct. Anal. 140 (1996), 300-311.

[BaMi03] B. Bagchi amd G. Misra, *The homogeneous shifts*, J. Funct. Anal. 204 (2003), 293-319.

[Ba77] J. Ball, *Rota's theorem for general functional Hilbert spaces*, Proc. Amer. Math. Soc. 64 (1977), 55-61.

[Ba78] J. Ball, *Factorization and model theory for contraction operators with unitary part*, Mem. Amer. Math. Soc. 13 (1978), no. 198.

[Ba79] J. Ball, *A lifting theorem for operator models of finite rank on multiply-connected domains*, J. Operator Theory 1 (1979), 3-25.

[BaBo13a] J. Ball and V. Bolotnikov, *Weighted Bergman spaces: shift-invariant subspaces and input/state/output linear systems*, Integral Equations Operator Theory, 76 (2013), 301-356.

[BaBo13b] J. Ball and V. Bolotnikov, *A Beurling type theorem in weighted Bergman spaces*, C. R. Math. Acad. Sci. Paris 351 (2013), 433-436.

[BaHe83] J. Ball and J. W. Helton, *A Beurling-Lax theorem for the Lie group U(m,n) which contains most classical interpolation theory*, J. Operator Theory 9 (1983), 107-142.

[BaKr87] J. Ball and T. Kriete, *Operator-valued Nevanlinna-Pick kernels and the functional models for contraction operators*, Integral Equations Operator Theory 10 (1987), 17-61.

[BaTV01] J. Ball, T. Trent and V. Vinnikov, *Interpolation and commutant lifting for multipliers on reproducing kernel Hilbert spaces*, Operator theory and analysis (Amsterdam, 1997), 89–138, Oper. Theory Adv. Appl., 122, Birkhäuser, Basel, 2001.

[BaV05] J. Ball and V. Vinnikov, *Lax-Phillips scattering and conservative linear systems: a Cuntz-algebra multidimensional setting*, Mem. Amer. Math. Soc. 178 (2005), no. 837.

[Ba08] C. Barbian, *Beurling-type representation of invariant subspaces in reproducing kernel Hilbert spaces*, Integral Equations Operator Theory 61 (2008), 299-323.

[Ba11] C. Barbian, *A characterization of multiplication operators on reproducing kernel Hilbert spaces*, J. Operator Theory 65 (2011), 235-240.

[Bea88] B. Beauzamy, *Introduction to operator theory and invariant subspaces*, North-Holland Mathematical Library, 42. North-Holland Publishing Co., Amsterdam, 1988.

[BenTi07a] C. Benhida and D. Timotin, *Characteristic functions for multicontractions and automorphisms of the unit ball*, Integral Equations Operator Theory 57 (2007), 153-166.

[BenTi07b] C. Benhida and D. Timotin, *Some automorphism invariance properties for multicontractions*, Indiana Univ. Math. J. 56 (2007), 481-499.

[Beu49] A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. 81 (1949), 239-255.

[BEsSa05] T. Bhattacharyya, J. Eschmeier and J. Sarkar, *Characteristic function of a pure commuting contractive tuple*, Integral Equations Operator Theory 53 (2005), 23-32.

[BKMi13] S. Biswas, D. Keshari and G. Misra, *Infinitely divisible metrics and curvature inequalities for operators in the Cowen-Douglas class*, J. Lond. Math. Soc. 88 (2013), 941-956.

[Bo03] V. Bolotnikov, *Interpolation for multipliers on reproducing kernel Hilbert spaces*, Proc. Amer. Math. Soc. 131 (2003), 1373-1383.

[BuMa84] J. Burbea and P. Masani, *Banach and Hilbert spaces of vector-valued functions. Their general theory and applications to holomorphy*, Research Notes in Mathematics 90, Boston-London-Melbourne, Pitman Advanced Publishing Program (1984).

[Car62] L. Carleson, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math. (2) 76 (1962), 547-559.

[CaCl95] J. Carlson and D. Clark, *Cohomology and extensions of Hilbert modules*, J. Funct. Anal. 128 (1995) 278-306.

[CaCl97] J. Carlson and D. Clark, *Projectivity and extensions of Hilbert modules over $A(\mathbb{D}^n)$* , Michigan Math. J. 44 (1997) 365-373.

[ChPa11] I. Chalendar and J. Partington, *Modern approaches to the invariant-subspace problem*, Cambridge Tracts in Mathematics, 188. Cambridge University Press, Cambridge, 2011.

[CDSaS14] A. Chattopadhyay, B. K. Das, J. Sarkar and S. Sarkar, *Wandering subspaces of the Bergman space and the Dirichlet space over polydisc*, Integral Equations Operator Theory 79 (2014), 567-577.

[Ch09] L. Chen, *On unitary equivalence of quasi-free Hilbert modules*, Studia Math. 195 (2009), 87-98.

[ChDoGu11] L. Chen, R. Douglas and K. Guo, *On the double commutant of Cowen-Douglas operators*, J. Funct. Anal. 260 (2011), 1925-1943.

[XCDo92] X. Chen and R. G. Douglas, *Localization of Hilbert modules*, Mich. Math. J. 39 (1992), 443 - 454

[XCGu03] X. Chen and K. Guo, *Analytic Hilbert Modules*, Chapman & Hall/CRC Research Notes in Mathematics, 433. Chapman & Hall/CRC, Boca Raton, FL, 2003.

[CoDo78] M. Cowen and R. Douglas, *Complex geometry and operator theory*, Acta Math. 141 (1978), 187–261.

[CoDo83] M. Cowen and R. Douglas, *On operators possessing an open set of eigen- values*, Memorial Conf. for Fejer-Riesz, Colloq. Math. Soc. J. Bolyai, Vol. 35, pp. 323–341, North-Holland, Amsterdam, 1983.

[Cu81] R. Curto, *Fredholm invertible n-tuples of operators. The deformation problem*, Trans. Amer. Math. Soc. 266 (1981) 129-159.

[Cu88] R. Curto, *Applications of several complex variables to multiparameter spectral theory*, Surveys of Some Recent Results in Operator Theory, Vol. II, J.B. Conway and B.B. Morrel, editors, Longman Publishing Co., London (1988); pp. 25–90.

[CuHe85] R. Curto and D. Herrero, *On closures of joint similarity orbits*, Integral Equations Operator Theory 8 (1985), 489-556.

[CuSal84] R. Curto and N. Salinas, *Generalized Bergman kernels and the Cowen-Douglas theory*, Amer J. Math. 106 (1984), 447-488.

[CuVa93] R. Curto and F.-H. Vasilescu, *Standard operator models in the polydisc*, Indiana Univ. Math. J. 42 (1993), 791-810.

[CuVa95] R. Curto and F.-H. Vasilescu, *Standard operator models in the polydisc. II*, Indiana Univ. Math. J. 44 (1995), 727-746.

[DaDo05] K. Davidson and R. Douglas, *The generalized Berezin transform and commutator ideals*, Pacific J. Math. 222 (2005), 29–56.

[DiEs06] M. Didas and J. Eschmeier, *Unitary extensions of Hilbert $A(\mathbb{D})$-modules split*, J. Funct. Anal. 238 (2006), 565-577.

[Do86] R. Douglas, *Hilbert modules over function algebras*, Advances in invariant subspaces and other results of operator theory (Timisoara and Herculane, 1984), 125139, Oper. Theory Adv. Appl., 17, Birkhauser, Basel, 1986.

[Do88] R. Douglas, *On Silov resolution of Hilbert modules*, Special classes of linear operators and other topics (Bucharest, 1986), 5160, Oper. Theory Adv. Appl., 28, Birkhuser, Basel, 1988.

[Do09] R. Douglas, *Operator theory and complex geometry*, Extracta Math. 24 (2009), 135-165.

[Do11] R. Douglas, *Variations on a theme of Beurling*, New York J. Math. 17A (2011), 1-10.

[Do14] R. Douglas, *Connections of the corona problem with operator theory and complex geometry*, to appear in the Proceedings of the Fields Institute.

[DoEs12] R. Douglas and J. Eschmeier, *Spectral inclusion theorems. Mathematical methods in systems*, optimization, and control, 113-128, Oper. Theory Adv. Appl., 222, Birkhauser/Springer Basel AG, Basel, 2012.

[DoFo76] R. Douglas and C. Foias, *A homological view in dilation theory*, preprint series in mathematics, No.15/1976, Institutul De Matematica, Bucuresti.

[DoKKSa12] R. Douglas, Y. Kim, H. Kwon and J. Sarkar, *Curvature invariant and generalized canonical operator models - I*, Operator Theory Advances and Applications, 221 (2012), 293–304.

[DoKKSa14] R. Douglas, Y. Kim, H. Kwon and J. Sarkar, *Curvature invariant and generalized canonical operator models - II*, J. Funct. Anal. 266 (2014), 2486-2502.

[DoMi03] R. Douglas and G. Misra, *Quasi-free resolutions of Hilbert modules*, Integral Equations Operator Theory 47 (2003), 435–456.

[DoMi05] R. Douglas and G. Misra, *On quasi-free Hilbert modules*, New York J. Math. 11 (2005), 547–561.

[DoMiSa12] R. Douglas, G. Misra and J. Sarkar, *Contractive Hilbert modules and their dilations*, Israel Journal of Math. 187 (2012), 141–165.

[DoMiVa00] R. Douglas, G. Misra and C. Varughese, *On quotient modules—the case of arbitrary multiplicity*, J. Funct. Anal. 174 (2000), 364–398.

[DoPa89] R. Douglas and V. Paulsen, *Hilbert Modules over Function Algebras*, Research Notes in Mathematics Series, 47, Longman, Harlow, 1989.

[DoPaSY95] R. Douglas, V. Paulsen, C.-H. Sah and K. Yan, . *Algebraic reduction and rigidity for Hilbert modules*, Amer. J. Math. 117 (1995), 75-92.

[DoPaY89] R. Douglas, V. Paulsen and K. Yan, *Operator theory and algebraic geometry*, Bull. Amer. Math. Soc. (N.S.) 20 (1989), 67-71.

[DoPuW12] R. Douglas, M. Putinar and K. Wang, *Reducing subspaces for analytic multipliers of the Bergman space*, J. Funct. Anal. 263 (2012), 1744-1765.

[DoSa08] R. Douglas and J. Sarkar, *On unitarily equivalent submodules*, Indiana Univ. Math. J. 57 (2008), 2729–2743.

[DoSa10a] R. Douglas and J. Sarkar, *Some Remarks on the Toeplitz Corona problem*, Hilbert spaces of analytic functions, CRM Proc. Lecture Notes, 51 (2010) , Amer. Math. Soc., Providence, RI, 81–89.

[DoSa10b] R. Douglas and J. Sarkar, *A note on semi-Fredholm Hilbert modules*, Operator Theory Advances and Applications, 202 (2010), 143–150.

[DoSa11] R. Douglas and J. Sarkar, *Essentially Reductive Weighted Shift Hilbert Modules*, Journal of Operator Theory, 65 (2011), 379–401.

[DoSuZ11] R. Douglas, S. Sun and D. Zheng, *Multiplication operators on the Bergman space via analytic continuation*, Adv. Math. 226 (2011) 541-583.

[DoY92] R. Douglas and K. Yan, *A multi-variable Berger-Shaw theorem*, J. Operator Theory 27 (1992), 205-217.

[DoYa93] R. Douglas and K. Yan, *Hilbert-Samuel polynomials for Hilbert modules*, Indiana Univ. Math. J. 42 (1993), 811–820.

[DrMc05] M. Dritschel and S. McCullough, *The failure of rational dilation on a triply connected domain*, J. Amer. Math. Soc. 18 (2005), 873-918.

[Dr78] S. Drury, *A generalization of von Neumann's inequality to the complex ball*, Proc. Amer. Math. Soc., 68 (1978), 300-304.

[Es07a] J. Eschmeier, *On the Hilbert-Samuel Multiplicity of Fredholm Tuples*, Indiana Univ. Math. J., 56 (2007), 1463–1477.

[Es07b] J. Eschmeier, *Samuel multiplicity and Fredholm theory*, Math. Ann. 339 (2007), 21-35.

[Es08a] J. Eschmeier, *Samuel multiplicity for several commuting operators*, J. Operator Theory 60 (2008), 399-414.

[Es08b] J. Eschmeier, *Fredholm spectrum and growth of cohomology groups*, Studia Math. 186 (2008), 237-249.

[EsP96] J. Eschmeier and M. Putinar, *Spectral decompositions and analytic sheaves*, London Mathematical Society Monographs. New Series, 10. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996.

[EsP02] J. Eschmeier and M. Putinar, *Spherical contractions and interpolation problems on the unit ball*, J. Reine Angew. Math. 542 (2002), 219-236.

[EsSc14] J. Eschmeier and J. Schmitt, *Cowen-Douglas operators and dominating sets*, to appear in J. of Operator Theory.

[Ei95] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, New York, 1995.

[Fa03] X. Fang, *Hilbert polynomials and Arveson's curvature invariant*, J. Funct. Anal. 198 (2003), 445-464.

[Fa06] X. Fang, *The Fredholm index of a pair of commuting operators*, Geom. Funct. Anal. 16 (2006), 367-402.

[Fa08] X. Fang, *Estimates for Hilbertian Koszul homology*, J. Funct. Anal. 255 (2008), 1-12.

[Fa09] X. Fang, *The Fredholm index of a pair of commuting operators*, II. J. Funct. Anal. 256 (2009), 1669-1692.

[FeSt72] C. Fefferman and E. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), 137-193.

[FoFr90] C. Foias and A. Frazho, *The commutant lifting approach to interpolation problems*, Operator Theory: Advances and Applications, 44. Birkhauser Verlag, Basel, 1990.

[Fuh12] P. Fuhrmann, *A polynomial approach to linear algebra*, Second edition. Universitext. Springer, New York, 2012.

[GH94] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994.

[GRSu02] D. Greene, S. Richter and C. Sundberg, *The structure of inner multipliers on spaces with complete Nevanlinna-Pick kernels*, J. Funct. Anal. 194 (2002), 311-331.

[Gu99] K. Guo, *Normal Hilbert modules over the ball algebra $A(B)$*, Studia Math. 135 (1999), 1-12.

[GuH11] K. Guo and H. Huang, *Multiplication operators defined by covering maps on the Bergman space: The connection between operator theory and von Neumann algebras*, J. Funct. Anal. 260 (2011) 1219-1255.

[Ha61] P. Halmos, *Shifts on Hilbert spaces*, J. Reine Angew. Math. 208 (1961), 102-112.

[Ha82] P. Halmos, *A Hilbert space problem book*, Second edition, Graduate Texts in Mathematics, 19. Springer-Verlag, New York-Berlin, 1982.

[HePe04] H. Hedenmalm and Y Perdomo, *Mean value surfaces with prescribed curvature form*, J. Math. Pures Appl., 83 (2004), 1075-1107.

[He64] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York/London, 1964.

[Ho67] L. Hormander, *Generators for some rings of analytic functions*, Bull. Amer. Math. Soc. 73 (1967) 943-949.

[III10] K. J. Izuchi, K. H. Izuchi and Y. Izuchi, *Wandering subspaces and the Beurling type Theorem I*, Arch. Math. (Basel) 95 (2010), 439-446.

[JeLu79] N. Jewell and A. Lubin, *Commuting weighted shifts and analytic function theory in several variables*, J. Operator Theory 1 (1979), 207-223.

[JGuJ05] C. Jiang, X. Guo and K. Ji, *K-group and similarity classification of operators*, J. Funct. Anal. 225 (2005), 167-192.

[JJ07] C. Jiang and K. Ji, *Similarity classification of holomorphic curves*, Adv. Math. 215 (2007), 446-468.

[JWa06] C. Jiang and Z. Wang, *Structure of Hilbert Space Operators*, World Scientific Publishing Co. Pvt. Ltd., Hackensack, NJ, 2006.

[Ju05] M. Jury, *Invariant subspaces for a class of complete Pick kernels*, Proc. Amer. Math. Soc. 133 (2005), 3589-3596.

[KMi09] A. Koranyi amd G. Misra, *A classification of homogeneous operators in the Cowen-Douglas class*, Integral Equations Operator Theory 63 (2009), 595-599.

[KMi11] A. Koranyi and G. Misra, *A classification of homogeneous operators in the Cowen-Douglas class*, Adv. Math. 226 (2011), 5338-5360.

[La59] P. Lax, *Translation invariant spaces*, Acta Math. 101 (1959) 163-178.

[Li88] Q. Lin, *Operator theoretical realization of some geometric notions*, Trans. Amer. Math. Soc. 305 (1988), 353-367.

[Lu76] A. Lubin, *Models for commuting contractions*, Michigan Math. J. 23 (1976), 161-165.

[Lu77] A. Lubin, *Weighted shifts and products of subnormal operators*, Indiana Univ. Math. J. 26 (1977), 839-845.

[Ma85] V. Mandrekar, *The validity of Beurling theorems in polydiscs*, Proc. Amer. Math. Soc. 103 (1988), 145-148.

[Ma85] M. Martin, *Hermitian geometry and involutive algebras*, Math. Z. 188 (1985), 359-382.

[Mc96] J. McCarthy, *Boundary values and Cowen-Douglas curvature*, J. Funct. Anal. 137 (1996), 1-18.

[SMR02] S. McCullough and S. Richter, *Bergman-type reproducing kernels, contractive divisors, and dilations*, J. Funct. Anal. 190 (2002), 447-480.

[SMT00] S. McCullough and T. Trent, *Invariant subspaces and Nevanlinna-Pick kernels*, J. Funct. Anal. 178 (2000), 226-249.

[Mi84] G. Misra, *Curvature inequalities and extremal properties of bundle shifts*, J. Operator Theory, 11 (1984), 305 – 317.

[MiS90] G. Misra and N. Sastry, *Bounded modules, extremal problems, and a curvature inequality*, J. Funct. Anal. 88 (1990), 118 – 134.

[Mu07] V. Muller, *Spectral theory of linear operators and spectral systems in Banach algebras*, Operator Theory Advances and Appl., 139, Birkhauser, Basel, 2007.

[MuVa93] V. Muller and F.-H. Vasilescu, *Standard models for some commuting multioperators*, Proc. Amer. Math. Soc. 117 (1993), 979-989.

[NaFo70] B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space*, North Holland, Amsterdam, 1970.

[Ni02] N. Nikolski, *Operators, functions, and systems: an easy reading.* Vol. 1 and Vol. 2, Mathematical Surveys and Monographs, 92, 93. American Mathematical Society, Providence, RI, 2002.

[Ol06] A. Olofsson, *A characteristic operator function for the class of n-hypercontractions*, J. Funct. Anal., 236 (2006), 517-545.

[Pa70] S. Parrott, *Unitary dilations for commuting contractions*, Pacific J. Math., 34 (1970), 481–490.

[Po89] G. Popescu, *Characteristic functions for infinite sequences of noncommuting operators*, J. Operator Th., 22 (1989), 51-71.

[Po99] G. Popescu, *Poisson transforms on some C^*-algebras generated by isometries*, J. Funct. Anal. 161 (1999), 27-61.

[Po06] G. Popescu, *Operator theory on noncommutative varieties*, Indiana Univ. Math. J. 55 (2006), 389-442.

[Po10] G. Popescu, *Operator theory on noncommutative domains*, Mem. Amer. Math. Soc. 205 (2010), no. 964.

[Po11] G. Popescu, *Joint similarity to operators in noncommutative varieties*, Proc. Lond. Math. Soc. (3) 103 (2011), 331-370.

[P94] M. Putinar, *On the rigidity of Bergman submodules*, Amer. J. Math. 116 (1994), 1421-1432.

[RaRo73] H. Radjavi and P. Rosenthal, *Invariant subspaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 77, Springer-Verlag, New York-Heidelberg, 1973.

[R88] S. Richter, *Unitary equivalence of invariant subspaces of Bergman and Dirichlet spaces*, Pacific J. Math. 133 (1988), 151-156.

[RSu10] S. Richter and C. Sundberg, *Joint extensions in families of contractive commuting operator tuples*, J. Funct. Anal. 258 (2010), 3319-3346.

[RRov97] M. Rosenblum and J. Rovnyak, *Hardy classes and operator theory*, Corrected reprint of the 1985 original. Dover Publications, Mineola, NY, 1997.

[Ro60] G.-C Rota, *On models for linear operators*, Comm. Pure Appl. Math. 13 (1960), 469–472.

[Ru69] W. Rudin, *Function Theory in Polydiscs*, Benjamin, New York 1969.

[Ru80] W. Rudin, *Function Theory in the Unit Ball of $\mathbb{C}^n$*, Springer-Verlag, New York (1980).

[Sa13a] J. Sarkar, *Submodules of the Hardy module over polydisc*, arXiv:1304.1564, to appear in Israel Journal of Mathematics.

[Sa13b] J. Sarkar, *An invariant subspace theorem and invariant subspaces of analytic reproducing kernel Hilbert spaces - I*, preprint. arXiv:1309.2384.

[Sa13c] J. Sarkar, *An invariant subspace theorem and invariant subspaces of analytic reproducing kernel Hilbert spaces - II*, preprint. arXiv:1310.1014.

[Sa14a] J. Sarkar, *Applications of Hilbert Module Approach to Multivariable Operator Theory*, Handbook of Operator Theory, to appear.

[Sa14b] J. Sarkar, *Operator Theory on Symmetrized Bidisc*, To appear in Indiana University Mathematics Journal.

[Sa14c] J. Sarkar, *Wold decomposition for doubly commuting isometries*, Linear Algebra Appl. 445 (2014), 289-301.

[SaASW13] J. Sarkar, A. Sasane and B. Wick, *Doubly commuting submodules of the Hardy module over polydiscs*, Studia Mathematica, 217 (2013), 179–192.

[Sh01] S. Shimorin, *Wold-type decompositions and wandering subspaces for operators close to isometries*, J. reine angew. Math. 531 (2001) 147-189.

[St80] D. Stegenga, *Multipliers of the Dirichlet Space*, Illinois J. Math. 24(1980), 113–139.

[Ta70a] J. Taylor, *The analytic-functional calculus for several commuting operators*, Acta Math. 125 (1970), 138.

[Ta70b] J. Taylor, *A joint spectrum for several commuting operators*, J. Funct. Anal. 6 (1970), 172-191.

[Uc90] M. Uchiyama, *Curvatures and similarity of operators with holomorphic eigenvectors*, Trans. Amer. Math. Soc. 319 (1990), 405–415.

[Va74] N. Varopoulos, *On an inequality of von Neumann and an application of the metric theory of tensor products to operators theory*, J. Funct. Anal. 16 (1974), 83-100.

[Va82] F.-H. Vasilescu, *Analytic functional calculus and spectral decompositions*, Translated from the Romanian. Mathematics and its Applications (East European Series), 1. D. Reidel Publishing Co., Dordrecht; Editura Academiei Republicii Socialiste Romnia, Bucharest, 1982.

[Va92] F.-H. Vasilescu, *An operator-valued Poisson kernel*, J. Funct. Anal. 110 (1992), 47-72.

[Ve72] U. Venugopalkrishna, *Fredholm operators associated with strongly pseudoconvex domains in* $\mathbb{C}^n$, J. Functional Analysis 9 (1972), 349–373.

[Jv51] J. von Neumann, *Eine Spektraltheorie fur allgemeine Operatoren eines unitdren Raumes*, Math. Nachr. 4 (1951), 258–281.

[We80] R. O. Wells, *Differential analysis on complex manifolds, Graduate Texts in Mathematics*, Springer-Verlag, New York-Berlin, 1980.

[Wo38] H. Wold, *A study in the analysis of stationary time series*, Almquist and Wiksell, Uppsala, 1938.

[Wol97] R. Wolff, *Spectra of analytic Toeplitz tuples on Hardy spaces*, Bull. London Math. Soc. 29 (1997), 65-72.

[Ya92] K. Yan, *Equivalence of homogeneous principal Hardy submodules*, Indiana Univ. Math. J. 41 (1992), 875-884.

[Y05] R. Yang, *Hilbert-Schmidt submodules and issues of unitary equivalence*, J. Operator Theory 53 (2005) 169-184.

[Zh91] K. Zhu, *Mobius invariant Hilbert spaces of holomorphic functions in the unit ball of* $\mathbb{C}^n$, Trans. Amer. Math. Soc. 323 (1991), 823–842.

[Zh00] K. Zhu, *Operators in Cowen-Douglas classes*, Illinois J. Math.44 (2000), 767–783.

[Zh08] K. Zhu and R. Zhao, *Theory of Bergman spaces on the unit ball*, Memoires de la SMF 115 (2008).

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