APPROXIMATION, INTERPOLATION, AND LIFTING ON THE UNIT BALL

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Abstract. We solve the Nevanlinna-Pick interpolation problem on the open unit ball of $\mathbb{C}^n$. Our solutions signify the role of inner functions on the unit ball, objects whose existence was once in doubt, and to which Aleksandrov, Rudin, Sibony, and others made fundamental contributions. The results also reveal the importance of extremal functions, which emerge as natural analogues of finite Blaschke products in the unit ball. This viewpoint is illustrated by a Carathéodory approximation theorem and a unit ball analogue of Pick's theorem: every solvable interpolation problem admits an extremal function solution. We also solve the commutant lifting problem, where both inner functions and extremal functions play a fundamental role. These results resolve several well-known problems on the unit ball.

Contents

1. Introduction

For each $n \geq 1$, let $\mathbb{B}^n$ denote the open unit ball in $\mathbb{C}^n$:

$$\mathbb{B}^n = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{i=1}^{n} |z_i|^2 < 1 \right\}.$$

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We write $\mathbb{B}^1 = \mathbb{D}$. Let $H^\infty(\mathbb{B}^n)$ denote the Banach space of all scalar-valued bounded analytic functions on $\mathbb{B}^n$. The space $H^\infty(\mathbb{B}^n)$, endowed with the sup norm $\|\cdot\|_\infty$, is a commutative Banach algebra. Denote the closed unit ball of $H^\infty(\mathbb{B}^n)$ by

$$\mathcal{S}(\mathbb{B}^n) = \{\varphi \in H^\infty(\mathbb{B}^n) : \|\varphi\|_\infty \leq 1\}.$$

The elements of $\mathcal{S}(\mathbb{B}^n)$ are called *Schur functions.*

By *interpolation data* (or simply *data*), we mean two sets $\mathcal{Z}$ and $\mathcal{W}$, where $\mathcal{Z} = \{z_i\}_{i=1}^m$ is a set of m distinct points in $\mathbb{B}^n$, called the interpolation nodes, and $\mathcal{W} = \{w_i\}_{i=1}^m$ is a collection of m scalars in $\mathbb{D}$, called the target values.

Problem 1 (Nevanlinna–Pick interpolation problem)**.** Given data $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subseteq \mathbb{D}$, determine when there exists a function $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that

$$\varphi(z_i) = w_i,$$

for all $i = 1, \ldots, m$.

If such a function $\varphi \in \mathcal{S}(\mathbb{B}^n)$ exists, then we say that the interpolation data $\mathcal{Z}$ and $\mathcal{W}$ solve the *Nevanlinna–Pick interpolation problem*, or simply *the interpolation problem.* In this case, φ is called an *interpolant*, and we say that φ *interpolates* $\mathcal{Z}$ and $\mathcal{W}$. To avoid trivialities, we always assume that the number of nodes satisfies

$$m > 1.$$

The *extremal problem* imposes additional restrictions on the interpolants:

Problem 2 (Extremal problem)**.** Determine when the data $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subseteq \mathbb{D}$ solve the interpolation problem with the property that every $\varphi \in \mathcal{S}(\mathbb{B}^n)$ interpolating $\mathcal{Z}$ and $\mathcal{W}$ satisfies

$$\|\varphi\|_\infty = 1.$$

The interpolation problems described above have long been of central importance in several areas of mathematics and certain branches of engineering. The problem of characterizing interpolation on $\mathbb{B}^n$ has remained open for many years, and in this paper, we provide a complete solution. Our approach, however, goes considerably further. We establish the significance of extremal functions as analogues of finite Blaschke products on $\mathbb{B}^n$ and develop an approximation theory that in particular extends the classical Carathéodory theorem from $\mathbb{D}$ to $\mathbb{B}^n$. This is developed, in particular, as a consequence of our solution to the Pick problem on the unit ball. We show that inner functions provide a powerful tool in interpolation and play a fundamental role in our development of the theory. Their importance in one-variable function theory is well known; however, on $\mathbb{B}^n$, even their existence was uncertain until the pioneering work of Aleksandrov [4] (also see Rudin [41]).

We briefly review the classical results on interpolation problems. When $n = 1$, Problem 1 was solved independently by Nevanlinna [34] and Pick [36] in the 1910s:

Theorem 1.1 (Nevanlinna–Pick interpolation theorem on $\mathbb{D}$)**.** *There exists $\varphi \in \mathcal{S}(\mathbb{D})$ such that $\varphi(z_i) = w_i$ for all $i = 1, \ldots, m$, if and only if the $m \times m$ Pick matrix*

$$\left[\frac{1 - w_i\overline{w_j}}{1 - z_i\overline{z_j}}\right]_{m\times m},$$

is positive semi-definite.

Now we turn to the extremal problem, which is especially intriguing due to its close connection with classical function theory. We collect all the extremal solutions:

$$\mathcal{E}(\mathbb{B}^n) = \{\varphi \in \mathcal{S}(\mathbb{B}^n) : \varphi \text{ solves some extremal problem}\}.$$

Functions in this class are called *extremal functions*. In the case $n = 1$, an interpolation problem is extremal if and only if its solution is unique and is given by a finite Blaschke product (cf. [3, page 77]). In particular, we have

$$\mathcal{E}(\mathbb{D}) = \{\text{finite Blaschke product}\}. \tag{1.1}$$

Recall that a *finite Blaschke product* is a function of the form

$$b = \prod_{j=1}^{p} \frac{\overline{\alpha_j}}{|\alpha_j|} \frac{z - \alpha_j}{1 - \overline{\alpha}_j z},$$

where $\alpha_1, \dots, \alpha_p \in \mathbb{D}$ and $p \in \mathbb{N}$. Finite Blaschke products play a fundamental role in function theory. We highlight two results that illustrate their significance and are particularly relevant in our several variables context. The first is a classical result of Pick:

Theorem 1.2 (Pick's theorem on $\mathbb{D}$)**.** *Suppose the data $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{D}$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subseteq \mathbb{D}$ solve the interpolation problem. Then there exists*

$$b \in \mathcal{E}(\mathbb{D}),$$

that interpolates $\mathcal{Z}$ and $\mathcal{W}$.

By (1.1), we know that b is a finite Blaschke product. The second is a classical result of Carathéodory [9]:

Theorem 1.3 (Carathéodory's theorem on $\mathbb{D}$)**.** *Let φ be a nonzero function in $\mathcal{S}(\mathbb{D})$. Then there exists a sequence of finite Blaschke products $\{b_k\}_k \subseteq \mathcal{E}(\mathbb{D})$ such that*

$$\lim_{k\to\infty} b_k = \varphi,$$

uniformly on compact subsets of $\mathbb{D}$.

A very closely related instance concerns inner functions on $\mathbb{B}^n$, specifically when $n > 1$. Let $\mathbb{S}^n = \partial\mathbb{B}^n$. A function $\varphi \in H^\infty(\mathbb{B}^n)$ is said to be *inner* if

$$|\varphi(\zeta)| = 1,$$

for almost every $\zeta \in \mathbb{S}^n$ (in the sense of K-limits; see Rudin [43] or Section 4). We write

$$\mathcal{I}(\mathbb{B}^n) = \{\varphi \in H^\infty(\mathbb{B}^n) : \varphi \text{ is inner}\}.$$

In one variable, finite Blaschke products are the simplest examples of inner functions. In contrast, on $\mathbb{B}^n$, $n > 1$, there is no comparable description or understanding of finite Blaschke products, let alone inner functions in general. Indeed, for a long time, it was widely conjectured that no nonconstant inner functions exist on $\mathbb{B}^n$, $n > 1$. This belief persisted until 1981, when Aleksandrov [4] exhibited the existence of such functions (they are pathological); shortly thereafter, Löw [28] independently established their existence (also see Hakim and Sibony [20]; and see Rudin's historical comments in [42]). Even more surprisingly, despite their pathological nature, such functions exist in enormous abundance (see Aleksandrov [4] and Rudin [42]):

Theorem 1.4 (Aleksandrov-Rudin theorem on $\mathbb{B}^n$)**.** *Let $\varphi \in \mathcal{S}(\mathbb{B}^n)$ be a nonzero function. Then there exists a sequence $\{\varphi_k\} \subseteq \mathcal{I}(\mathbb{B}^n)$ such that*

$$\lim_{k\to\infty} \varphi_k = \varphi,$$

uniformly on compact subsets of $\mathbb{B}^n$, and

$$\lim_{k\to\infty} \int_{\mathbb{S}^n} \varphi_k f \, d\sigma = \int_{\mathbb{S}^n} \varphi f \, d\sigma,$$

for all $f \in L^1(\mathbb{S}^n)$, where $d\sigma$ denotes the normalized surface measure on $\mathbb{S}^n$.

In contrast to the scarcity of concrete examples of inner functions, there is an abundance of extremal functions [27]. We view extremal functions as the natural analogues of finite Blaschke products on B^n. Indeed, when $n = 1$, as pointed out in (1.1), these two classes coincide. From this perspective, we establish the following generalization of Pick's result from $\mathbb{D}$ to $\mathbb{B}^n$ (see Theorem 2.1):

Theorem 1. Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subseteq \mathbb{D}$ be data with $\mathcal{W} \neq \{0\}$. If $\mathcal{Z}$ and $\mathcal{W}$ solve the interpolation problem, then there exists an extremal solution

$$\varphi \in \mathcal{E}(\mathbb{B}^n),$$

that interpolates the data $\mathcal{Z}$ and $\mathcal{W}$.

As an application of this result, we establish the following extension of Carathéodory's theorem from $\mathbb{D}$ to $\mathbb{B}^n$. Note that this result is also in the spirit of Aleksandrov and Rudin's theorem on $\mathbb{B}^n$, except that the class of inner functions is replaced by the class of extremal functions (see Theorem 2.2):

Theorem 2. Let $\varphi \in \mathcal{S}(\mathbb{B}^n)$ be nonzero. Then there is a sequence $\{\varphi_k\} \subseteq \mathcal{E}(\mathbb{B}^n)$ satisfying

$$\lim_{k\to\infty} \varphi_k = \varphi,$$

uniformly on compact subsets of $\mathbb{B}^n$. Furthermore,

$$\lim_{k\to\infty} \int_{\mathbb{S}^n} \varphi_k f \, d\sigma = \int_{\mathbb{S}^n} \varphi f \, d\sigma,$$

for all $f \in L^1(\mathbb{S}^n)$.

The Aleksandrov-Rudin theorem, together with the above yields:

$$\overline{\mathcal{I}(\mathbb{B}^n)}^{(w^*, L^\infty(\mathbb{S}^n))} = \overline{\mathcal{E}(\mathbb{B}^n)}^{(w^*, L^\infty(\mathbb{S}^n))} = \mathcal{S}(\mathbb{B}^n).$$

Therefore, we now have two distinguished classes of functions that approximate Schur functions: $\mathcal{I}(\mathbb{B}^n)$ and $\mathcal{E}(\mathbb{B}^n)$. Theorem 1 highlights the role of the class of extremal functions in the interpolation problem, while the following result does the same for the class of inner functions (see Theorem 7.5):

Theorem 3. Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subseteq \mathbb{D}$ be interpolation data. Then $\mathcal{Z}$ and $\mathcal{W}$ solve the interpolation problem if and only if there exists a sequence of inner functions $\{\varphi_k\} \subseteq \mathcal{I}(\mathbb{B}^n)$ such that

$$\lim_{k\to\infty} \varphi_k(z_j) = w_j,$$

for all $j = 1, \ldots, m$.

We remark that part of this paper, including several results, examples, and counterexamples, relies on the theory of inner functions developed by Aleksandrov [4] and Rudin [42]. Extremal functions play a similarly important role throughout the paper.

In addition to the above, we present several other characterizations of the interpolation problem, which are collected in Theorem 7. The significance of the preceding result lies in the way it highlights the inherent complexity of the interpolation problem and its deep connection to the subtleties of function theory on $\mathbb{B}^n$, $n > 1$.

Another approach to our solution of the interpolation problem follows in the footsteps of Sarason [45]: we first establish a commutant lifting theorem for $H^2(\mathbb{B}^n)$, the Hardy space over $\mathbb{B}^n$, and then use it to resolve the interpolation problem. We further characterize those interpolation problems that are extremal. It should be noted that the commutant lifting theorem for $H^2(\mathbb{B}^n)$, in the case $n > 1$, is itself an important and challenging problem. Recall that $H^2(\mathbb{B}^n)$ consists of all analytic functions $f : \mathbb{B}^n \to \mathbb{C}$ satisfying

$$\|f\| := \Big(\sup_{0<r<1} \int_{\mathbb{S}^n} |f(r\zeta_1, \ldots, r\zeta_n)|^2 \, d\sigma \Big)^{\frac{1}{2}} < \infty.$$

Each $\varphi \in H^\infty(\mathbb{B}^n)$ defines a multiplication operator T_φ on $H^2(\mathbb{B}^n)$ given by

$$T_\varphi f = \varphi f \qquad (f \in H^2(\mathbb{B}^n)),$$

with the property that $\|T_\varphi\| = \|\varphi\|_\infty$. The operator T_φ is also known as the *analytic Toeplitz operator* with symbol φ. The coordinate functions $\varphi = z_i$ give rise to the special multiplication operators T_{z_i}, $i = 1, \ldots, n$. We have the fundamental commutation property:

$$\{T_{z_1}, \ldots, T_{z_n}\}' = \{T_\varphi : \varphi \in H^\infty(\mathbb{B}^n)\}.$$

A closed subspace $\mathcal{Q}$ of $H^2(\mathbb{B}^n)$ is called a *quotient module* [11, 12, 19] if for each $i = 1, \ldots, n$,

$$T_{z_i}^* \mathcal{Q} \subseteq \mathcal{Q},$$

Given a quotient module $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ and a function $\varphi \in H^\infty(\mathbb{B}^n)$, we define S_φ on $\mathcal{Q}$ by

$$S_\varphi = P_{\mathcal{Q}} T_\varphi|_{\mathcal{Q}},$$

where $P_{\mathcal{Q}}$ denotes the orthogonal projection of $H^2(\mathbb{B}^n)$ onto $\mathcal{Q}$ [12]. We always assume that $\mathcal{Q} \neq \{0\}$. Of particular interest are the operators S_{z_i}, where

$$S_{z_i} f = P_{\mathcal{Q}}(z_i f),$$

for all $f \in \mathcal{Q}$ and $i = 1, \ldots, n$. We say that an operator $X \in \mathcal{B}(\mathcal{Q})$ is a *module map* if

$$X S_{z_i} = S_{z_i} X,$$

for all $i = 1, \ldots, n$. Given a Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(\mathcal{H})$ the space of all bounded linear operators on $\mathcal{H}$, and we set

$$\mathcal{B}_1(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \|T\| \leq 1\}.$$

With this terminology and structure in place, we can now formulate the commutant lifting problem as follows:

Problem 3. Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{B}^n)$, and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. When does there exist a Schur function $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that

$$X = S_\varphi?$$

Equivalently, this is a question about the commutativity of the following diagram, together with the additional requirement that $\varphi \in \mathcal{S}(\mathbb{B}^n)$:

$$\begin{tikzcd} H^2(\mathbb{B}^n) \arrow[r, "T_\varphi"] & H^2(\mathbb{B}^n) \arrow[d, "i_{\mathcal{Q}}^*"] \\ \mathcal{Q} \arrow[u, "i_{\mathcal{Q}}"] \arrow[r, "X"'] & \mathcal{Q} \end{tikzcd}$$

where $i_{\mathcal{Q}} : \mathcal{Q} \longrightarrow H^2(\mathbb{B}^n)$ denotes the inclusion map (here we note that $S_\varphi = i_{\mathcal{Q}}^* T_\varphi i_{\mathcal{Q}}$). If such an operator $X \in \mathcal{B}_1(\mathcal{Q})$ admits a representation $X = S_\varphi$ for some $\varphi \in \mathcal{S}(\mathbb{B}^n)$, then we say that X admits a *lift*, or that X admits a *lift to* T_φ, or simply that X is *liftable*.

When $n = 1$, Sarason's seminal work asserts that every such module map $X \in \mathcal{B}_1(\mathcal{Q})$ admits a lift [45]. Moreover, the lift is *norm preserving*: there exists $\varphi \in \mathcal{S}(\mathbb{D})$ such that $X = S_\varphi$ and

$$\|X\| = \|\varphi\|_\infty. \tag{1.2}$$

Sarason applied his lifting theorem to certain finite-dimensional quotient modules, thereby recovering the Nevanlinna–Pick interpolation theorem. Now we turn to several variables. In what follows, unless otherwise stated, the case of $n = 1$ will also be included in the discussion.

First we establish the essential tools that would be needed for the lifting theorem. For $S \subseteq L^2(\mathbb{S}^n)$, let S^{conj} denote the set of all conjugate functions of elements of S:

$$S^{conj} = \{\overline{f} : f \in S\},$$

where $\overline{f}$ denotes the conjugate of f. In view of K-limits, we shall, when convenient, identify $H^2(\mathbb{B}^n)$ with $H^2(\mathbb{S}^n)$ without further explanation (cf. Section 4), where

$$H^2(\mathbb{S}^n) = \overline{\mathbb{C}[z_1, \ldots, z_n]}^{L^2(\mathbb{S}^n)}.$$

Next, we define the space of "mixed functions" (also see [13]):

$$\mathcal{M}(\mathbb{S}^n) := L^2(\mathbb{S}^n) \ominus [H^2(\mathbb{S}^n) + H^2(\mathbb{S}^n)^{conj}].$$

Note that $\mathcal{M}(\mathbb{T}) = \{0\}$, where $\mathbb{S}^1 = \partial\mathbb{B} = \mathbb{T}$. This space admits a convenient decomposition in terms of harmonic homogeneous polynomials in the sense of Rudin [43, Chapter 12], which will be outlined in Section 13. Define the space of Hardy functions "vanishing at 0" as

$$H_0^2(\mathbb{S}^n) = H^2(\mathbb{S}^n) \ominus \{1\}.$$

Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{B}^n)$, which, in view of K-limits, we may regard as a closed subspace of $H^2(\mathbb{S}^n)$, as noted above. We consider $\mathcal{Q}^{conj}$, $\mathcal{M}(\mathbb{S}^n)$, and $H_0^2(\mathbb{S}^n)$ as subspaces of $L^1(\mathbb{S}^n)$ and define

$$\mathcal{M}_{\mathcal{Q}} = \mathcal{Q}^{conj} \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H_0^2(\mathbb{S}^n)].$$

where $\dot{+}$ denotes the skew sum in the Banach space $L^1(\mathbb{S}^n)$.

We now return to the inner and extremal functions. Throughout the paper, $\mathcal{A}(\mathbb{B}^n)$ denotes either $\mathcal{I}(\mathbb{B}^n)$ or $\mathcal{E}(\mathbb{B}^n)$. Let $S \subseteq L^1(\mathbb{S}^n)$, $\varphi \in \mathcal{S}(\mathbb{B}^n)$, and let $\{\varphi_k\}$ be a sequence in $\mathcal{A}(\mathbb{B}^n)$. We say that

$$w^* - \lim_{k \to \infty} \varphi_k = \varphi \text{ on } S,$$

if for every $f \in S$, we have

$$\lim_{k\to\infty} \int_{\mathbb{S}^n} \varphi_k f d\sigma = \int_{\mathbb{S}^n} \varphi f d\sigma.$$

The following is a summary of all the commutant lifting theorems obtained in this paper. These results also apply to the case $n = 1$ (but there will be exceptions like Theorem 11.5), and hence many of them are new even in the single-variable setting.

Theorem 4. Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{B}^n)$, and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. Set

$$\psi = X(P_{\mathcal{Q}}1).$$

The following conditions are equivalent:

(1) (Commutant lifting) X admits a lift.

(2) (Perturbations) There exists $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that

$$\psi - \varphi \in \mathcal{Q}^{\perp}.$$

Moreover, in this case $X = S_\varphi$.

(3) (Qualitative property) $X_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|.\|_1) \longrightarrow \mathbb{C}$ is a contraction, where

$$X_{\mathcal{Q}} f = \int_{\mathbb{S}^n} \psi f d\sigma \qquad (f \in \mathcal{M}_{\mathcal{Q}}).$$

(4) (Quantitative property) $\text{dist}_{L^1(\mathbb{S}^n)}\Big(\frac{\bar{\psi}}{\|\psi\|_2^2}, \widetilde{\mathcal{M}_{\mathcal{Q}}}\Big) \geq 1$, where

$$\widetilde{\mathcal{M}_{\mathcal{Q}}} = [\mathcal{Q}^{conj} \ominus \{\bar{\psi}\}] \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H_0^2(\mathbb{S}^n)].$$

(5) (Inner functions - I) There exists a sequence $\{\varphi_k\} \subseteq \mathcal{A}(\mathbb{B}^n)$ such that

$$w^* - \lim_{k\to\infty} \varphi_k = \psi \text{ on } \mathcal{M}_{\mathcal{Q}}.$$

(6) (Inner functions - II) There exists a sequence $\{\varphi_k\} \subseteq \mathcal{A}(\mathbb{B}^n)$ such that

$$w^* - \lim_{k\to\infty} \varphi_k = \psi \text{ on } \mathcal{Q}^{conj}.$$

In the context of the above theorem, we further point out that Lemma 5.2 yields the following identity:

$$\ker X_{\mathcal{Q}} = \widetilde{\mathcal{M}_{\mathcal{Q}}}.$$

The equivalence of (1) and (2) is less involved, yet it plays a central role in establishing the remaining equivalence conditions as well as several other results in this paper. Moreover, the perturbation property in (2) is connected, perhaps coincidentally, with Rudin's perturbation theorem for inner functions [42], a connection that we will explore further in Section 11.

The above quantitative and qualitative properties of lifting also appear in the polydisc setting [13], which we will elaborate on in Section 13.

Following Sarason (1.2), a natural question is whether module maps on quotient modules of $H^2(\mathbb{B}^n)$, $n > 1$, admit norm-preserving liftings. The situation in several variables, however, changes drastically. Using a function-theoretic result of Rudin concerning the Smirnov class for $n > 1$, we obtain the following somewhat unexpected result (see Corollary 9.4):

Theorem 5. Let X be a nonconstant module map acting on a finite-dimensional quotient module $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$. If

$$n > 1,$$

then X does not admit a norm-preserving lift.

In fact, we have the following characterization of norm-preserving liftings of module maps on finite-dimensional quotient modules (see Theorem 9.5 and Corollaries 9.6 and 9.7):

Theorem 6. Suppose $n \geq 1$. Let $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ be a finite-dimensional quotient module and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a nonconstant module map. Consider the functional $X_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1) \to \mathbb{C}$ as in the qualitative part of Theorem 4. The following are equivalent:

(1) X has a norm preserving lift.
(2) $\|X\| = \|X_{\mathcal{Q}}\|$.
(3) $n = 1$.

A key ingredient in the proof of the above theorem, and indeed of many subsequent results in this paper, is the norm formula for the functional $X_{\mathcal{Q}}$. Namely, in the setting of Theorem 4, if $X \in \mathcal{B}_1(\mathcal{Q})$ admits a lift, then (see Theorem 4.5)

$$\|X_{\mathcal{Q}}\| = \inf\{\|\varphi\|_\infty : \varphi \in \mathcal{S}(\mathbb{B}^n) \text{ and } X = S_\varphi\}.$$

In particular, $\|X_{\mathcal{Q}}\|$ equals the minimum possible $H^\infty(\mathbb{B}^n)$-norm among all symbols representing X. This identity is intriguing and shows that the choice of the functional $X_{\mathcal{Q}}$ is appropriate and holds further potential for related problems.

Now we turn to a detailed presentation of our solution to the interpolation problem. It is important to emphasize that a Pick matrix-type characterization for Schur function interpolants is generally not expected in several variables, since such criteria are intrinsically tied to complete Nevanlinna–Pick kernels [2, 32, 37]. In contrast, the Szegö kernel K_n on $\mathbb{B}^n$, which is central in the Schur function framework, is not a complete Nevanlinna–Pick kernel except for the case of $n = 1$. Recall that $K_n : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$ is defined by

$$K_n(z, w) = (1 - \langle z, w\rangle)^{-n},$$

where $\langle z, w\rangle = \sum_{i=1}^n z_i \bar{w}_i$ for all $z, w \in \mathbb{B}^n$. For each $w \in \mathbb{B}^n$, the *Szegö kernel function* $K_n(\cdot, w) \in H^2(\mathbb{B}^n)$ is defined by

$$(K_n(\cdot, w))(z) = K_n(z, w),$$

for all $z \in \mathbb{B}^n$. Given interpolation data $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$ and $\{w_i\}_{i=1}^m \subseteq \mathbb{D}$, define m-dimensional quotient module $\mathcal{Q}_{\mathcal{Z}} \subseteq H^2(\mathbb{B}^n)$ by

$$\mathcal{Q}_{\mathcal{Z}} = \text{span}\{K_n(\cdot, z_j) : j = 1, \ldots, m\}.$$

Also, define a function

$$\psi_{\mathcal{Z},\mathcal{W}} = \sum_{j=1}^m c_j K_n(\cdot, z_j). \tag{1.3}$$

where the scalars $\{c_j\}_{j=1}^m$ are given by

$$\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} K_n(z_1, z_1) & \ldots & K_n(z_1, z_m) \\ \vdots & \ddots & \vdots \\ K_n(z_m, z_1) & \ldots & K_n(z_m, z_m) \end{bmatrix}^{-1} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}.$$

The inverse makes sense, since the matrix in question is a Gram matrix. There are two notable features that immediately make this function particularly distinguished:

(1) $\psi_{\mathcal{Z},\mathcal{W}}$ interpolates the prescribed data; that is, for all $i = 1, \ldots, m$,

$$\psi_{\mathcal{Z},\mathcal{W}}(z_i) = w_i.$$

(2) $\psi_{\mathcal{Z},\mathcal{W}}$ is rational, with all poles lying outside $\overline{\mathbb{B}^n}$.

This function plays a significant role in a number of interpolation results developed in this paper, which are summarized in the following theorem:

Theorem 7. Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subseteq \mathbb{D}$ be interpolation data. Consider the function

$$\psi_{\mathcal{Z},\mathcal{W}} = \sum_{j=1}^m c_j K_n(\cdot, z_j),$$

as defined in (1.3), where the scalars $\{c_1, \ldots, c_m\}$ are given by

$$\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \big[K_n(z_i, z_j)\big]_{m\times m}^{-1} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix},$$

and define

$$\widetilde{\mathcal{M}_{\mathcal{Q}_\mathcal{Z}}} = [\mathcal{Q}_\mathcal{Z}^{conj} \ominus \{\overline{\psi_{\mathcal{Z},\mathcal{W}}}\}]\dot{+}[\mathcal{M}(\mathbb{S}^n)\dot{+}H_0^2(\mathbb{S}^n)].$$

The following conditions are equivalent:

(1) (Interpolation) $\mathcal{Z}$ and $\mathcal{W}$ solve the interpolation problem.

(2) (Qualitative property) $\Pi_{\mathcal{Z},\mathcal{W}} : (\mathcal{M}_{\mathcal{Q}_\mathcal{Z}}, \|\cdot\|_1) \to \mathbb{C}$ is a contraction, where

$$\Pi_{\mathcal{Z},\mathcal{W}} f = \int_{\mathbb{S}^n} \psi_{\mathcal{Z},\mathcal{W}} f \, d\sigma \qquad (f \in \mathcal{M}_{\mathcal{Q}_\mathcal{Z}}),$$

and

$$\mathcal{M}_{\mathcal{Q}_\mathcal{Z}} = \mathcal{Q}_\mathcal{Z}^{conj}\dot{+}[\mathcal{M}(\mathbb{S}^n)\dot{+}H_0^2(\mathbb{S}^n)].$$

(3) (Quantitative property - I) $\mathrm{dist}_{L^1(\mathbb{S}^n)}\Big(\frac{\overline{\psi_{\mathcal{Z},\mathcal{W}}}}{\|\psi_{\mathcal{Z},\mathcal{W}}\|_2^2}, \widetilde{\mathcal{M}_{\mathcal{Q}_\mathcal{Z}}}\Big) \geq 1.$

(4) (Quantitative property - II) $\mathrm{dist}_{L^1(\mathbb{S}^n)}\left(\frac{1}{\left(\sum_{i=1}^m |w_i|^2\right)^2} \sum_{j=1}^m \overline{w_j} \overline{K_n(\cdot, z_j)}, \widetilde{\mathcal{M}_{\mathcal{Q}_\mathcal{Z}}}\right) \geq 1.$

(5) (Inner functions) There exists a sequence of inner functions $\{\varphi_k\} \subseteq \mathcal{I}(\mathbb{B}^n)$ such that for each $j = 1, \ldots, m$,

$$\lim_{k\to\infty} \varphi_k(z_j) = w_j.$$

The distance formula appearing in the quantitative part of the above theorem, as well as in Theorem 4, is reminiscent of the classical Nehari theorem [33] (see the discussion following Corollary 7.3).

The fact that the distance appearing in the above interpolation theorem is equal to one has an important implication. In one variable, extremal problem is closely related to finite Blaschke products (see (1.1)). In several variables, however, finite Blaschke products remain elusive. Nevertheless, we have the following result (see Theorem 10.1):

Theorem 8. Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{B}^n$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subset \mathbb{D}$ be interpolation data. Consider the notation introduced in Theorem 7. Then the following are equivalent:

(1) $\mathcal{Z}$ and $\mathcal{W}$ admit an extremal solution.

(2) $\|\Pi_{\mathcal{Z},\mathcal{W}}\| = 1$.

(3) $\mathrm{dist}_{L^1(\mathbb{S}^n)}\Big(\frac{\overline{\psi}}{\|\psi\|_2^2}, \widetilde{\mathcal{M}_{\mathcal{Q}_\mathcal{Z}}}\Big) = 1.$

We adopt a normalization technique and apply our lifting theorem to obtain the following result, which directly connects solutions to the interpolation problem with solutions to the extremal problem (see Corollary 10.3). We consider the notation introduced in Theorem 7:

Theorem 9. Let $m > 1$. Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{B}^n$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subset \mathbb{D}$ be interpolation data, with $\mathcal{W} \neq \{0\}$, and suppose that $\mathcal{Z}$ and $\mathcal{W}$ solve the interpolation problem. Define

$$\lambda_{\mathcal{Z},\mathcal{W}} = \inf\{\|\eta\|_\infty : \eta \in \mathcal{S}(\mathbb{B}^n) \text{ interpolates } \mathcal{Z} \text{ and } \mathcal{W}\}.$$

Then there exists an extremal function $\varphi \in \mathcal{E}(\mathbb{B}^n)$ such that

$$\lambda_{\mathcal{Z},\mathcal{W}}\varphi,$$

interpolates $\mathcal{Z}$ and $\mathcal{W}$. Furthermore, $\lambda_{\mathcal{Z},\mathcal{W}}\varphi$ is a minimum-norm interpolant for the data $\mathcal{Z}$ and $\mathcal{W}$.

Kosiński and Zwonek [27, Theorem 2] provide a certain description of the solution set for the three-point extremal problem. Consequently, whenever a three-point interpolation problem is solvable, Theorem 8 yields an explicit interpolant obtained by scaling a function from the extremal solution set of Zwonek and Kosiński by a factor in $(0, 1]$ (see Theorem 12.1).

We remark that the theory of Hilbert function spaces in several variables is of different levels of intricacy on $\mathbb{B}^n$ than on the polydisc $\mathbb{D}^n$. While solutions to the interpolation and commutant lifting problems in the polydisc setting have been developed in the recent paper [13], as well as in several earlier works that established results of considerable depth and influence [1, 6], the literature contains virtually no comparable attempts, let alone partial results, within the framework of Schur functions on $\mathbb{B}^n$ (however, see [14, 27]). This disparity also reflects the greater complexity of the function theory associated with Hilbert function spaces on the ball (see Section 13 for more comments).

In particular, the connection between quotient modules of $H^2(\mathbb{B}^n)$ and inner functions developed here reveals a fundamental distinction between $\mathbb{B}^n$ and $\mathbb{D}^n$. A central theme of this paper is the study of the interplay between interpolation problems, inner and extremal functions, and their approximation properties on $\mathbb{B}^n$. This interplay reflects the inherent complexity of Hilbert function space theory in the setting of the unit ball.

The remainder of the paper is organized as follows. Section 2 is devoted to the Pick and Carathéodory theorems on $\mathbb{B}^n$. Sections 3, 4, 5, and 6 present several characterizations of commutant lifting. Section 7 addresses solutions to the interpolation problem, while Section 8 provides some solutions to the Carathéodory-Fejér interpolation problem. Norm-preserving lifting is discussed in Section 9, and Section 10 is devoted to solutions to the extremal problem. Sections 11 and 12 are mostly examples of lifting and interpolation. The paper concludes with a final remarks section in Section 13.

2. Pick and Carathéodory theorems on $\mathbb{B}^n$

In this section, we establish two results that extend the classical theorems of Pick and Carathéodory from $\mathbb{D}$ to $\mathbb{B}^n$. We begin by recalling the extremal problem (see also Section 1).

Given interpolation data $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subseteq \mathbb{D}$, we say that $\mathcal{Z}$ and $\mathcal{W}$ solve the *extremal problem* (or, more specifically, the *m-point extremal problem*) if:

(1) (Interpolation) $\mathcal{Z}$ and $\mathcal{W}$ solve the interpolation problem.

(2) (Extremal) If $\psi \in \mathcal{S}(\mathbb{B}^n)$ interpolates $\mathcal{Z}$ and $\mathcal{W}$, then
$$\|\psi\|_\infty = 1.$$

For each natural number $m \in \mathbb{N}$, we denote by $\mathcal{E}_m(\mathbb{B}^n)$ the set of all solutions to the m-point extremal problems:
$$\mathcal{E}_m(\mathbb{B}^n) = \{\varphi \in \mathcal{S}(\mathbb{B}^n) : \varphi \text{ solves some } m\text{-point extremal problem}\}.$$
Define
$$\mathcal{E}(\mathbb{B}^n) = \bigcup_{m\in\mathbb{N}} \mathcal{E}_m(\mathbb{B}^n).$$
Elements of $\mathcal{E}(\mathbb{B}^n)$ are called *extremal functions.* The classical one-variable case $n = 1$ is of particular interest. In this case, we know that (recall (1.1))
$$\mathcal{E}(\mathbb{D}) = \{\text{finite Blaschke products}\}.$$
In fact, in the classical one-variable setting, an interpolation problem is extremal if and only if the associated Pick matrix is positive semidefinite but not positive definite [3, Chapter 6]. Moreover, in this case, the extremal solution is unique [3, Theorem 6.4].

There is another interpretation of the set $\mathcal{E}(\mathbb{D})$ due to Pick: If the data $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{D}$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subseteq \mathbb{D}$ solve the interpolation problem, then there exists
$$b \in \mathcal{E}(\mathbb{D}),$$
such that b interpolates $\mathcal{Z}$ and $\mathcal{W}$. As noted above, φ is a finite Blaschke product (see Theorem 1.2). Finally, we recall the classical Carathéodory theorem (see Theorem 1.3): Let $\varphi \in \mathcal{S}(\mathbb{D})$. Then there exists a sequence $\{b_k\} \subseteq \mathcal{E}(\mathbb{D})$ such that
$$\lim_{k\to\infty} b_k = \varphi,$$
uniformly on compact subsets of $\mathbb{D}$. Again, as noted above, b_k's are all finite Blaschke products. We now consider the general case $n \geq 1$. Here, in contrast to the classical one-variable case, there is no satisfactory analogue of finite Blaschke products. We therefore replace finite Blaschke products by the class of extremal functions. Viewed in this light, the following theorem may be regarded as an extension of Pick's theorem from $\mathbb{D}$ to $\mathbb{B}^n$.

Theorem 2.1. *Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subseteq \mathbb{D}$ be data with $\mathcal{W} \neq \{0\}$. If $\mathcal{Z}$ and $\mathcal{W}$ solve the interpolation problem, then there exists an extremal function*
$$\varphi \in \mathcal{E}(\mathbb{B}^n),$$
that interpolates $\mathcal{Z}$ and $\mathcal{W}$.

Proof. Without loss of generality, assume that the interpolation problem is not extremal. Define
$$\mathcal{S}_{\mathcal{Z},\mathcal{W}} = \{\varphi \in \mathcal{S}(\mathbb{B}^n) : \varphi \text{ interpolates } \mathcal{Z} \text{ and } \mathcal{W}\}.$$
Since $\mathcal{Z}$ and $\mathcal{W}$ solve the interpolation problem, by our assumption, it follows that
$$\mathcal{S}_{\mathcal{Z},\mathcal{W}} \neq \emptyset.$$
Pick
$$z_{m+1} \in \mathbb{B}^n \setminus \{z_1, \ldots, z_m\},$$
and define
$$M = \sup_{\varphi\in\mathcal{S}_{\mathcal{Z},\mathcal{W}}} |\varphi(z_{m+1})|.$$

Clearly, $M \leq 1$ (as $\mathcal{S}_{\mathcal{Z},\mathcal{W}} \subseteq \mathcal{S}(\mathbb{B}^n)$). We claim that

$$M > 0.$$

Since the interpolation problem in non-extremal, there exists $\varphi \in \mathcal{S}_{\mathcal{Z},\mathcal{W}}$ such that

$$\|\varphi\|_\infty < 1.$$

If $\varphi(z_{m+1}) \neq 0$, then $M > 0$. Therefore, assume that

$$\varphi(z_{m+1}) = 0.$$

Choose a polynomial $p \in \mathbb{C}[z_1, \ldots, z_n]$ such that

$$\|p\|_\infty = 1,$$

and

$$p(z_i) = 0,$$

for all $i = 1, \ldots, m$, and

$$p(z_{m+1}) \neq 0.$$

Since $\|\varphi\|_\infty < 1$, there exists $t > 0$ such that

$$\|\widetilde{\varphi}\|_\infty < 1,$$

where

$$\widetilde{\varphi} := \varphi + tp$$

Clearly, $\widetilde{\varphi}$ interpolates $\mathcal{Z}$ and $\mathcal{W}$; that is, $\widetilde{\varphi} \in \mathcal{S}_{\mathcal{Z},\mathcal{W}}$. Also, $\widetilde{\varphi}(z_{m+1}) = tp(z_{m+1})$ implies that $\widetilde{\varphi}(z_{m+1}) \neq 0$. Then

$$M \geq \left|\widetilde{\varphi}(z_{m+1})\right| > 0,$$

proving the claim that $M > 0$. There exists a sequence $\{\varphi_r\} \subseteq \mathcal{S}_{\mathcal{Z},\mathcal{W}}$ such that

$$\lim_{r\to\infty} |\varphi_r(z_{m+1})| = M.$$

Since $\|\varphi_r\|_\infty \leq 1$, for $r \geq 1$, the sequence $\{\varphi_r\}$ is uniformly bounded. Hence, by Montel's theorem, there exists a subsequence $\{\varphi_{r_k}\}$ that converges uniformly on compact subsets of $\mathbb{B}^n$ to some function θ. By Weierstrass' theorem, it follows that θ is analytic on $\mathbb{B}^n$. On the other hand, since $|\varphi_{r_k}(z)| \leq 1$ for all k, we have $|\theta(z)| \leq 1$ for all $z \in \mathbb{B}^n$, that is,

$$\theta \in \mathcal{S}(\mathbb{B}^n).$$

Moreover, $\{\varphi_{r_k}\} \subseteq \mathcal{S}_{\mathcal{Z},\mathcal{W}}$ implies

$$\varphi_{r_k}(z_j) = w_j,$$

for all $k \in \mathbb{N}$ and for all $j = 1, \ldots, m$. Since φ_{r_k} converges pointwise to θ, it follows that

$$\varphi_{r_k}(z_j) \to \theta(z_j),$$

and therefore

$$\theta(z_j) = w_j,$$

for all $j = 1, \ldots, m$. This proves that $\theta \in \mathcal{S}_{\mathcal{Z},\mathcal{W}}$. Let

$$w_{m+1} := \theta(z_{m+1}),$$

and define data $\widetilde{\mathcal{Z}}$ and $\widetilde{\mathcal{W}}$ by

$$\widetilde{\mathcal{Z}} = \mathcal{Z} \cup \{z_{m+1}\},$$

and

$$\widetilde{\mathcal{W}} = \mathcal{W} \cup \{w_{m+1}\}.$$

Clearly, $\theta \in \mathcal{S}(\mathbb{B}^n)$ interpolates $\widetilde{\mathcal{Z}}$ and $\widetilde{\mathcal{W}}$. We claim that $\widetilde{\mathcal{Z}}$ and $\widetilde{\mathcal{W}}$ solve the extremal problem. Suppose $\eta \in \mathcal{S}(\mathbb{B}^n)$ interpolates $\widetilde{\mathcal{Z}}$ and $\widetilde{\mathcal{W}}$. We claim that $\|\eta\|_\infty = 1$. Suppose, to the contrary, that

$$\|\eta\|_\infty < 1.$$

Once again, choose a polynomial $p \in \mathbb{C}[z_1, \ldots, z_n]$ such that

$$\|p\|_\infty = 1,$$

and

$$p(z_i) = 0,$$

for all $i = 1, \ldots, m$, and

$$p(z_{m+1}) \neq 0.$$

On the other hand,

$$\varphi_{r_k}(z_{m+1}) \to \theta(z_{m+1}) = w_{m+1},$$

together with

$$|\varphi_{r_k}(z_{m+1})| \to M,$$

implies

$$|w_{m+1}| = |\theta(z_{m+1})| = M.$$

Therefore,

$$\left| \frac{w_{m+1}}{p(z_{m+1})} \right| > 0.$$

Since $1 - \|\eta\|_\infty > 0$, we may choose $\lambda > 0$ so that

$$0 < \lambda \left| \frac{w_{m+1}}{p(z_{m+1})} \right| < 1 - \|\eta\|_\infty.$$

Set

$$\lambda_0 = \lambda \frac{w_{m+1}}{p(z_{m+1})},$$

and define

$$\psi = \eta + \lambda_0 p.$$

For $i = 1, \ldots, m$, we have $\psi(z_i) = w_i$, and also

$$\|\psi\|_\infty \leq \|\eta\|_\infty + |\lambda_0| < \|\eta\|_\infty + 1 - \|\eta\|_\infty = 1.$$

Thus $\psi \in \mathcal{S}_{\mathcal{Z},\mathcal{W}}$. Moreover, we have

$$\begin{aligned}
\big|\psi(z_{m+1})\big| &= \big|\eta(z_{m+1}) + \lambda_0 p(z_{m+1})\big| \\
&= \big|w_{m+1} + \lambda_0 p(z_{m+1})\big| \\
&= \big|w_{m+1} + \lambda w_{m+1}\big| \\
&= (1+\lambda)\big|w_{m+1}\big| \\
&> \big|w_{m+1}\big| \\
&= |\theta(z_{m+1})|.
\end{aligned}$$

Therefore, we have

$$M = \sup_{\varphi \in \mathcal{S}_{\mathcal{Z},\mathcal{W}}} |\varphi(z_{m+1})| = |\theta(z_{m+1})| < \big|\psi(z_{m+1})\big|,$$

which contradicts to the fact that $\psi \in \mathcal{S}_{\mathcal{Z},\mathcal{W}}$. Therefore,

$$\|\eta\|_\infty = 1.$$

By construction, $\theta \in \mathcal{S}_{\widetilde{\mathcal{Z}},\widetilde{\mathcal{W}}}$. Since the data $\widetilde{\mathcal{Z}}$ and $\widetilde{\mathcal{W}}$ define an extremal problem, it follows that θ is an extremal solution. In particular, $\theta \in \mathcal{E}_{m+1}(\mathbb{B}^n)$. Consequently, $\theta \in \mathcal{E}(\mathbb{B}^n)$ and θ interpolates $\mathcal{Z}$ and $\mathcal{W}$. This completes the proof of the theorem. $\square$

The "one-point extension" approach adapted in the proof bears some resemblance to an argument of Garnett [17, p. 133, Corollary 1.9]. It is therefore worthwhile to compare the present theorem with Garnett's result. Such a comparison is especially intriguing because Garnett's argument is based on the theory of dual extremal problems, a line of investigation originating in the work of Havinson [21, 22], Macintyre and Rogosinski [29], and Rogosinski and Shapiro [40].

We now prove a Carathéodory theorem on the ball. Recall that a sequence of functions $\{\varphi_k\} \subseteq \mathcal{E}(\mathbb{B}^n)$ is said to converge to a function φ in the weak*-topology of $L^\infty(\mathbb{S}^n)$ if

$$\lim_{k\to\infty} \int_{\mathbb{S}^n} \varphi_k f d\sigma = \int_{\mathbb{S}^n} \varphi f d\sigma,$$

for all $f \in L^1(\mathbb{S}^n)$.

Theorem 2.2. *Let $\theta \in \mathcal{S}(\mathbb{B}^n)$ be nonzero. Then there is a sequence $\{\varphi_k\} \subseteq \mathcal{E}(\mathbb{B}^n)$ such that*

$$\lim_{k\to\infty} \varphi_k = \theta,$$

uniformly on compact subsets of $\mathbb{B}^n$. Moreover, this convergence also holds in the weak-topology of $L^\infty(\mathbb{S}^n)$.*

Proof. Fix a countable dense subset $\{z_j : j \in \mathbb{N}\}$ of $\mathbb{B}^n$. For each $j \in \mathbb{N}$, define

$$w_j = \theta(z_j).$$

Since $\theta \in \mathcal{S}(\mathbb{B}^n)$, we know that

$$\{w_j : j \in \mathbb{N}\} \subseteq \mathbb{D}.$$

For each $m \geq 1$, consider the interpolation data $\mathcal{Z}_m$ and $\mathcal{W}_m$ defined by

$$\mathcal{Z}_m = \{z_j\}_{j=1}^m,$$

and

$$\mathcal{W}_m = \{w_j\}_{j=1}^m.$$

Evidently, $\mathcal{Z}_m$ and $\mathcal{W}_m$ solve the interpolation problem, since θ itself is an interpolant function. By Theorem 2.1, for each $m \in \mathbb{N}$, there exists a function $\varphi_m \in \mathcal{E}_m(\mathbb{B}^n) \subseteq \mathcal{E}(\mathbb{B}^n)$ such that φ_m interpolates $\mathcal{Z}_m$ and $\mathcal{W}_m$. Given that

$$\|\varphi_m\|_\infty = 1,$$

for $m \in \mathbb{N}$, we conclude that $\{\varphi_m : m \geq 1\}$ is a uniformly bounded sequence of analytic functions. Therefore, by Montel's theorem, there is a subsequence $\{\varphi_{m_k}\}$ and a function $\eta : \mathbb{B}^n \to \mathbb{C}$ such that

$$\lim_{k\to\infty} \varphi_{m_k} = \eta,$$

uniformly on compact subsets of $\mathbb{B}^n$. As noted in the proof of Theorem 2.1, we have $\eta \in \mathcal{S}(\mathbb{B}^n)$, and

$$\lim_{k\to\infty} \varphi_{m_k}(z) = \eta(z),$$

for all $z \in \mathbb{B}^n$.

Fix $j \in \mathbb{N}$. Since $\{m_k\}$ is a strictly increasing sequence of naturals, there exists a $k_0 \in \mathbb{N}$ such that

$$m_k \geq j,$$

for all $k \geq k_0$. We know that the function φ_{m_k} interpolates the data $\mathcal{Z}_{m_k}$ and $\mathcal{W}_{m_k}$. For each $k \geq k_0$, in particular, since $1 \leq j \leq m_k$, it follows that

$$\varphi_{m_k}(z_j) = w_j.$$

Since $w_j = \theta(z_j)$, we have

$$\varphi_{m_k}(z_j) = \theta(z_j),$$

for all $k \geq k_0$. Thus, passing to the limit (noting that k_0 depends only on j) we obtain

$$\lim_{k\to\infty} \varphi_{m_k}(z_j) = \theta(z_j).$$

Therefore, we conclude that

$$\theta(z_j) = \eta(z_j),$$

for all $j \in \mathbb{N}$. Since $\{z_j : j \in \mathbb{N}\}$ is dense in $\mathbb{B}^n$, by continuity of η and θ, it follows that

$$\eta = \theta.$$

The first part of the theorem follows by relabeling the subsequence φ_{m_k} as φ_k for all $k \in \mathbb{N}$. Given the proof of the first part, the second part of the result is now easy and follows exactly as in part (b) of [42, Theorem 5.3]. This completes the proof of the theorem. $\square$

In view of the identification of $\mathcal{E}(\mathbb{D})$ with the set of all finite Blaschke products in the classical $n = 1$ case, as pointed out in (1.1), the above result can be seen as a unit ball version of the Carathéodory theorem. This can also be considered as an extremal function version of the Aleksandrov-Rudin theorem (see Theorem 1.4 or [42, Theorem 5.3]).

3. Characterizations by perturbations

We now turn to quotient modules of $H^2(\mathbb{B}^n)$ and present a criterion for commutant lifting. Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{B}^n)$. In the context of Problem 3, observe that if $X := S_\varphi$ for some $\varphi \in \mathcal{S}(\mathbb{B}^n)$, then

$$\|X\| = \|S_\varphi\| = \|P_\mathcal{Q} T_\varphi|_\mathcal{Q}\| \leq \|\varphi\|_\infty \leq 1,$$

that is, $X \in \mathcal{B}_1(\mathcal{Q})$. Moreover, it is evident that $XS_{z_i} = S_{z_i}X$ for all $i = 1, \dots, n$. Thus, S_φ is a contractive module map on $\mathcal{Q}$. Therefore, the commutant lifting problem is concerned with establishing the converse: whether a contractive module map on $\mathcal{Q}$ necessarily arises as a compressed multiplication operator by a Schur class function.

In this section, we present a solution to this problem that is technically less involved, yet it will play an important role in addressing the relatively more involved results that follow. At the same time, the solution provided here reveals a perturbation property that is of independent interest. To avoid trivialities in the lifting problem, we assume throughout the paper that all quotient modules $\mathcal{Q}$ and module maps $X \in \mathcal{B}(\mathcal{Q})$ under consideration are nonzero; that is,

$$\mathcal{Q} \neq \{0\} \text{ and } X \neq 0. \tag{3.1}$$

We begin with some notation. Given n commuting operators $\{T_i\}_{i=1}^n$ acting on a Hilbert space $\mathcal{H}$, and $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$, we write

$$T^k = T_1^{k_1} \cdots T_n^{k_n}.$$

Similarly, we write $z^k = z_1^{k_1} \cdots z_n^{k_n}$.

Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{B}^n)$; that is, $\mathcal{Q}$ is a closed subspace of $H^2(\mathbb{B}^n)$ satisfying $z_i \mathcal{Q}^\perp \subseteq \mathcal{Q}^\perp$ for all $i = 1, \ldots, n$. We give a special attention to the function $P_{\mathcal{Q}} 1 \in \mathcal{Q}$. This function is of particular interest, as it is a joint cyclic vector in the sense of quotient modules. That is,

$$\mathcal{Q} = \overline{\text{span}}\{S_z^k(P_{\mathcal{Q}}1) : k \in \mathbb{Z}_+^n\}. \tag{3.2}$$

To see this, for each $k \in \mathbb{Z}_+^n$, we compute

$$S_z^k P_{\mathcal{Q}} 1 = P_{\mathcal{Q}} T_z^k|_{\mathcal{Q}} P_{\mathcal{Q}} 1 = P_{\mathcal{Q}} T_z^k P_{\mathcal{Q}} 1 = P_{\mathcal{Q}} T_z^k 1 = P_{\mathcal{Q}} z^k.$$

The claim now follows from the fact that

$$H^2(\mathbb{B}^n) = \overline{\text{span}}\{z^k : k \in \mathbb{Z}_+^n\}.$$

A closed subspace $\mathcal{S} \subseteq H^2(\mathbb{B}^n)$ is called a *submodule* if $\mathcal{S}^\perp$ is a quotient module of $H^2(\mathbb{B}^n)$. Equivalently, $\mathcal{S}$ is a joint invariant subspace under coordinate functions:

$$z_i \mathcal{S} \subseteq \mathcal{S},$$

for all $i = 1, \ldots, n$. We state a well-known result: Let $\mathcal{S}$ be a closed subspace of $H^2(\mathbb{B}^n)$. Then $\mathcal{S}$ is a submodule if and only if

$$\varphi \mathcal{S} \subseteq \mathcal{S}, \tag{3.3}$$

for all $\varphi \in H^\infty(\mathbb{B}^n)$. It is well known that the structure of submodules and quotient modules of $H^2(\mathbb{B}^n)$ is complicated, and unlike the case $n = 1$ (which is covered by Beurling's theorem and the classical Sz.-Nagy–Foiaș model theory [31]), no satisfactory characterization or understanding is available. This complexity presents a major obstacle in understanding the several-variable commutant lifting theorem as well as the interpolation problem (and vice versa as well). Our first characterization of commutant lifting for contractive module maps relies on perturbations of certain natural functions and establishes a connection with submodules of $H^2(\mathbb{B}^n)$.

We present the lifting theorem for a more general framework. We consider subalgebras of $H^\infty(\mathbb{B}^n)$ for lifting maps; more specifically, we consider the ball algebra $A(\mathbb{B}^n)$, since this closed subalgebra of $H^\infty(\mathbb{B}^n)$ will be used in subsequent results. Recall that $A(\mathbb{B}^n)$ is the Banach algebra consisting of all analytic functions on $\mathbb{B}^n$ that admit a continuous extension to $\overline{\mathbb{B}^n}$. We set

$$\mathcal{A} = H^\infty(\mathbb{B}^n) \text{ or } A(\mathbb{B}^n).$$

Note that the statement that a contractive module map X acting on a quotient module admits a lift means, as usual, the existence of a Schur function $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that $X = S_\varphi$. If φ lies merely in $\mathcal{A}$ (and is not necessarily a Schur function), then we say that X admits a lift to $\mathcal{A}$ (in this case, X is not necessarily contractive).

Let $X \in \mathcal{B}(\mathcal{Q})$ be a module map. Define

$$\psi_X = X(P_{\mathcal{Q}} 1).$$

In what follows, we will suppress the subscript and simply write ψ (as X will be clear from the context). This function

$$\psi = X(P_{\mathcal{Q}}1) \in \mathcal{Q},$$

will play a central role throughout the developments that follow. Before we proceed, we point out that ψ is never the zero function:

$$\psi \neq 0.$$

Indeed, assume that for a nonzero quotient module $\mathcal{Q}$ (as we always assume, see (3.2)) of $H^2(\mathbb{B}^n)$, and for a module map $X \in \mathcal{B}(\mathcal{Q})$, suppose for contradiction that

$$\psi = X(P_{\mathcal{Q}}1) = 0.$$

Since X is a module map on $\mathcal{Q}$, it follows that

$$XS_z^k = S_z^k X,$$

and hence

$$X(S_z^k P_{\mathcal{Q}}1) = 0,$$

$k \in \mathbb{Z}_+^n$. By (3.2)

$$\mathcal{Q} = \overline{\text{span}}\{S_z^k P_{\mathcal{Q}}1 : k \in \mathbb{Z}_+^n\},$$

which immediately implies that $X = 0$, contradicting our assumption that $X \neq 0$. Therefore, we always have $\psi \neq 0$. This observation will also be important in the interpolation problem in Section 7. For now, we prove a lifting characterization in terms of perturbation.

Theorem 3.1. *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{B}^n)$, and let $X \in \mathcal{B}(\mathcal{Q})$ be a module map. Then X admits a lift to $\mathcal{A}$ if and only if there exists $\varphi \in \mathcal{A}$ such that*

$$\psi - \varphi \in \mathcal{Q}^\perp.$$

Moreover, in this case $X = S_\varphi$.

Proof. Suppose X admits a lift. That is, there exists $\varphi \in \mathcal{A}$ such that

$$X = S_\varphi.$$

Then

$$\psi = X(P_{\mathcal{Q}}1) = S_\varphi(P_{\mathcal{Q}}1) = P_{\mathcal{Q}}T_\varphi P_{\mathcal{Q}}1.$$

Since $\mathcal{Q}^\perp$ is a submodule, it follows that (see (3.3))

$$\varphi\mathcal{Q}^\perp \subseteq \mathcal{Q}^\perp,$$

and hence $P_{\mathcal{Q}}T_\varphi P_{\mathcal{Q}} = P_{\mathcal{Q}}T_\varphi$. This implies

$$\psi = P_{\mathcal{Q}}\varphi,$$

so we conclude that

$$\psi - \varphi = P_{\mathcal{Q}}\varphi - \varphi = -P_{\mathcal{Q}^\perp}\varphi.$$

This implies

$$\psi - \varphi \in \mathcal{Q}^\perp.$$

Conversely, assume that $\psi - \varphi \in \mathcal{Q}^\perp$. Therefore, $P_{\mathcal{Q}}(\psi - \varphi) = 0$. Since $\psi \in \mathcal{Q}$, we must have $P_{\mathcal{Q}}\psi = \psi$, and hence

$$\psi = P_{\mathcal{Q}}\varphi.$$

Pick $k \in \mathbb{Z}_+^n$. As X is a module map, $XS_{z_i} = S_{z_i}X$ for all $i = 1, \ldots, n$, and hence

$$XS_z^k = S_z^k X.$$

Therefore

$$XS_z^kP_{\mathcal{Q}}1 = S_z^kXP_{\mathcal{Q}}1 = S_z^k\psi.$$

Moreover, since $\varphi \in H^\infty(\mathbb{B}^n)$ is analytic, it follows that T_φ is an analytic Toeplitz operator. As a result,

$$T_z^kT_\varphi = T_\varphi T_z^k,$$

which implies $S_z^kS_\varphi = S_\varphi S_z^k$ for all $k \in \mathbb{Z}_+^n$. Therefore, for all $k \in \mathbb{Z}_+^n$, we have

$$\begin{aligned} X(S_z^kP_{\mathcal{Q}}1) &= S_z^k\psi \\ &= S_z^kP_{\mathcal{Q}}\varphi \\ &= S_z^kS_\varphi P_{\mathcal{Q}}1 \\ &= S_\varphi(S_z^kP_{\mathcal{Q}}1). \end{aligned}$$

In view of (3.2), it follows that $X = S_\varphi$, which completes the proof of the theorem. □

We now address the commutant lifting problem in its common form. Note that no specific algebraic property of the algebra $\mathcal{A}$ has been used in the proof of the above theorem. The following result may be proved in exactly the same manner as Theorem 3.1; one simply replaces $\mathcal{A}$ by $\mathcal{S}(\mathbb{B}^n)$ throughout the argument:

Corollary 3.2. *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{B}^n)$, and let $X \in \mathcal{B}(\mathcal{Q})$ be a module map. Then X admits a lift if and only if there exists $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that*

$$\psi - \varphi \in \mathcal{Q}^\perp.$$

Moreover, in this case $X = S_\varphi$.

We now specialize to the ball algebra. Define

$$\mathcal{SB}(\mathbb{B}^n) = \mathcal{S}(\mathbb{B}^n) \cap A(\mathbb{B}^n).$$

In the setting of the above theorem, if, in addition, the lifting symbol φ is also in the ball algebra (that is, if $\varphi \in \mathcal{SB}(\mathbb{B}^n)$), then we say that X admits a lift to $\mathcal{SB}(\mathbb{B}^n)$. Again, the following holds with a proof similar to that of Theorem 3.1:

Corollary 3.3. *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{B}^n)$, and let $X \in \mathcal{B}(\mathcal{Q})$ be a module map. Then X admits a lift to $\mathcal{SB}(\mathbb{B}^n)$ if and only if there exists $\varphi \in \mathcal{SB}(\mathbb{B}^n)$ such that*

$$\psi - \varphi \in \mathcal{Q}^\perp.$$

Moreover, in this case X lifts to S_φ.

As already noted for $n = 1$, every contractive module map admits a lifting. However, as we will see (cf. Section 11), the situation changes significantly in higher dimensions: unlike the one-variable case, where every contractive module map lifts by Sarason's theorem [45], there exist contractive module maps that may or may not admit liftings. Thus, the classification of liftings in several variables becomes a nontrivial and meaningful problem.

Before concluding this section, we note that the perturbation problem above is, perhaps coincidentally, connected with another perturbation phenomenon for inner functions studied by Rudin [42], a connection that we will discuss further in Section 11.

4. Qualitative characterizations

In this section, we go deeper into the structure of the Hardy space $H^2(\mathbb{B}^n)$. All notations and preliminaries presented at the beginning of this section are standard for $H^2(\mathbb{B}^n)$. For further details, we refer the reader to the classic [43] (also see [26]).

Let $\zeta \in \mathbb{S}^n$ and let $\alpha > 1$. In analogy with the notion of an angular region in one variable, define $\mathcal{D}_\alpha(\zeta)$ by (see [43, 5.4.1])

$$\mathcal{D}_\alpha(\zeta) = \Big\{ z \in \mathbb{B}^n : |1 - \langle z, \zeta \rangle| < \frac{\alpha}{2}(1 - \|z\|^2) \Big\}.$$

A function $f : \mathbb{B}^n \to \mathbb{C}$ is said to have $K-$limit c at $\zeta \in \mathbb{S}^n$ if, for every $\alpha > 1$,

$$\lim_{\substack{z \to \zeta \\ z \in \mathcal{D}_\alpha(\zeta)}} f(z) = c.$$

If this limit exists, we write (see [43, 5.4.6])

$$(K - \lim f)(\zeta) = c.$$

The notion of K-limits gives boundary values of the Hardy functions as follows (see [43, 5.6.6]): For each $f \in H^2(\mathbb{B}^n)$, the function f^* is defined almost everywhere on $\mathbb{S}^n$, where

$$f^*(\zeta) = (K - \lim f)(\zeta), \tag{4.1}$$

for $\zeta \in \mathbb{S}^n$ a.e. Define

$$H^2(\mathbb{S}^n) = \overline{\{ f|_{\mathbb{S}^n} : f \in A(\mathbb{B}^n) \}}^{L^2(\mathbb{S}^n)},$$

and

$$H^\infty(\mathbb{S}^n) = H^2(\mathbb{S}^n) \cap L^\infty(\mathbb{S}^n).$$

By the corollary preceding Theorem 5.6.9 of [43], the map $f \mapsto f^*$ is a linear isometry from $H^2(\mathbb{B}^n)$ onto $H^2(\mathbb{S}^n)$. In view of this, we will often identify $H^2(\mathbb{B}^n)$ with $H^2(\mathbb{S}^n)$ via the unitary operator $U : H^2(\mathbb{B}^n) \to H^2(\mathbb{S}^n)$, where

$$Uf = f^*,$$

for all $f \in H^2(\mathbb{B}^n)$. Note that the set of polynomials

$$\{ p^* : p \in \mathbb{C}[z_1, \dots z_n] \},$$

is dense in $H^2(\mathbb{S}^n)$. We also remark that if $K - \lim f$ exists for a function f, then

$$K - \lim \bar{f} = \overline{K - \lim f}.$$

The key tool in the above identification is the use of Poisson integral techniques, which rely on the Poisson kernel. The Poisson kernel on the unit ball is defined by [43, Definition 3.3.1]

$$P(z, \zeta) = \frac{(1 - \|z\|^2)^n}{|1 - \langle z, \zeta \rangle|^{2n}}.$$

for all $z \in \mathbb{B}^n$ and $\zeta \in \mathbb{S}^n$. The Poisson integral $\mathcal{P}[\varphi]$ of a function $\varphi \in L^1(\mathbb{S}^n)$ is defined by [43, page 41]

$$\mathcal{P}[\varphi](z) = \int_{\mathbb{S}^n} P(z, \zeta) \varphi(\zeta) d\sigma(\zeta),$$

for all $z \in \mathbb{B}^n$. For each $f \in H^2(\mathbb{B}^n)$, we claim that

$$f = \mathcal{P}[f^*]. \tag{4.2}$$

Indeed, since $f^* \in H^2(\mathbb{S}^n)$, [43, Theorem 5.6.8(b)] implies

$$\mathcal{P}[f^*] \in H^2(\mathbb{B}^n).$$

Then

$$g := \mathcal{P}[f^*] \in H^2(\mathbb{B}^n),$$

and (again, see [43, Theorem 5.6.8(b)])

$$g^* = (\mathcal{P}[f^*])^* = f^*,$$

a.e. on $\mathbb{S}^n$. Thus, $g^*, f^* \in H^2(\mathbb{S}^n)$ and $g^* = f^*$. Since $Uh = h^*$ (recall the unitary $U : H^2(\mathbb{B}^n) \to H^2(\mathbb{S}^n)$ above), it follows $g = f$, that is, $f = \mathcal{P}[f^*]$. This completes the proof of the claim. We also have the following:

$$\lim_{r \to 1} \int_{\mathbb{S}^n} |f^* - f_r|^2 d\sigma = 0,$$

where, f_r, $0 < r < 1$, is defined by

$$f_r(\zeta) = f(r\zeta),$$

for $\zeta \in \mathbb{S}^n$ a.e. Note that $r\zeta = (r\zeta_1, \ldots, r\zeta_n)$. The above claim follows from (4.2), together with an application of [43, Theorem 3.3.4(b)].

Now we set up the notations needed for the lifting theorem. Although some of these have already appeared in the introduction, we present them again here with the necessary elaboration. Recall that the space of mixed functions is defined by

$$\mathcal{M}(\mathbb{S}^n) = L^2(\mathbb{S}^n) \ominus [H^2(\mathbb{S}^n) + H^2(\mathbb{S}^n)^{conj}].$$

From the definition of $\mathcal{M}(\mathbb{S}^n)$ it is clear that $\mathcal{M}(\mathbb{S}^n)$ is self-adjoint, that is,

$$f \in \mathcal{M}(\mathbb{S}^n) \Leftrightarrow \overline{f} \in \mathcal{M}(\mathbb{S}^n).$$

Moreover, we note that if $\mathcal{S}$ is a closed subspace of $L^2(\mathbb{S}^n)$ then $\mathcal{S}^{conj}$ is also a closed subspace of $L^2(\mathbb{S}^n)$. Recall that $H^2_0(\mathbb{S}^n)$ is the closed subspace of $H^2(\mathbb{S}^n)$ consisting of functions vanishing at 0, that is,

$$H^2_0(\mathbb{S}^n) = \{f^* \in H^2(\mathbb{S}^n) : f \in H^2(\mathbb{B}^n), f(0) = 0\}$$

Equivalently, we have

$$H^2_0(\mathbb{S}^n) = \{f^* \in H^2(\mathbb{S}^n) : \langle f, 1 \rangle_{H^2(\mathbb{B}^n)} = 0\} = H^2(\mathbb{S}^n) \ominus \{1\}.$$

Similarly, if S is a subset of $H^2(\mathbb{S}^n)$, then we represent S^{conj} as

$$S^{conj} = \{\overline{f} : f \in S\}.$$

Fix a quotient module $\mathcal{Q}$ of $H^2(\mathbb{S}^n)$ and define

$$\mathcal{M}_{\mathcal{Q}} = \mathcal{Q}^{conj} \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H^2_0(\mathbb{S}^n)].$$

where $\dot{+}$ denotes the skew sum in the Banach space $L^1(\mathbb{S}^n)$. Note that here we are using the identification of $H^2(\mathbb{B}^n)$ with $H^2(\mathbb{S}^n)$ via the K-limits of the Hardy functions (via the unitary operator U defined as above). In other words, we consider $\mathcal{M}_{\mathcal{Q}}$ as a subspace of $L^1(\mathbb{S}^n)$:

$$(\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1) \subset L^1(\mathbb{S}^n).$$

Now if we consider $\mathcal{M}_{\mathcal{Q}}$ to be a subspace of the Hilbert space $L^2(\mathbb{S}^n)$, then the skew sum mentioned above becomes an orthogonal sum. We keep this observation as a lemma for future reference:

Lemma 4.1. *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{B}^n)$. Then $\mathcal{M}(\mathbb{S}^n)$, $H_0^2(\mathbb{S}^n)$, and $\mathcal{Q}^{conj}$ are pairwise orthogonal closed subspaces of $L^2(\mathbb{S}^n)$. Moreover,*

$$H_0^2(\mathbb{S}^n) \perp H^2(\mathbb{S}^n)^{conj}.$$

Remark 4.2. Using the identification of $H^2(\mathbb{B}^n)$ and $H^2(\mathbb{S}^n)$, we have

$$H^2(\mathbb{S}^n) + H^2(\mathbb{S}^n)^{conj} = H_0^2(\mathbb{S}^n) \oplus H^2(\mathbb{S}^n)^{conj},$$

which readily implies that $H^2(\mathbb{S}^n) + H^2(\mathbb{S}^n)^{conj}$ is a closed subspace of $L^2(\mathbb{S}^n)$.

We also recall the duality of the L^p-spaces in our present context. For each $\varphi \in L^\infty(\mathbb{S}^n)$, define $\chi_\varphi \in (L^1(\mathbb{S}^n))^*$ by

$$\chi_\varphi f = \int_{\mathbb{S}^n} f\varphi \, d\sigma,$$

for all $f \in L^1(\mathbb{S}^n)$. This yields the duality (a linear isometry and bijective map)

$$(L^1(\mathbb{S}^n))^* \cong L^\infty(\mathbb{S}^n). \tag{4.3}$$

This duality will play a key role in the quantitative lifting theorem:

Theorem 4.3. *Let $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ be a quotient module and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. Define $X_\mathcal{Q} : \mathcal{M}_\mathcal{Q} \to \mathbb{C}$ by*

$$X_\mathcal{Q} f = \int_{\mathbb{S}^n} \psi f d\sigma,$$

for all $f \in \mathcal{M}_\mathcal{Q}$, where

$$\psi = X(P_\mathcal{Q} 1).$$

Then X admits a lift if and only if $X_\mathcal{Q} : (\mathcal{M}_\mathcal{Q}, \|.\|_1) \to \mathbb{C}$ is a contraction.

Proof. First, we assume that $X \in \mathcal{B}_1(\mathcal{Q})$ admits a lift. That is, there exists $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that $X = S_\varphi$. By Corollary 3.2, we know that

$$\psi - \varphi \in \mathcal{Q}^\perp.$$

Since φ is a Schur function, by taking K-limit, we obtain

$$\|\varphi\|_{H^\infty(\mathbb{S}^n)} \leq 1.$$

By the duality (4.3), we know that $\chi_\varphi \in (L^1(\mathbb{S}^n))^*$, where

$$\chi_\varphi f = \int_{\mathbb{S}^n} \varphi f \, d\sigma,$$

for all $f \in L^1(\mathbb{S}^n)$. We also know (by the duality) that

$$\|\chi_\varphi\| = \|\varphi\|_{H^\infty(\mathbb{S}^n)} \leq 1.$$

We claim that

$$\chi_\varphi|_{\mathcal{M}_\mathcal{Q}} = X_\mathcal{Q}. \tag{4.4}$$

To prove this, first we pick $f \in \mathcal{Q}^{conj}$. We have

$$\begin{aligned} X_{\mathcal{Q}} f &= \int_{\mathbb{S}^n} \psi f d\sigma \\ &= \int_{\mathbb{S}^n} \psi \overline{\overline{f}} d\sigma \\ &= \langle \psi, \overline{f} \rangle_{H^2(\mathbb{S}^n)} \\ &= \langle \psi, \overline{f} \rangle_{H^2(\mathbb{B}^n)}. \end{aligned}$$

Since $\psi - \varphi \in \mathcal{Q}^{\perp}$, it follows that $\langle \psi - \varphi, \overline{f} \rangle_{H^2(\mathbb{B}^n)} = 0$, and hence

$$\langle \psi, \overline{f} \rangle_{H^2(\mathbb{B}^n)} = \langle \varphi, \overline{f} \rangle_{H^2(\mathbb{B}^n)}.$$

This implies

$$\begin{aligned} X_{\mathcal{Q}} f &= \langle \varphi, \overline{f} \rangle_{H^2(\mathbb{B}^n)} \\ &= \langle \varphi, \overline{f} \rangle_{H^2(\mathbb{S}^n)} \\ &= \int_{\mathbb{S}^n} \varphi f d\sigma \\ &= \chi_{\varphi} f, \end{aligned}$$

that is,

$$X_{\mathcal{Q}}|_{\mathcal{Q}^{conj}} = \chi_{\varphi}|_{\mathcal{Q}^{conj}}.$$

For $f \in \mathcal{M}(\mathbb{S}^n)$, we have

$$X_{\mathcal{Q}} f = \int_{\mathbb{S}^n} \psi f d\sigma = \langle f, \overline{\psi} \rangle_{H^2(\mathbb{S}^n)} = 0 = \chi_{\varphi} f.$$

This shows that

$$X_{\mathcal{Q}}|_{\mathcal{M}(\mathbb{S}^n)} = \chi_{\varphi}|_{\mathcal{M}(\mathbb{S}^n)}.$$

Finally, if $f \in H^2_0(\mathbb{S}^n)$, then, as $\mathcal{Q}^{conj} \perp H^2_0(\mathbb{S}^n)$, we have

$$X_{\mathcal{Q}} f = \int_{\mathbb{S}^n} \psi f d\sigma = 0,$$

and on the other hand, as $\varphi \in H^2(\mathbb{B}^n)$ and $f \in H^2_0(\mathbb{S}^n)$, we have

$$\chi_{\varphi} f = \int_{\mathbb{S}^n} \varphi f d\sigma = 0 = X_{\mathcal{Q}} f.$$

Therefore,

$$X_{\mathcal{Q}}|_{H^2_0(\mathbb{S}^n)} = \chi_{\varphi}|_{H^2_0(\mathbb{S}^n)} = 0,$$

and proves the claim that $X_{\mathcal{Q}} = \chi_{\varphi}|_{\mathcal{M}_{\mathcal{Q}}}$. By using this, we have

$$\|X_{\mathcal{Q}}\| \leq \|\chi_{\varphi}\| = \|\varphi\|_{\infty} \leq 1,$$

which proves that $X_{\mathcal{Q}}$ is a contractive functional on $\mathcal{M}_{\mathcal{Q}}$.
For the converse direction, assume that $X_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|.\|_1) \longrightarrow \mathbb{C}$ is a contractive functional. By the Hahn-Banach extension theorem, there exists $\theta \in L^{\infty}(\mathbb{S}^n)$ such that

$$\chi_{\theta}|_{\mathcal{M}_{\mathcal{Q}}} = X_{\mathcal{Q}},$$

and

$$\|\theta\|_{\infty} = \|X_{\mathcal{Q}}\| \leq 1.$$

For $f \in \mathcal{Q}^{conj}$, as $\chi_\theta f = X_{\mathcal{Q}} f$, we have

$$\int_{\mathbb{S}^n} \theta f d\sigma = \int_{\mathbb{S}^n} \psi f d\sigma,$$

and hence

$$\int_{\mathbb{S}^n} (\theta - \psi) f d\sigma = 0. \tag{4.5}$$

We claim that $\theta \in H^2(\mathbb{S}^n)$. To this end, we first note that $\theta \in L^2(\mathbb{S}^n)$. Let $f \in \mathcal{M}(\mathbb{S}^n)$. As $\mathcal{M}(\mathbb{S}^n)$ is self-adjoint, we have $\overline{f} \in \mathcal{M}(\mathbb{S}^n)$, and hence

$$\begin{aligned}
\langle \theta, f\rangle_{L^2(\mathbb{S}^n)} &= \int_{\mathbb{S}^n} \theta \overline{f} d\sigma \\
&= \chi_\theta \overline{f} \\
&= X_{\mathcal{Q}} \overline{f} \\
&= \int_{\mathbb{S}^n} \psi \overline{f}\, d\sigma \\
&= \langle \psi, f\rangle_{L^2(\mathbb{S}^n)} \\
&= 0.
\end{aligned}$$

This implies

$$\theta \in L^2(\mathbb{S}^n) \ominus \mathcal{M}(\mathbb{S}^n).$$

In view of Remark 4.2, we know that the subspace $H^2(\mathbb{S}^n) + H^2(\mathbb{S}^n)^{conj}$ is closed in $L^2(\mathbb{S}^n)$, which yields

$$\begin{aligned}
L^2(\mathbb{S}^n) \ominus \mathcal{M}(\mathbb{S}^n) &= L^2(\mathbb{S}^n) \ominus \Big(L^2(\mathbb{S}^n) \ominus [H^2(\mathbb{S}^n) + H^2(\mathbb{S}^n)^{conj}] \Big) \\
&= H^2(\mathbb{S}^n) + H^2(\mathbb{S}^n)^{conj},
\end{aligned}$$

and consequently

$$\theta \in H^2(\mathbb{S}^n) + H^2(\mathbb{S}^n)^{conj}.$$

There exists $\eta_1 \in H^2(\mathbb{B}^n)^{conj}$ and $\eta_2 \in H^2(\mathbb{B}^n)$ such that (recall that η_j^* denotes the K-limit function of η_j, $j = 1, 2$)

$$\theta = {\eta_1}^* + {\eta_2}^*.$$

Define

$$\varphi_1 = \eta_1 - \eta_1(0),$$

and

$$\varphi_2 = \eta_2 + \eta_1(0).$$

Then $\varphi_1^* \in H_0^2(\mathbb{S}^n)^{conj}$ and $\varphi_2^* \in H^2(\mathbb{S}^n)$, and

$$\theta = \varphi_1^* + \varphi_2^*.$$

As $\varphi_1^* \in H_0^2(\mathbb{S}^n)^{conj}$, we have $\overline{\varphi_1^*} \in H_0^2(\mathbb{S}^n)$, and hence

$$\begin{aligned}\langle \theta, \varphi_1^* \rangle_{L^2(\mathbb{S}^n)} &= \int_{\mathbb{S}^n} \theta \overline{\varphi_1^*} d\sigma \\ &= \chi_\theta(\overline{\varphi_1^*}) \\ &= X_{\mathcal{Q}}(\overline{\varphi_1^*}) \\ &= \int_{\mathbb{S}^n} \psi^* \overline{\varphi_1^*} d\sigma \\ &= \langle \overline{\varphi_1^*}, \overline{\psi^*} \rangle_{L^2(\mathbb{S}^n)} \\ &= 0,\end{aligned}$$

which implies

$$\langle \theta, \varphi_1^* \rangle_{L^2(\mathbb{S}^n)} = 0.$$

Now as $\theta = \varphi_1^* + \varphi_2^*$ and $\langle \varphi_1^*, \varphi_2^* \rangle_{L^2(\mathbb{S}^n)} = 0$, we have

$$0 = \langle \theta, \varphi_1^* \rangle_{L^2(\mathbb{S}^n)} = \langle \varphi_1^* + \varphi_2^*, \varphi_1^* \rangle_{L^2(\mathbb{S}^n)} = \|\varphi_1^*\|^2_{L^2(\mathbb{S}^n)},$$

that is, $\varphi_1^* = 0$, and hence

$$\theta = \varphi_2^* \in H^2(\mathbb{S}^n).$$

We also have $\theta \in L^\infty(\mathbb{S}^n)$, so that

$$\varphi_2^* \in L^\infty(\mathbb{S}^n) \cap H^2(\mathbb{S}^n).$$

As $\varphi_2 \in H^2(\mathbb{B}^n)$, (4.2) implies that

$$\varphi_2 = \mathcal{P}[\varphi_2^*],$$

where $\mathcal{P}$ denotes the Poisson integral (see [43, page 41]). In other words, for any $z \in \mathbb{B}^n$, we have

$$\varphi_2(z) = \mathcal{P}[\varphi_2^*](z) = \int_{\mathbb{S}^n} P(z, \zeta) \varphi_2^*(\zeta) d\sigma(\zeta),$$

where

$$P(z, \zeta) = \frac{(1 - \|z\|^2)^n}{|1 - \langle z, \zeta \rangle|^{2n}}.$$

for all $z \in \mathbb{B}^n$ and $\zeta \in \mathbb{S}^n$, is the Poisson kernel. For $z \in \mathbb{B}^n$, we compute

$$\begin{aligned}|\varphi_2(z)| &= \Big| \int_{\mathbb{S}^n} P(z, \zeta) \varphi_2^*(\zeta) d\sigma(\zeta) \Big| \\ &\le \int_{\mathbb{S}^n} P(z, \zeta) |\varphi_2^*(\zeta)| d\sigma(\zeta) \\ &\le \|\varphi_2^*\|_{L^\infty(\mathbb{S}^n)},\end{aligned}$$

which proves that $\varphi_2 \in H^\infty(\mathbb{B}^n)$, and

$$\|\varphi_2\|_{H^\infty(\mathbb{B}^n)} \le \|\varphi_2^*\|_{L^\infty(\mathbb{S}^n)} = \|\theta\|_{L^\infty(\mathbb{S}^n)} \le 1.$$

We recall from (4.5) that

$$\int_{\mathbb{S}^n} (\theta - \psi) f d\sigma = 0,$$

for all $f \in \mathcal{Q}^{conj}$. Now $\theta = \varphi_2^*$, for $\varphi_2 \in H^\infty(\mathbb{B}^n)$ implies

$$\varphi_2 - \psi \in \mathcal{Q}^\perp.$$

Finally, since $\|\varphi_2\|_\infty \leq 1$, by Corollary 3.2, we conclude that X admits a lift, namely S_θ. In summary, we have proved the existence of a function $\theta \in \mathcal{S}(\mathbb{B}^n)$ such that

$$\begin{cases} \chi_\theta|_{\mathcal{M}_\mathcal{Q}} = X_\mathcal{Q}, \\ X = S_\theta, \text{ and} \\ \|\theta\|_\infty = \|X_\mathcal{Q}\| \leq 1. \end{cases} \tag{4.6}$$

This completes the proof of the theorem. ☐

The kernel space of the functional $X_\mathcal{Q}$ will play an important role in obtaining a quantitative characterization of possible liftings.

Similar qualitative characterization of lifting also holds on the polydisc [13]. However, the proof of the above result differs from the approach used in the polydisc case. Specifically, we avoided the use of an orthonormal basis for $L^2(\mathbb{S}^n)$, which was a key technique in the polydisc setting (also see Section 13 for further discussion). Moreover, we have the following remark:

Remark 4.4. In the proof of the sufficiency part of the theorem, although $X_\mathcal{Q}$ is assumed to be contractive, we do not use the contractivity of X to obtain a lift. Instead, we show that the contractivity of $X_\mathcal{Q}$ forces X to admit a lift, and hence X is a contraction.

Given a contractive module map $X \in \mathcal{B}_1(\mathcal{Q})$, it is natural to ask for the infimum of the norms of all symbols $\varphi \in \mathcal{S}(\mathbb{B}^n)$ satisfying

$$X = S_\varphi.$$

The following result shows that this quantity is precisely $\|X_\mathcal{Q}\|$. This not only highlights the power of the functional $X_\mathcal{Q}$ but also shows the accuracy and effectiveness of its construction.

Theorem 4.5. *Let $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ be a quotient module, and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map that admits a lift. Then the functional $X_\mathcal{Q} : (\mathcal{M}_\mathcal{Q}, \|\cdot\|_1) \to \mathbb{C}$ from Theorem 4.3 satisfies*

$$\|X_\mathcal{Q}\| = \inf\{\|\varphi\|_\infty : \varphi \in \mathcal{S}(\mathbb{B}^n) \text{ and } X = S_\varphi\}.$$

Proof. If $X = S_\varphi$ for some $\varphi \in \mathcal{S}(\mathbb{B}^n)$, by the duality (4.3), we have

$$\|\chi_\varphi\| = \|\varphi\|_\infty \leq 1.$$

and by the necessary part of Theorem 4.3 (or see (4.4)), we know that

$$\chi_\varphi|_{\mathcal{M}_\mathcal{Q}} = X_\mathcal{Q}.$$

Therefore, we have

$$\|X_\mathcal{Q}\| = \|\chi_\varphi|_{\mathcal{M}_\mathcal{Q}}\| \leq \|\chi_\varphi\| = \|\varphi\|_\infty.$$

On the other hand, since X admits a lift, $X_\mathcal{Q}$ is a contraction. From this point, we follow the proof of the sufficient part of Theorem 4.3 and arrive at identity (4.6): $X = S_\theta$ and $\chi_\theta|_{\mathcal{M}_\mathcal{Q}} = X_\mathcal{Q}$ for some $\theta \in \mathcal{S}(\mathbb{B}^n)$ with

$$\|\theta\|_\infty = \|X_\mathcal{Q}\| \leq 1.$$

Also, we have $X_\mathcal{Q} f = \int_{\mathbb{S}^n} \theta f d\sigma$ for all $f \in \mathcal{M}_\mathcal{Q}$, and hence

$$\int_{\mathbb{S}^n} (\psi - \theta) f d\sigma = 0,$$

for all $f \in \mathcal{M}_\mathcal{Q}$. In particular,

$$\psi - \theta \in \mathcal{Q}^\perp,$$

and thus, by Corollary 3.2, it follows that $X = S_\theta$. This, together with $\|\theta\|_\infty = \|X_{\mathcal{Q}}\|$, immediately yields the desired norm formula. □

In a sense, $\|X_{\mathcal{Q}}\|$ represents the optimal lifting norm for X. The above norm identity will play a crucial role in the subsequent developments of this paper.

5. Quantitative characterizations

We use all the characterizations obtained thus far in this paper to propose a quantitative characterization of lifting. Here, "quantitative" refers to a distance formula that bears a resemblance to the classical Nehari theorem [33].

Given a normed linear space V, a nonempty set $S \subseteq V$ and $x \in V$, the distance from x to S is defined by

$$\mathrm{dist}_V(x, S) = \inf\{\|x - y\| : y \in S\}.$$

The following lemma is key to our quantitative analysis of lifting:

Lemma 5.1. *Let V be normed linear space, η be a nonzero functional in V^*, and let $x \in V$. Then*

$$\mathit{dist}_V(x, \ker \eta) = \frac{|\eta(x)|}{\|\eta\|}.$$

Proof. Let $y \in \ker \eta$. Then

$$|\eta(x)| = |\eta(x - y)| \leq \|\eta\|\|x - y\|.$$

and hence

$$\frac{|\eta(x)|}{\|\eta\|} \leq \|x - y\|,$$

so that

$$\mathrm{dist}_X(x, \ker \eta) \geq \frac{|\eta(x)|}{\|\eta\|}.$$

For the reverse inequality, we recall that $\|\eta\| = \sup_{y \neq 0} \frac{|\eta(y)|}{\|y\|}$. Therefore, there exists a sequence $\{y_n\} \subseteq X$ such that

$$\eta(y_n) \neq 0,$$

for all n, and

$$\frac{|\eta(y_n)|}{\|y_n\|} \to \|\eta\|.$$

Since

$$x - \frac{\eta(x)}{\eta(y_n)} y_n \in \ker \eta,$$

we have

$$\begin{aligned}
\mathrm{dist}_V(x, \ker \eta) &\leq \left\| x - x - \frac{\eta(x)}{\eta(y_n)} y_n \right\| \\
&= \left\| \frac{\eta(x)}{\eta(y_n)} y_n \right\| \\
&= \frac{|\eta(x)|}{|\eta(y_n)|} \|y_n\| \\
&\longrightarrow \frac{|\eta(x)|}{\|\eta\|},
\end{aligned}$$

and hence $\operatorname{dist}_V(x, \ker \eta) \leq \frac{|\eta(x)|}{\|\eta\|}$. □

Given a quotient module $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ and a module map $X \in \mathcal{B}_1(\mathcal{Q})$, we have defined $\psi = X(P_{\mathcal{Q}}1)$, and

$$\mathcal{M}_{\mathcal{Q}} = \mathcal{Q}^{conj} \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H_0^2(\mathbb{S}^n)].$$

Under the given assumptions, also recall that $X_{\mathcal{Q}} : \mathcal{M}_{\mathcal{Q}} \to \mathbb{C}$ is defined by

$$X_{\mathcal{Q}} f = \int_{\mathbb{S}^n} \psi f \, d\sigma,$$

for all $f \in \mathcal{M}_{\mathcal{Q}}$ (cf. Theorem 4.3). Define

$$\widetilde{\mathcal{M}_{\mathcal{Q}}} = [\mathcal{Q}^{conj} \ominus \{\overline{\psi}\}] \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H_0^2(\mathbb{S}^n)].$$

Lemma 5.2. *Consider the functional $X_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1) \to \mathbb{C}$. Then* $\ker X_{\mathcal{Q}} = \widetilde{\mathcal{M}_{\mathcal{Q}}}$.

Proof. We write $X_{\mathcal{Q}} f = \int_{\mathbb{S}^n} \psi f \, d\sigma$ as

$$X_{\mathcal{Q}} f = \langle f, \overline{\psi} \rangle_{L^2(\mathbb{S}^n)},$$

for all $f \in \mathcal{M}_{\mathcal{Q}}$, and consequently

$$\ker X_{\mathcal{Q}} = [\mathcal{Q}^{conj} \ominus \{\overline{\psi}\}] \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H_0^2(\mathbb{S}^n)],$$

which completes the proof of the lemma. □

In view of the above and Lemma 5.1, we have the following characterization of lifting. We again emphasize that, to avoid triviality, we always assume that the module maps $X \in \mathcal{B}(\mathcal{Q})$ are nonzero.

Theorem 5.3. *Let $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ be a quotient module, and let $X \in \mathcal{B}_1(\mathcal{Q})$. Then X admits a lift if and only if*

$$dist_{L^1(\mathbb{S}^n)}\Big(\frac{\overline{\psi}}{\|\psi\|_2^2}, \widetilde{\mathcal{M}_{\mathcal{Q}}}\Big) \geq 1.$$

Proof. Lemma 5.2 yields

$$\ker X_{\mathcal{Q}} = \widetilde{\mathcal{M}_{\mathcal{Q}}}.$$

Thus

$$\operatorname{dist}_{L^1(\mathbb{S}^n)}\Big(\frac{\overline{\psi}}{\|\psi\|_2^2}, \widetilde{\mathcal{M}_{\mathcal{Q}}}\Big) = \operatorname{dist}_{L^1(\mathbb{S}^n)}\Big(\frac{\overline{\psi}}{\|\psi\|_2^2}, \ker X_{\mathcal{Q}}\Big).$$

Observe that

$$X_{\mathcal{Q}}\Big(\frac{\overline{\psi}}{\|\psi\|_2^2}\Big) = \int_{\mathbb{S}^n} \psi \frac{\overline{\psi}}{\|\psi\|_2^2} d\sigma = 1,$$

We then have, by Lemma 5.1, that

$$\operatorname{dist}_{L^1(\mathbb{S}^n)}\Big(\frac{\overline{\psi}}{\|\psi\|_2^2}, \widetilde{\mathcal{M}_{\mathcal{Q}}}\Big) = \frac{1}{\|X_{\mathcal{Q}}\|}.$$

Again, Theorem 4.3 states that X admits a lift if and only if

$$\|X_{\mathcal{Q}}\| \leq 1.$$

In view of the above, this condition is equivalent to requiring that

$$\operatorname{dist}_{L^1(\mathbb{S}^n)}\Big(\frac{\overline{\psi}}{\|\psi\|_2^2}, \widetilde{\mathcal{M}_{\mathcal{Q}}}\Big) \geq 1.$$

This completes the proof of the theorem. $\square$

In the above proof, we have highlighted the key identity (see Lemma 5.2)

$$\ker X_{\mathcal{Q}} = [\mathcal{Q}^{conj} \ominus \{\overline{\psi}\}] \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H^2_0(\mathbb{S}^n)],$$

which holds for a general quotient module $\mathcal{Q}$ and module map $X \in \mathcal{B}_1(\mathcal{Q})$. This same identity also holds in the polydisc setting. In a sense, the above identity provides deeper insight into the structure of the functional $X_{\mathcal{Q}}$.

6. Characterizations by approximations

Recall from Section 2 that

$$\mathcal{E}(\mathbb{B}^n) = \{\varphi \in \mathcal{S}(\mathbb{B}^n) : \varphi \text{ is a solution to some extremal problem}\}.$$

In this section, we establish a connection between the lifting of contractive module maps and the classes of inner functions and extremal solutions. We begin by recalling the Aleksandrov-Rudin theorem (see Theorem 1.4): Let $\varphi \in \mathcal{S}(\mathbb{B}^n)$. Then there is a sequence of inner functions $\{\varphi_k\} \subseteq \mathcal{I}(\mathbb{B}^n)$ such that

$$\lim_{k\to\infty} \varphi_k = \varphi,$$

uniformly on compact subsets of $\mathbb{B}^n$. Moreover,

$$w^*\!-\lim_{k\to\infty} \varphi_k = \varphi \text{ on } L^1(\mathbb{S}^n),$$

where $w^*\!-\lim_{k\to\infty} \varphi_k = \varphi$ on a set $S \subseteq L^1(\mathbb{S}^n)$, we mean that

$$\lim_{k\to\infty} \int_{\mathbb{S}^n} \varphi_k f \, d\sigma = \int_{\mathbb{S}^n} \varphi f \, d\sigma,$$

for all $f \in S$. In Theorem 2.2, we proved the same result for the class of extremal solutions. Motivated by the roles of these two classes and their approximation properties for Schur functions, we shall use the notation

$$\mathcal{A}(\mathbb{B}^n),$$

to denote either $\mathcal{I}(\mathbb{B}^n)$ or $\mathcal{E}(\mathbb{B}^n)$. We refer to $\mathcal{A}(\mathbb{B}^n)$ as the class of approximation functions. Combining the Aleksandrov-Rudin theorem (Theorem 1.4) with Theorem 2.2, we obtain the following result.

Theorem 6.1. *Let $\varphi \in \mathcal{S}(\mathbb{B}^n)$ be nonzero. Then there is a sequence $\{\varphi_k\} \subseteq \mathcal{A}(\mathbb{B}^n)$ such that*

$$\lim_{k\to\infty} \varphi_k = \varphi,$$

uniformly on compact subsets of $\mathbb{B}^n$. Moreover,

$$w^*\!-\lim_{k\to\infty} \varphi_k = \varphi \text{ on } L^1(\mathbb{S}^n).$$

In other words, the last assertion of the above theorem asserts that

$$\overline{\mathcal{A}(\mathbb{B}^n)}^{w^*} = \mathcal{S}(\mathbb{B}^n).$$

With this, we continue in the setting of Theorem 4.3 and present yet another characterization of lifting in terms of inner functions. Given a quotient module $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ and a module map $X \in \mathcal{B}_1(\mathcal{Q})$, we recall that

$$X_{\mathcal{Q}} f = \int_{\mathbb{S}^n} \psi f \, d\sigma,$$

for $f \in \mathcal{M}_\mathcal{Q}$, defines a functional $X_\mathcal{Q} : (\mathcal{M}_\mathcal{Q}, \|\cdot\|_1) \longrightarrow \mathbb{C}$, where $\psi = X(P_\mathcal{Q}1)$ and

$$\mathcal{M}_\mathcal{Q} = \mathcal{Q}^{conj} \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H_0^2(\mathbb{S}^n)].$$

We connect the function ψ with the class of $\mathcal{A}(\mathbb{B}^n)$ functions as follows:

Theorem 6.2. *Let $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$, and $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map, and let $\psi = X(P_\mathcal{Q}1)$. Then X admits a lift if and only if there is a sequence of functions $\{\varphi_k\} \subseteq \mathcal{A}(\mathbb{B}^n)$ such that*

$$w^*\!-\lim_{k\to\infty} \varphi_k = \psi \text{ on } \mathcal{M}_\mathcal{Q}.$$

Proof. Suppose X admits a lift. From the proof of Theorem 4.3, and specifically by equation (4.4), there exists $\theta \in \mathcal{S}(\mathbb{B}^n)$ such that for $f \in \mathcal{M}_\mathcal{Q}$, we have

$$X_\mathcal{Q} f = \int_{\mathbb{S}^n} \theta f \, d\sigma,$$

for all $f \in \mathcal{M}_\mathcal{Q}$. By Theorem 6.1, there exists a sequence of functions $\{\varphi_k\} \subseteq \mathcal{A}(\mathbb{B}^n)$ such that

$$w^*\!-\lim_{k\to\infty} \varphi_k = \theta \text{ on } L^1(\mathbb{S}^n).$$

In particular, $w^* - \lim_{k\to\infty} \varphi_k = \theta$ on $\mathcal{M}_\mathcal{Q}$. Moreover, for each $f \in \mathcal{M}_\mathcal{Q}$, since

$$X_\mathcal{Q} f = \int_{\mathbb{S}^n} \psi f \, d\sigma = \int_{\mathbb{S}^n} \theta f \, d\sigma,$$

it follows that

$$\lim_{k\to\infty} \int_{\mathbb{S}^n} \varphi_k f \, d\sigma = \int_{\mathbb{S}^n} \psi f \, d\sigma,$$

for all $f \in \mathcal{M}_\mathcal{Q}$, that is, $w^*\!- \lim_{k\to\infty} \varphi_k = \psi$ on $\mathcal{M}_\mathcal{Q}$.
Conversely, assume that $w^*\!- \lim_{k\to\infty} \varphi_k = \psi$ on $\mathcal{M}_\mathcal{Q}$. For $f \in \mathcal{M}_\mathcal{Q}$, we have

$$\begin{aligned} |X_\mathcal{Q} f| &= \Big| \int_{\mathbb{S}^n} \psi f \, d\sigma \Big| \\ &= \Big| \lim_{k\to\infty} \int_{\mathbb{S}^n} \varphi_k f \, d\sigma \Big| \\ &\le \lim_{k\to\infty} \int_{\mathbb{S}^n} |\varphi_k f| d\sigma \\ &\le \|f\|_1, \end{aligned}$$

that is, $X_\mathcal{Q}$ on $(\mathcal{M}_\mathcal{Q}, \|\cdot\|_1)$ is a contraction, and therefore, by Theorem 4.3, it follows that X admits a lift. $\square$

Now, we examine the structure of $\mathcal{M}_\mathcal{Q}$ more closely in order to prove a more economical version of the above theorem:

Corollary 6.3. *Let $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$, and $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map, and let $\psi = X(P_\mathcal{Q}1)$. Then X admits a lift if and only if there is a sequence of functions $\{\varphi_k\} \subseteq \mathcal{A}(\mathbb{B}^n)$ such that*

$$w^* - \lim_{k\to\infty} \varphi_k = \psi \text{ on } \mathcal{Q}^{conj}.$$

Proof. The necessary part is a special case of Theorem 6.2 (note that $\mathcal{Q}^{conj} \subseteq \mathcal{M}_{\mathcal{Q}}$). For sufficiency, suppose there exists a sequence of functions $\{\varphi_k\} \subseteq \mathcal{A}(\mathbb{B}^n)$ such that $\lim_{k\to\infty} \varphi_k = \psi$ on $\mathcal{Q}^{conj}$. If we can prove that

$$w^* - \lim_{k\to\infty} \varphi_k = \psi \text{ on } \mathcal{M}_{\mathcal{Q}},$$

then the sufficiency part of this corollary will follow from Theorem 6.2. To prove the claim, pick any $g \in \mathcal{M}_{\mathcal{Q}} = \mathcal{Q}^{conj} \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H_0^2(\mathbb{S}^n)]$. Then, there exist $g_1 \in \mathcal{Q}^{conj}$, $g_2 \in \mathcal{M}(\mathbb{S}^n)$, and $g_3 \in H_0^2(\mathbb{S}^n)$ such that

$$g = g_1 + g_2 + g_3.$$

Fix a k. Since $g_2 \in \mathcal{M}(\mathbb{S}^n)$, we have

$$\int_{\mathbb{S}^n} \varphi_k g_2 \, d\sigma = \langle \varphi_k, \overline{g_2} \rangle_{L^2(\mathbb{S}^n)} = 0 = \langle \psi, \overline{g_2} \rangle = \int_{\mathbb{S}^n} \psi g_2 d\sigma,$$

and also

$$\int_{\mathbb{S}^n} \varphi_k g_3 d\sigma = \langle \varphi_k g_3, 1 \rangle = \varphi_k(0) g_3(0) = 0,$$

as $g_3 \in H_0^2(\mathbb{S}^n)$. Also, since

$$\int_{\mathbb{S}^n} \psi g_3 d\sigma = \langle g_3, \overline{\psi} \rangle = 0,$$

we have

$$\int_{\mathbb{S}^n} \varphi_k g d\sigma = \int_{\mathbb{S}^n} \varphi_k (g_1 + g_2 + g_3) d\sigma = \int_{\mathbb{S}^n} \varphi_k g_1 d\sigma.$$

In view of our assumption, as $g_1 \in \mathcal{Q}^{conj}$, we have

$$\int_{\mathbb{S}^n} \varphi_k g_1 d\sigma \to \int_{\mathbb{S}^n} \psi g_1 d\sigma,$$

and consequently

$$\int_{\mathbb{S}^n} \varphi_k g d\sigma = \int_{\mathbb{S}^n} \varphi_k g_1 d\sigma \to \int_{\mathbb{S}^n} \psi g_1 d\sigma = \int_{\mathbb{S}^n} \psi g d\sigma.$$

This completes the proof of the corollary. $\square$

We now focus on lifting of module maps on finite-dimensional quotient modules. We specialize to a weak topology on $L^\infty(\mathbb{S}^n)$. Let $S \subseteq L^1(\mathbb{S}^n)$. For each $f \in S$, define $\hat{f} \in (L^\infty(\mathbb{S}^n))^*$ by

$$\hat{f}(\varphi) = \int_{\mathbb{S}^n} \varphi f d\sigma,$$

for all $\varphi \in L^\infty(\mathbb{S}^n)$, and set

$$\hat{S} := \{\hat{f} : f \in S\} \subseteq (L^\infty(\mathbb{S}^n))^*.$$

We let τ_S denote the weak-topology on $L^\infty(\mathbb{S}^n)$ generated by the family of functionals $\hat{S}$. In other words,

$$(L^\infty(\mathbb{S}^n), \tau_S),$$

is the smallest topology on $L^\infty(\mathbb{S}^n)$ for which each functional $\hat{f} : L^\infty(\mathbb{S}^n) \to \mathbb{C}$, $f \in S$, is continuous. Therefore, we have the following: Let $\{\varphi_i\}$ be a net in $L^\infty(\mathbb{S}^n)$. Then

$$\{\varphi_i\} \longrightarrow \varphi \text{ in } (L^\infty(\mathbb{S}^n), \tau_S),$$

if and only if

$$\hat{f}(\varphi_i) \longrightarrow \hat{f}(\varphi),$$

for all $f \in S$. The following result establishes a connection between the weak topology introduced above and the lifting on finite-dimensional quotient modules:

Theorem 6.4. *Let $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ be a finite-dimensional quotient module, and $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. Then X admits a lift if and only if there is a sequence of functions $\{\varphi_k\} \subseteq \mathcal{S}(\mathbb{B}^n)$ such that*

$$\varphi_k \longrightarrow \psi \textit{ in } (L^\infty(\mathbb{S}^n), \tau_{\mathcal{Q}^{conj}}),$$

where

$$\psi = X(P_{\mathcal{Q}}1).$$

Proof. Since $\mathcal{Q}$ is finite-dimensional, it follows that (see [19, Corollary 3.4])

$$\mathcal{Q} \subseteq H^\infty(\mathbb{B}^n).$$

In particular, we have $\psi \in L^\infty(\mathbb{S}^n)$. Now, convergence

$$\varphi_k \longrightarrow \psi \text{ in } (L^\infty(\mathbb{S}^n), \tau_{\mathcal{Q}^{cconj}}),$$

is equivalent to the convergence of

$$\hat{f}\varphi_k) \longrightarrow \hat{f}(\psi),$$

for all $f \in \mathcal{Q}^{cconj}$. Equivalently, we have

$$\int_{\mathbb{S}^n} \varphi_k f d\sigma \longrightarrow \int_{\mathbb{S}^n} \psi f d\sigma,$$

for all $f \in \mathcal{Q}^{conj}$. The result now follows from Corollary 6.3. □

Consider the set of all inner functions as a subset of $L^\infty(\mathbb{S}^n)$. Let $\mathcal{Q}$ be a finite-dimensional quotient module and let $X \in \mathcal{B}_1(\mathcal{Q})$. Then the above result can be stated as follows: X admits a lift if and only if

$$\psi = X(P_{\mathcal{Q}}1) \in \overline{\mathcal{A}(\mathbb{B}^n)}^{(L^\infty(\mathbb{S}^n), \tau_{\mathcal{Q}})},$$

where $\mathcal{A}(\mathbb{B}^n)$ stands for either the class $\mathcal{I}(\mathbb{B}^n)$ of inner functions or the class $\mathcal{E}(\mathbb{B}^n)$ of extremal solutions.

7. Interpolation

When $n = 1$, Sarason [45] showed that the solution to the classical interpolation problem follows from his commutant lifting theorem. In a similar spirit, we apply the characterization of lifting to solve the interpolation problem for Schur functions on $\mathbb{B}^n$. In addition, we apply approximation results to give analogous characterizations of solvability of the interpolation problem.

Recall that $H^2(\mathbb{B}^n)$ is a reproducing kernel Hilbert space corresponding to the Szegö kernel on $\mathbb{B}^n$ defined as

$$K_n(z, w) = \frac{1}{(1 - \langle z, w \rangle)^n},$$

for all $z, w \in \mathbb{B}^n$. Given m distinct points $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$, define

$$\mathcal{Q}_{\mathcal{Z}} = \text{span } \{K_n(\cdot, z_i) : i = 1, \dots, m\}.$$

Clearly, $\mathcal{Q}_{\mathcal{Z}}$ is an m-dimensional subspace of $H^2(\mathbb{B}^n)$. Since

$$T_{z_j}^* K_n(\cdot, v) = \overline{v_j} K_n(\cdot, v), \tag{7.1}$$

for all $j = 1, \ldots, n$, and $v = (v_1, \ldots, v_n) \in \mathbb{B}^n$, it follows that $\mathcal{Q}_{\mathcal{Z}}$ is also a quotient module of $H^2(\mathbb{B}^n)$. Observe that

$$\mathcal{Q}_{\mathcal{Z}} \subset A(\mathbb{B}^n) \subset H^\infty(\mathbb{B}^n).$$

Fix interpolation data $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subseteq \mathbb{D}$. Following Sarason, we define $X_{\mathcal{Z},\mathcal{W}} \in \mathcal{B}(\mathcal{Q}_{\mathcal{Z}})$ by

$$X_{\mathcal{Z},\mathcal{W}}^* K_n(\cdot, z_j) = \overline{w_j} K_n(\cdot, z_j),$$

for all $j = 1, \ldots, m$. By (7.1), it follows that $X_{\mathcal{Z},\mathcal{W}}$ is a module map. Following the construction of ψ in Theorem 4.3, define

$$\psi = X_{\mathcal{Z},\mathcal{W}}(P_{\mathcal{Q}_{\mathcal{Z}}} 1).$$

On the other hand, since $\{z_i\}_{i=1}^m$ is a set of distinct points, the $m \times m$ matrix

$$\big[K_n(z_i, z_j)\big]_{m \times m} = \begin{bmatrix} K_n(z_1, z_1) & \ldots & K_n(z_1, z_m) \\ \vdots & \ddots & \vdots \\ K_n(z_m, z_1) & \ldots & K_n(z_m, z_m) \end{bmatrix},$$

is invertible. Indeed, we know that

$$K_n(z_i, z_j) = \langle K_n(\cdot, z_j), K_n(\cdot, z_i) \rangle,$$

for all $i, j = 1, \ldots, m$, and that the set $\{K_n(\cdot, z_i)\}_{i=1}^m$ consists of linearly independent vectors. Since the Szegö kernel is positive definite, the above matrix is an invertible Gram matrix. Define $\psi_{\mathcal{Z},\mathcal{W}} \in \mathcal{Q}_{\mathcal{Z}}$ by

$$\psi_{\mathcal{Z},\mathcal{W}} = \sum_{j=1}^m c_j K_n(\cdot, z_j), \tag{7.2}$$

where $c_1, \ldots, c_m$ are given by

$$\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \big[K_n(z_i, z_j)\big]_{m \times m}^{-1} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}.$$

Note the important fact that

$$\psi_{\mathcal{Z},\mathcal{W}}(z_j) = w_j, \tag{7.3}$$

for all $j = 1, \ldots, m$.

In the discussion preceding Theorem 3.1, we have already observed that

$$\psi = X(P_{\mathcal{Q}} 1) \neq 0,$$

where $\mathcal{Q}$ is a quotient module of $H^2(\mathbb{B}^n)$ and $X \in \mathcal{B}(\mathcal{Q})$ is a module map. We reiterate that throughout this discussion we are working under the standing assumption that both the quotient module $\mathcal{Q}$ and the module map X are nonzero. Before relating the above ψ to the function $\psi_{\mathcal{Z},\mathcal{W}}$ introduced in (7.2), we first investigate the case when $\psi_{\mathcal{Z},\mathcal{W}} = 0$. Suppose

$$\psi_{\mathcal{Z},\mathcal{W}} = 0.$$

Then

$$\sum_{j=1}^{m} c_j K_n(\cdot, z_j) = 0.$$

Since the set $\{K_n(., z_j) : 1 \leq j \leq m\}$ is linearly independent, we have $c_j = 0$ for $j = 1, \ldots, m$. Since

$$\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \left[K_n(z_i, z_j)\right]^{-1}_{m \times m} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix},$$

we conclude that

$$\begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} = 0,$$

that is, $\mathcal{W} = \{0\}$. This proves the following:

$$\psi_{\mathcal{Z},\mathcal{W}} = 0 \Longleftrightarrow \mathcal{W} = \{0\}.$$

On the other hand, if $\mathcal{W} = \{0\}$, then the interpolation problem corresponding to the data $\mathcal{Z} = \{z_1, \ldots, z_m\} \subset \mathbb{B}^n$ and $\mathcal{W}$ is always solvable, with the zero function serving as a solution. Therefore, without loss of generality, we shall henceforth assume that the interpolation data $\mathcal{Z}$ and $\mathcal{W}$ satisfy

$$\psi_{\mathcal{Z},\mathcal{W}} \neq 0,$$

equivalently, that

$$\mathcal{W} \neq \{0\}.$$

So far, we have constructed two different kinds of functions, namely ψ and $\psi_{\mathcal{Z},\mathcal{W}}$, from two distinct perspectives. However, they are the same:

Proposition 7.1. *Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subseteq \mathbb{D}$ be interpolation data. Then*

$$\psi = \psi_{\mathcal{Z},\mathcal{W}},$$

where $\psi = X_{\mathcal{Z},\mathcal{W}}(P_{\mathcal{Q}_{\mathcal{Z}}}1)$ and $\psi_{\mathcal{Z},\mathcal{W}}$ is the function defined in (7.2).

Proof. Note that since $\psi \in \mathcal{Q}_{\mathcal{Z}}$, there exist scalars $\alpha_1, \ldots \alpha_m$ such that

$$\psi = \sum_{i=1}^{m} \alpha_i K_n(\cdot, z_i).$$

For each $j = 1, \ldots, m$, we have

$$\begin{aligned}
\psi(z_j) &= \langle \psi, K_n(\cdot, z_j)\rangle \\
&= \langle X_{\mathcal{Z},\mathcal{W}}(P_{\mathcal{Q}_{\mathcal{Z}}}1), K_n(\cdot, z_j)\rangle \\
&= \langle P_{\mathcal{Q}_{\mathcal{Z}}}1, X^*_{\mathcal{Z},\mathcal{W}} K_n(\cdot, z_j)\rangle \\
&= \langle P_{\mathcal{Q}_{\mathcal{Z}}}1, \overline{w_j} K_n(\cdot, z_j)\rangle \\
&= w_j \langle 1, P_{\mathcal{Q}_{\mathcal{Z}}} K_n(\cdot, z_j)\rangle \\
&= w_j \langle 1, K_n(\cdot, z_j)\rangle \\
&= w_j K_n(z_j, 0) \\
&= w_j,
\end{aligned}$$

that is, $w_j = \psi(z_j)$, which implies

$$w_j = \sum_{i=1}^{m} \alpha_i K_n(z_j, z_i).$$

In other words, we have

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} = \big[K_n(z_i, z_j)\big]_{m\times m}^{-1} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix},$$

and consequently, by comparing with the identity preceding (7.3), we conclude that $c_j = \alpha_j$. Therefore, we obtain the desired identity: $\psi_{\mathcal{Z},\mathcal{W}} = \psi$. $\square$

We are now ready for a solution to the interpolation problem:

Theorem 7.2. *Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subseteq \mathbb{D}$ be interpolation data. Define*

$$\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}} = \mathcal{Q}_{\mathcal{Z}}^{conj} \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H_0^2(\mathbb{S}^n)],$$

and

$$\psi_{\mathcal{Z},\mathcal{W}} = \sum_{j=1}^{m} c_j K_n(\cdot, z_j),$$

where the scalars $\{c_1, \ldots, c_m\}$ are given by

$$\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \big[K_n(z_i, z_j)\big]_{m\times m}^{-1} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}.$$

Then $\mathcal{Z}$ and $\mathcal{W}$ solve the interpolation problem if and only if

$$\Pi_{\mathcal{Z},\mathcal{W}} f = \int_{\mathbb{S}^n} \psi_{\mathcal{Z},\mathcal{W}} f \, d\sigma \qquad (f \in \mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}),$$

defines a contraction $\Pi_{\mathcal{Z},\mathcal{W}} : (\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}, \|\cdot\|_1) \to \mathbb{C}$.

Proof. Note that $\varphi \in \mathcal{S}(\mathbb{B}^n)$ interpolates $\mathcal{Z}$ and $\mathcal{W}$ if and only if

$$S_\varphi = X_{\mathcal{Z},\mathcal{W}}. \tag{7.4}$$

To see this, first observe that $S_\varphi^* = T_\varphi^*|_{\mathcal{Q}}$, and by the reproducing property, we have

$$T_\varphi^* K_n(\cdot, z_i) = \overline{\varphi(z_i)} K_n(\cdot, z_i),$$

for all $i = 1, \ldots, m$. Since $\mathcal{Q}_{\mathcal{Z}} = \operatorname{span}\{K_n(\cdot, z_i) : i = 1, \ldots, m\}$, it then follows that

$$S_\varphi^* K_n(\cdot, z_i)) = \overline{\varphi(z_i)} K_n(\cdot, z_i),$$

for all $i = 1, \ldots, m$. On the other hand, by the definition of $X_{\mathcal{Z},\mathcal{W}}$ on $\mathcal{Q}_{\mathcal{Z}}$, we have

$$X_{\mathcal{Z},\mathcal{W}}^* K_n(\cdot, z_i) = \overline{w_i} K_n(\cdot, z_i),$$

for all $i = 1, \ldots, m$. Hence $S_\varphi = X_{\mathcal{Z},\mathcal{W}}$, proving (7.4). This yields the following interpretation: the data $\mathcal{Z}$ and $\mathcal{W}$ are solved by an interpolant $\varphi \in \mathcal{S}(\mathbb{B}^n)$ if and only if $X_{\mathcal{Z},\mathcal{W}} \in \mathcal{B}(\mathcal{Q}_{\mathcal{Z}})$ admits a lift, namely $X_{\mathcal{Z},\mathcal{W}} = S_\varphi$. By Theorem 4.3, we know that the operator $X_{\mathcal{Z},\mathcal{W}}$ on $\mathcal{Q}_{\mathcal{Z}}$ admits a lift if and only if the functional $X_{\mathcal{Q}_{\mathcal{Z}}}$ is a contraction on $(\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}, \|\cdot\|_1)$, where

$$X_{\mathcal{Q}_{\mathcal{Z}}} f = \int_{\mathbb{S}^n} \psi f \, d\sigma.$$

for all $f \in \mathcal{M}_{\mathcal{Q}_\mathcal{Z}}$, and

$$\psi = X_{\mathcal{Z},\mathcal{W}}(P_{\mathcal{Q}_\mathcal{Z}}1).$$

The result now follows from the fact that $\psi_{\mathcal{Z},\mathcal{W}} = \psi$, as observed in Proposition 7.1. □

We shall frequently appeal to the observation in (7.4) in the arguments that follow. By Proposition 7.1, we know that $\psi_{\mathcal{Z},\mathcal{W}} = \psi = X_{\mathcal{Z},\mathcal{W}}(P_{\mathcal{Q}_\mathcal{Z}}1)$. This, together with Theorem 5.3 yields the following:

Corollary 7.3. *In the setting of Theorem 7.2, define*

$$\widetilde{\mathcal{M}_{\mathcal{Q}_\mathcal{Z}}} = [\mathcal{Q}_\mathcal{Z}^{conj} \ominus \{\overline{\psi_{\mathcal{Z},\mathcal{W}}}\} \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H_0^2(\mathbb{S}^n)].$$

Then $\mathcal{Z}$ and $\mathcal{W}$ solve the interpolation problem if and only if

$$dist_{L^1(\mathbb{S}^n)}\Big(\frac{\overline{\psi_{\mathcal{Z},\mathcal{W}}}}{\|\psi_{\mathcal{Z},\mathcal{W}}\|_2^2}, \widetilde{\mathcal{M}_{\mathcal{Q}_\mathcal{Z}}}\Big) \geq 1.$$

The above distance formula is reminiscent of the Nehari theorem [33], although it arises in a different context, namely that of Hankel operators. Since relatively little is known about Hankel operators on $\mathbb{B}^n$, it may be worthwhile to explore whether these results can be connected to the classical theory of Hankel operators, should such a connection exist (see [5, 47] for results on Hankel operators in several variables.)

A more refined analysis of the distance formula provides yet another characterization of interpolation. In the following, given a set $\{w_i\}_{i=1}^m \subseteq \mathbb{D}$, we write $w = (w_1, \dots, w_m) \in \mathbb{C}^m$, and use the standard notation:

$$\|w\| = (\sum_{i=1}^m |w_i|^2)^{\frac{1}{2}}.$$

Corollary 7.4. *In the setting of Theorem 7.2, assume that $\|w\| \neq 0$, and define*

$$\eta_{\mathcal{Z},\mathcal{W}} := \sum_{j=1}^m w_j K_n(\cdot, z_j).$$

Then $\mathcal{Z}$ and $\mathcal{W}$ solve the interpolation problem if and only if

$$dist_{L^1(\mathbb{S}_n)}\left(\frac{\overline{\eta_{\mathcal{Z},\mathcal{W}}}}{\|w\|^2}, \widetilde{\mathcal{M}_{\mathcal{Q}_\mathcal{Z}}}\right) \geq 1.$$

Proof. We recall the reproducing property that $\langle \psi_{\mathcal{Z},\mathcal{W}}, K_n(\cdot, z_i)\rangle = \psi_{\mathcal{Z},\mathcal{W}}(z_i)$ for all $i = 1,\dots,m$, and compute

$$\begin{aligned}
\Big\langle \psi_{\mathcal{Z},\mathcal{W}} - \frac{\|\psi_{\mathcal{Z},\mathcal{W}}\|^2}{\|w\|^2}\eta_{\mathcal{Z},\mathcal{W}}, \psi_{\mathcal{Z},\mathcal{W}}\Big\rangle &= \|\psi_{\mathcal{Z},\mathcal{W}}\|^2 - \Big\langle \frac{\|\psi_{\mathcal{Z},\mathcal{W}}\|^2}{\|w\|^2}\eta_{\mathcal{Z},\mathcal{W}}, \psi_{\mathcal{Z},\mathcal{W}}\Big\rangle \\
&= \|\psi_{\mathcal{Z},\mathcal{W}}\|^2 - \frac{\|\psi_{\mathcal{Z},\mathcal{W}}\|^2}{\|w\|^2}\Big\langle \sum_{j=1}^m w_j K_n(\cdot, z_j), \psi_{\mathcal{Z},\mathcal{W}}\Big\rangle \\
&= \|\psi_{\mathcal{Z},\mathcal{W}}\|^2 - \frac{\|\psi_{\mathcal{Z},\mathcal{W}}\|^2}{\|w\|^2}\sum_{j=1}^m w_j\Big\langle K_n(\cdot, z_j), \psi_{\mathcal{Z},\mathcal{W}}\Big\rangle \\
&= \|\psi_{\mathcal{Z},\mathcal{W}}\|^2 - \frac{\|\psi_{\mathcal{Z},\mathcal{W}}\|^2}{\|w\|^2}\sum_{j=1}^m w_j\overline{\psi_{\mathcal{Z},\mathcal{W}}(z_j)} \\
&= \|\psi_{\mathcal{Z},\mathcal{W}}\|^2 - \frac{\|\psi_{\mathcal{Z},\mathcal{W}}\|^2}{\|w\|^2}\sum_{j=1}^m |w_j|^2 \\
&= 0,
\end{aligned}$$

where the penultimate equality follows from the fact that $\psi_{\mathcal{Z},\mathcal{W}}(z_i) = w_i$ for all $i = 1,\dots,m$. This implies

$$\overline{\psi_{\mathcal{Z},\mathcal{W}}} - \frac{\|\psi_{\mathcal{Z},\mathcal{W}}\|^2}{\|w\|^2}\overline{\eta_{\mathcal{Z},\mathcal{W}}} \in \mathcal{Q}_{\mathcal{Z}}^{conj} \ominus \{\overline{\psi_{\mathcal{Z},\mathcal{W}}}\}.$$

At this point, we recall a basic property of the distance function in normed linear spaces. If S is a closed subspace of a normed linear space X, then

$$\operatorname{dist}(x, S) = \|[x]\|_{X/S} = \inf\{\|x - y\| : y \in S\},$$

where $[x]$ denotes the equivalence class of x in the quotient space X/S, corresponding to the equivalence relation $s \sim t$ if and only if $s - t \in S$. From this perspective, the above observation suggests that

$$\Big\|\Big[\overline{\psi_{\mathcal{Z},\mathcal{W}}}\Big]\Big\|_{L^1(\mathbb{S}^n)/\widetilde{\mathcal{M}_{\mathcal{Q}_\mathcal{Z}}}} = \Big\|\Big[\frac{\|\psi_{\mathcal{Z},\mathcal{W}}\|^2}{\|w\|^2}\overline{\eta_{\mathcal{Z},\mathcal{W}}}\Big]\Big\|_{L^1(\mathbb{S}^n)/\widetilde{\mathcal{M}_{\mathcal{Q}_\mathcal{Z}}}},$$

and hence

$$\operatorname{dist}_{L^1(\mathbb{S}_n)}\left(\frac{\overline{\eta_{\mathcal{Z},\mathcal{W}}}}{\|w\|^2}, \widetilde{\mathcal{M}_{\mathcal{Q}_\mathcal{Z}}}\right) = \operatorname{dist}_{L^1(\mathbb{S}_n)}\left(\frac{\overline{\psi_{\mathcal{Z},\mathcal{W}}}}{\|\psi_{\mathcal{Z},\mathcal{W}}\|_2^2}, \widetilde{\mathcal{M}_{\mathcal{Q}_\mathcal{Z}}}\right).$$

The result now follows from Corollary 7.3. □

We conclude this section by establishing a connection between interpolation theory and the class of inner functions $\mathcal{I}(\mathbb{B}^n)$. Recall the Aleksandrov-Ruding approximation result in Theorem 1.4: Given $\varphi \in \mathcal{S}(\mathbb{B}^n)$, nonzero, there exists a sequence $\{\varphi_k\} \subseteq \mathcal{I}(\mathbb{B}^n)$ such that

$$\lim_{k\to\infty} \varphi_k = \varphi,$$

uniformly on compact subsets of $\mathbb{B}^n$. The same theorem also shows that this convergence holds in the weak*-topology. We incorporate this observation into the following interpolation result.

Theorem 7.5. *Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subseteq \mathbb{D}$ be interpolation data. Then $\mathcal{Z}$ and $\mathcal{W}$ solve the interpolation problem if and only if there exists a sequence of inner functions $\{\varphi_k\} \subseteq \mathcal{I}(\mathbb{B}^n)$ such that*

$$\lim_{k\to\infty} \varphi_k(z_j) = w_j,$$

for all $j = 1, \dots, m$.

Proof. Suppose there is a function $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that $\varphi(z_j) = w_j$ for all $j = 1, \dots, m$. By Theorem 1.4, there exists a sequence of inner functions $\{\varphi_k\}$ such that

$$\lim_{k\to\infty} \varphi_k = \varphi,$$

uniformly on compact subsets of $\mathbb{B}^n$. Since $\mathcal{Z}$ is a finite set and φ interpolates $\mathcal{Z}$ and $\mathcal{W}$, the necessary part of the theorem follows. For the sufficient part, suppose there exists a sequence of inner functions $\{\varphi_k\}$ such that

$$\lim_{k\to\infty} \varphi_k(z_j) = w_j,$$

for all $j = 1, \dots, m$. Viewing $\{\varphi_k\}$ as a family of functions on $\mathbb{B}^n$, we observe that it is uniformly bounded, since $\|\varphi_k\|_\infty = 1$ for all k. Therefore, by Montel's theorem, there exist a function φ and a subsequence φ_{k_ℓ} such that

$$\lim_{l\to\infty} \varphi_{k_\ell} = \varphi,$$

uniformly on compact subsets of $\mathbb{B}^n$. As in the proof of Theorem 2.1, it follows that $\varphi \in \mathcal{S}(\mathbb{B}^n)$. Finally, for each $j = 1, \dots, m$,

$$\varphi(z_j) = \lim_{l\to\infty} \varphi_{k_\ell}(z_j) = w_j.$$

Therefore, $\varphi \in \mathcal{S}(\mathbb{B}^n)$ interpolates $\mathcal{Z}$ and $\mathcal{W}$. After relabeling φ_{k_ℓ} as φ_ℓ, the proof is complete. □

Another way to state the above result is as follows: $\mathcal{Z}$ and $\mathcal{W}$ solve the interpolation problem if and only if there exists a sequence of inner functions $\{\varphi_k\} \subseteq \mathcal{S}(\mathbb{B}^n)$ such that

$$\lim_{k\to\infty} \varphi_k(z_j) = \psi_{\mathcal{Z},\mathcal{W}}(z_j),$$

for all $j = 1, \dots, m$.

Recall that, in one variable, by Pick's theorem (see Theorem 1.2), whenever an interpolation problem is solvable, there exists a finite Blaschke product that serves as an interpolant. Consequently, the above result is immediate when $n = 1$. Clearly, for $n > 1$, this theorem can be viewed as a higher-dimensional analogue of this classical fact, where the role of a finite Blaschke product is replaced by a sequence of inner functions.

We remark that, much like the proof of our commutant lifting theorem, where we invoked the Hahn–Banach theorem, the proof of the above inner function-based interpolation result also relies on another classical result, namely, Montel's theorem.

8. The Carathéodory-Fejér interpolation

Recall the classical Carathéodory-Fejér interpolation problem: Given scalars $c_0, c_1, \dots, c_m$, find necessary and sufficient conditions for the existence of a Schur function $\varphi \in \mathcal{S}(\mathbb{D})$ such that

$$\frac{\varphi^{(j)}(0)}{j!} = c_j,$$

for all $j = 0, 1, \dots, m$. The Carathéodory-Fejér interpolation theorem states that such a Schur function exists if and only if the $(m+1) \times (m+1)$ matrix

$$\begin{bmatrix} c_0 & 0 & \cdots & 0 \\ c_1 & c_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ c_m & \cdots & c_1 & c_0 \end{bmatrix},$$

is a contraction on $\mathbb{C}^{m+1}$. We refer the reader to Sarason for a thorough historical account of the Carathéodory-Fejér interpolation problem. Although the problem was originally solved by Carathéodory [7, 8], its matrix-theoretic formulation and solution were developed by Toeplitz [48]. Since then, the problem has been studied by numerous mathematicians in a variety of contexts, and the literature continues to grow. We particularly highlight the classical contributions of Fejer [10], Fisher [15], Frobenius [16], Riesz [38, 39], Schur [46], and Szegö [18].

Similar to the interpolation problem, Sarason's lifting theorem approach also provides an alternative proof of this result. In this section, we apply the present lifting theorem to solve the same interpolation problem on the unit ball. Specifically, we formulate the Carathéodory-Fejér interpolation problem in the following way:

Problem 4. Fix $m \geq 1$. Let $p \in \mathcal{Q}_m$, that is, $p \in \mathbb{C}[z_1, \dots, z_n]$ is a polynomial of degree less or equal to m. The Carathéodory-Fejér interpolation problem seeks necessary and sufficient conditions for the existence of a Schur function $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that

$$p - \varphi \in \mathcal{Q}_m^{\perp}.$$

Given $m \in \mathbb{N}$, define the finite-dimensional quotient module $\mathcal{Q}_m$ as

$$\mathcal{Q}_m = \{p \in \mathbb{C}[z_1, \dots, z_n] : \deg p \leq m\}.$$

For a qualitative solution to this problem, we define a special subspace of the mixed space $\mathcal{M}(\mathbb{S}^n)$ as follows:

$$\mathcal{M}_n = \mathcal{Q}_m^{conj} \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H_0^2(\mathbb{S}^n)].$$

As usual, we treat $\mathcal{M}_m$ as a subspace of $L^1(\mathbb{S}^n)$.

Theorem 8.1. *Let $p \in \mathcal{Q}_m$. Define $X_p : \mathcal{M}_m \to \mathbb{C}$ by*

$$X_p f = \int_{\mathbb{S}^n} p f \, d\sigma,$$

for all $f \in \mathcal{M}_m$. Then there exists $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that

$$p - \varphi \in \mathcal{Q}_m^{\perp},$$

if and only if $X_p : (\mathcal{M}_m, \| \cdot \|_1) \to \mathbb{C}$ is a contraction.

Proof. Recall that $S_p = P_{\mathcal{Q}_m} T_p|_{\mathcal{Q}_m}$. Since $1 \in \mathcal{Q}_m$, we have

$$\psi = S_p(P_{\mathcal{Q}_m} 1) = P_{\mathcal{Q}_m} T_p 1 = P_{\mathcal{Q}_m} p = p,$$

and hence, by Theorem 3.2, S_p admits a lift if and only if there exists a $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that

$$p - \varphi \in \mathcal{Q}_m^{\perp}.$$

On the other hand, Theorem 4.3 implies that S_p admits a lift if and only if $X_{\mathcal{Q}_m} : (\mathcal{M}_{\mathcal{Q}_m}, \|\cdot\|_1) \to \mathbb{C}$ is a contraction, where

$$X_{\mathcal{Q}_m} f = \int_{\mathbb{S}^n} p f d\sigma,$$

for all $f \in \mathcal{M}_{\mathcal{Q}_m}$. Therefore, the result follows because $X_{\mathcal{Q}_m} = X_p$. □

Analogous to the Nevanlinna–Pick interpolation theorem, Theorem 7.3, the Carathéodory-Fejér interpolation theorem can also be formulated in terms of a distance formula, yielding a quantitative characterization of the solutions to this interpolation problem. The proof is similar to that of the quantitative characterization in the Nevanlinna–Pick case, and hence we omit the details.

Theorem 8.2. *Let p be a nonzero element in $\mathcal{Q}_m$. Define*

$$\widetilde{\mathcal{M}_p} = [\mathcal{Q}_m^{conj} \ominus \{\bar{p}\}] \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H_0^2(\mathbb{S}^n)].$$

Then there exists $\varphi \in \mathcal{S}(\mathbb{B}^n)$ if and only if

$$dist_{L^1(\mathbb{S}^n)}\Big(\frac{\overline{p}}{\|p\|_2^2}, \widetilde{\mathcal{M}_p}\Big) \geq 1.$$

Now we turn to approximate functions in $\mathcal{A}(\mathbb{B}^n)$. Recall that the class $\mathcal{A}(\mathbb{B}^n)$ of approximate functions refers to either the class $\mathcal{I}(\mathbb{B}^n)$ of inner functions or the class $\mathcal{E}(\mathbb{B}^n)$ of extremal solutions.

Theorem 8.3. *Let p be a nonzero element in $\mathcal{Q}_m$. Then there exists $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that*

$$p - \varphi \in \mathcal{Q}_m^{\perp},$$

if and only if there is a sequence of functions $\{\varphi_k\} \subseteq \mathcal{A}(\mathbb{B}^n)$ such that

$$\lim_{k\to\infty} \langle \varphi_k, z^l \rangle = \langle p, z^l \rangle,$$

for all $l \in \mathbb{Z}_+^n$ with $|l| \leq m$.

Proof. In the proof of Theorem 8.1, we have already noted that there exists a $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that $p - \varphi \in \mathcal{Q}_m^{\perp}$ if and only if the module map $S_p \in \mathcal{B}(\mathcal{Q}_m)$ admits a lift. On the other hand, by Corollary 6.3, S_p admits a lift if and only if there exists a sequence of functions $\{\varphi_k\} \subseteq \mathcal{S}(\mathbb{B}^n)$ such that

$$w^* - \lim_{k\to\infty} \varphi_k = p \text{ on } \mathcal{Q}_m^{conj}. \tag{8.1}$$

In view of the fact that $\mathcal{Q}_m$ is spanned by the monomials $\{z^l : |l| \leq m\}$, this is equivalent to

$$\lim_{k\to\infty} \int_{\mathbb{S}^n} \varphi_k \overline{z^l} d\sigma = \int_{\mathbb{S}^n} p \overline{z^l} d\sigma.$$

that is, $\lim_{k\to\infty} \langle \varphi_k, z^l \rangle = \langle p, z^l \rangle$ for all $l \in \mathbb{Z}_+^n$ with $|k| \leq m$. □

To further elaborate this result, we recall that if f on $\mathbb{B}^n$ is an analytic function, then f admits a power series representation $f(z) = \sum_k a_k z^k$ on $\mathbb{B}^n$, where (recall that the set of analytic monomials $\{z^k : k \in \mathbb{Z}_+^n\}$ is an orthogonal set in $H^2(\mathbb{B}^n)$, see [43, 1.4.8])

$$a_k = \frac{\langle f, z^k \rangle}{\langle z^k, z^k \rangle},$$

as well as

$$a_k = \frac{1}{k!} \frac{\partial^k f}{\partial z^k}\Big|_0,$$

for all $k \in \mathbb{Z}_+^n$.

Using the above observations, the Carathéodory-Fejér interpolation theorem can be reformulated in terms of the coefficients in the power series expansion of the corresponding functions as follows:

Corollary 8.4. *Let $p \in \mathcal{Q}_m$. Then there exists $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that*

$$p - \varphi \in \mathcal{Q}_m^\perp,$$

if and only if there is a sequence of functions $\{\varphi_k\} \subseteq \mathcal{A}(\mathbb{B}^n)$ such that

$$\lim_{k\to\infty} \frac{\partial^k \varphi_k}{\partial z^k}\Big|_0 = \frac{\partial^k p}{\partial z^k}\Big|_0,$$

for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq m$.

As in the proof of Theorem 7.5, one may also prove Theorem 8.3 using Montel's theorem. However, this approach is restricted to the class of inner functions, as is the case for Theorem 7.5.

9. Norm-preserving lifting: $n > 1$

In this section, we examine a phenomenon specific to the higher-dimensional case $n > 1$ in the context of norm-preserving liftings. We begin by recalling the classical commutant lifting theorem of Sarason [45]:

Theorem 9.1 (Sarason. $n = 1$)**.** *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{D})$, and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. Then there exists $\varphi \in \mathcal{S}(\mathbb{D})$ such that $X = S_\varphi$ and*

$$\|X\| = \|\varphi\|_\infty.$$

In other words, norm-preserving lifting always holds in the $n = 1$ case. Our approach to lifting in Theorem 4.3, when restricted to $n = 1$, recovers precisely Sarason's result, as in the polydisc case treated in [13, Theorem 9.2]. In this case, however, one obtains only the inequality

$$\|X\| \leq \|\varphi\|_\infty,$$

and not necessarily the norm-preserving aspect of lifting.

In stark contrast to Sarason, we show in this section that such a norm-preserving lifting phenomenon always fails for finite-dimensional quotient modules of $H^2(\mathbb{B}^n)$ whenever $n > 1$. First, we prove a lemma of general interest. We follow the notation introduced in Theorem 4.3.

Lemma 9.2. *Let $n \geq 1$. Let $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ be a quotient module and let $X \in \mathcal{B}(\mathcal{Q})$ be a module map. Define*

$$\psi = X(P_{\mathcal{Q}} 1),$$

and define $X_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1) \to \mathbb{C}$ by

$$X_{\mathcal{Q}} f = \int_{\mathbb{S}^n} \psi f \, d\sigma,$$

for all $f \in \mathcal{M}_{\mathcal{Q}}$. If $X_{\mathcal{Q}}$ is bounded, then

$$\|X\| \leq \|X_{\mathcal{Q}}\|.$$

Proof. Without loss of generality, assume that $X \neq 0$. First, suppose that $\|X\| = 1$. If $\|X_{\mathcal{Q}}\| < 1$, then Theorem 4.3 implies that X admits a lift, and (see (4.6)) there exists a function $\theta \in \mathcal{S}(\mathbb{B}^n)$ such that $X = S_\theta$ and

$$\|\theta\|_\infty = \|X_{\mathcal{Q}}\| < 1.$$

For $f \in clm_{\mathcal{Q}}$, we have

$$\|Xf\|_2 = \|S_\theta f\|_2 = \|P_{\mathcal{Q}}\theta f\|_2 \leq \|\theta f\|_2 \leq \|\theta\|_\infty \|f\|_2,$$

and hence $\|X\| \leq \|\theta\|_\infty < 1$, which contradicts the assumption $\|X\| = 1$. This implies $\|X_{\mathcal{Q}}\| \geq 1$, that is, $\|X\| \leq \|X_{\mathcal{Q}}\|$.
For the general case, assume that X is a nonzero module map, and define

$$A = \frac{1}{\|X\|} X.$$

Clearly, $A \in \mathcal{B}(\mathcal{Q})$ is a module map and $\|A\| = 1$. By the above case, we have

$$1 = \|A\| \leq \|A_{\mathcal{Q}}\|.$$

Now, for any $f \in \mathcal{M}_{\mathcal{Q}}$,

$$A_{\mathcal{Q}} f = \int_{\mathbb{S}^n} (AP_{\mathcal{Q}}1) f d\sigma = \int_{\mathbb{S}^n} \frac{1}{\|X\|} X(P_{\mathcal{Q}}1) f d\sigma = \frac{1}{\|X\|} X_{\mathcal{Q}} f,$$

which implies

$$A_{\mathcal{Q}} = \frac{1}{\|X\|} X_{\mathcal{Q}}.$$

Since $\|A_{\mathcal{Q}}\| \geq 1$, it follows that

$$\frac{\|X_{\mathcal{Q}}\|}{\|X\|} \geq 1,$$

that is, $\|X\| \leq \|X_{\mathcal{Q}}\|$. This completes the proof of the lemma. □

In the context of the above lemma, note that if $\mathcal{Q}$ is finite-dimensional, then $X_{\mathcal{Q}}$ is automatically bounded. We also point out a somewhat nontrivial feature of finite-dimensional quotient modules: If $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ is a finite-dimensional quotient module, then (cf. [19, Corollary 3.4] and [11, Chapter 2, Remark 2.5.5])

$$\mathcal{Q} \subset A(\mathbb{B}^n) \subset H^\infty(\mathbb{B}^n). \tag{9.1}$$

Before proceeding to the next result, we recall an interesting result of Rudin that highlights a significant distinction between the cases $n = 1$ and $n > 1$ [42, Theorem 4.6]: Let $n > 1$. If f is a nonzero function in $A(\mathbb{B}^n)$ and $u \in H^\infty(\mathbb{B}^n)$ is a nonconstant inner function, then

$$\frac{f}{u} \notin H^2(\mathbb{B}^n). \tag{9.2}$$

In fact, more is true: $\frac{f}{u}$ is not even in the Smirnov class $N_*(\mathbb{B}^n)$. In the following, we bring together Rudin's result with our commutant lifting theorem and the above lemma:

Theorem 9.3. *Assume $n > 1$. Let $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ be a finite-dimensional quotient module and let $X \in \mathcal{B}(\mathcal{Q})$ be a module map with $\|X\| = 1$. Then X admits a lift if and only if there exists a unimodular constant α such that*

$$X = \alpha I_{\mathcal{Q}}.$$

Proof. As in Lemma 9.2, define $X_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1) \to \mathbb{C}$ by

$$X_{\mathcal{Q}} f = \int_{\mathbb{S}^n} (X(P_{\mathcal{Q}}1)) f d\sigma,$$

for all $f \in \mathcal{M}_{\mathcal{Q}}$. Since $\mathcal{Q}$ is finite-dimensional, we know that $X_{\mathcal{Q}}$ is continuous. Moreover, as $\|X\| = 1$, Lemma 9.2 implies

$$\|X_{\mathcal{Q}}\| \geq 1.$$

Suppose X admits a lift. Theorem 4.3 ensures that $\|X_{\mathcal{Q}}\| \leq 1$ and thus we must have

$$\|X_{\mathcal{Q}}\| = 1.$$

By the sufficient part of the proof of Theorem 4.3, and more specifically, by (4.6), there exists $\theta \in \mathcal{S}(\mathbb{B}^n)$ such that $\|\theta\|_\infty = 1$ and

$$X = S_\theta.$$

Given that $\mathcal{Q}$ is finite-dimensional, we conclude that $X \in \mathcal{B}(\mathcal{Q})$ is norm-attaining. There exists $f \in \mathcal{Q}$, $\|f\|_2 = 1$, such that $\|Xf\|_2 = 1$, and hence

$$1 = \|Xf\|_2 = \|S_\theta f\|_2 = \|P_{\mathcal{Q}} \theta f\|_2 \leq \|\theta f\|_2 \leq 1.$$

Therefore, the above chain of inequalities reduces to equalities, and we conclude that

$$P_{\mathcal{Q}} \theta f = \theta f,$$

and, in particular,

$$\theta f \in \mathcal{Q}$$

Consequently,

$$Xf = S_\theta f = P_{\mathcal{Q}}(\theta f) = \theta f.$$

We claim that θ is a constant. We proceed as follows. Since

$$1 = \|Xf\|_2 = \|\theta f\|_2 = \|f\|_2,$$

it follows that

$$1 = \int_{\mathbb{S}^n} |f|^2 d\sigma = \int_{\mathbb{S}^n} |\theta f|^2 d\sigma,$$

that is,

$$\int_{\mathbb{S}^n} (1 - |\theta|^2)|f|^2 d\sigma = 0.$$

Since $\theta \in \mathcal{S}(\mathbb{B}^n)$, the integrand is nonnegative. Hence,

$$(1 - |\theta|^2)|f|^2 = 0,$$

on $\mathbb{S}^n$ a.e. Define

$$E_1 = \{\zeta \in \mathbb{S}^n : ((1 - |\theta|^2)|f|^2)(\zeta) \neq 0\}.$$

Clearly,

$$\sigma(E_1) = 0.$$

Define

$$E_2 = \{\zeta \in \mathbb{S}^n : f(\zeta) = 0\}.$$

Given that f is a nonzero analytic function in $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$, we have (in view of the K-limits, cf. [43, Theorem 5.5.9])

$$\sigma(E_2) = 0.$$

Let $E = E_1 \cup E_2$. Then

$$\sigma(E) = 0.$$

For any $\zeta \in E^c$, we have $f(\zeta) \neq 0$ and $((1-|\theta|^2)|f|^2)(\zeta) = 0$, which implies

$$|\theta|(\zeta) = 1,$$

for $\zeta \in E^c$. Since $\sigma(E) = 0$ as well, we conclude that θ is inner. Since $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ is finite-dimensional, by (9.1), it follows that

$$\mathcal{Q} \subset A(\mathbb{B}^n) \subset H^\infty(\mathbb{B}^n).$$

As noted earlier,

$$\theta f = Xf \in \mathcal{Q} \subseteq A(\mathbb{B}^n)$$

and hence $\theta f \in A(\mathbb{B}^n)$. On the other hand, writing $f = \frac{\theta f}{\theta}$, we conclude

$$\frac{\theta f}{\theta} = f \in H^2(\mathbb{B}^n).$$

If θ were nonconstant, this would contradict Rudin [42, Theorem 4.6] (or see the discussion preceding the statement of this theorem). Hence $\theta \equiv \alpha$ for some $\alpha \in \mathbb{T}$. Since $X = S_\theta$, it follows that $X = \alpha I_{\mathcal{Q}}$. The converse is immediate. □

As a result, for $n > 1$, we obtain the following result, which stands in sharp contrast to Sarason's conclusion in the one-variable case $n = 1$:

Corollary 9.4. *Assume $n > 1$. Let X be a nonconstant module map acting on a finite-dimensional quotient module $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$. Then X does not admit a norm-preserving lift.*

Proof. Let $A = \frac{X}{\|X\|}$. Then $A \in \mathcal{B}(\mathcal{Q})$ is a module map with $\|A\| = 1$. Suppose X has a norm-preserving lift, say $X = S_\theta$, $\theta \in \mathcal{S}(\mathbb{B}^n)$, with $\|\theta\|_\infty = \|X\|$. Define

$$\varphi = \frac{1}{\|X\|}\theta.$$

Clearly, $\varphi \in \mathcal{S}(\mathbb{B}^n)$ is a nonconstant function and $\|\varphi\|_\infty = 1$. For $f \in \mathcal{Q}$, we have

$$Af = \frac{1}{\|X\|} P_{\mathcal{Q}} \theta f = P_{\mathcal{Q}} \varphi f.$$

This shows that A admits a lift, namely, S_φ - which contradicts Theorem 9.3. □

However, quiet interestingly, module maps on infinite-dimensional quotient modules may admit norm-preserving lifts, as we will see in Theorem 11.7.

We now assume that $n \geq 1$. The following result stands in contrast to the previous corollary:

Theorem 9.5. *Let $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ be a quotient module, and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. Then X has a norm preserving lift if and only if*

$$\|X\| = \|X_{\mathcal{Q}}\|.$$

Proof. Suppose that X admits a norm-preserving lift:

$$X = S_\varphi \text{ with } \|X\| = \|\varphi\|_\infty,$$

for some $\varphi \in \mathcal{S}(\mathbb{B}^n)$. By Theorem 4.3 we know that $X_{\mathcal{Q}}$ is bounded. Then, by Theorem 4.5, we have

$$\|X_{\mathcal{Q}}\| \leq \|\varphi\|_\infty = \|X\|.$$

Together with Lemma 9.2, this implies that $\|X\| = \|X_{\mathcal{Q}}\|$. Conversely, suppose that $\|X\| = \|X_{\mathcal{Q}}\|$. Since $\|X\| \leq 1$, we have $\|X_{\mathcal{Q}}\| \leq 1$. By Theorems 4.3 and 4.5, we conclude that X admits a lift:

$$X = S_\varphi \text{ with } \|X_{\mathcal{Q}}\| = \|\varphi\|_\infty,$$

for some $\varphi \in \mathcal{S}(\mathbb{B}^n)$. Since $\|X\| = \|X_{\mathcal{Q}}\|$, it follows that X admits a norm-preserving lift. $\square$

Combining this with Corollary 9.4, we have the following:

Corollary 9.6 (On $n > 1$)**.** *Assume $n > 1$. Let $\mathcal{Q}$ be a finite-dimensional quotient module of $H^2(\mathbb{B}^n)$ and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a nonconstant module map. Then*

$$\|X\| < \|X_{\mathcal{Q}}\|.$$

Proof. If $\|X\| = \|X_{\mathcal{Q}}\|$, then by Theorem 9.5, X admits a norm preserving lift. This contradicts Corollary 9.4, which says that X does not admit a norm-preserving lift. Therefore, by Lemma 9.2, we conclude that $\|X\| < \|X_{\mathcal{Q}}\|$. $\square$

However, somewhat curiously, in the case $n = 1$, the above inequality becomes an equality:

Corollary 9.7 (On $n = 1$)**.** *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{D})$, and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a nonconstant module map. Then*

$$\|X\| = \|X_{\mathcal{Q}}\|.$$

Proof. Define

$$A = \frac{1}{\|X\|} X$$

Then $A \in \mathcal{B}(\mathcal{Q})$ is a module map with $\|A\| = 1$. Suppose $A = S_\theta$ for some $\theta \in \mathcal{S}(\mathbb{D})$. For each $f \in \mathcal{Q}$, we have

$$\|Af\|_2 = \|S_\theta f\|_2 = \|P_{\mathcal{Q}} \theta f\|_2 \leq \|\theta f\|_2 \leq \|\theta\|_\infty \|f\|_2,$$

that is, $\|A\| \leq \|\theta\|_\infty$. Therefore,

$$\|A\| \leq \inf\{\|\theta\|_\infty : A = S_\theta, \theta \in \mathcal{S}(\mathbb{D})\}.$$

On the other hand, by Sarason, Theorem 9.1, there exists $\varphi \in \mathcal{S}(\mathbb{D})$ such that

$$A = S_\varphi, \text{ and } \|\varphi\|_\infty = \|A\| = 1,$$

and hence

$$\|A\| = \inf\{\|\theta\|_\infty : A = S_\theta, \theta \in \mathcal{S}(\mathbb{D})\}.$$

Moreover, by the norm formula in Theorem 4.5,

$$\|A_{\mathcal{Q}}\| = \inf\{\|\theta\|_\infty : A = S_\theta, \theta \in \mathcal{S}(\mathbb{D})\},$$

where $A_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1) \to \mathbb{C}$ given by

$$A_{\mathcal{Q}} f = \int_{\mathbb{T}} f \psi \, d\sigma,$$

for all $f \in \mathcal{M}_{\mathcal{Q}}$ with $\psi = A P_{\mathcal{Q}} 1$. Therefore

$$\|A_{\mathcal{Q}}\| = \|A\| = 1.$$

Since $A(P_{\mathcal{Q}} 1) = \frac{1}{\|X\|} X(P_{\mathcal{Q}} 1)$, it follows that $A_{\mathcal{Q}} = \frac{1}{\|X\|} X_{\mathcal{Q}}$, and hence $\|X\| = \|X_{\mathcal{Q}}\|$. $\square$

Summarizing the preceding two results, we obtain the following dichotomy between the cases $n = 1$ and $n > 1$: Let $\mathcal{Q}$ be a finite-dimensional quotient module of $H^2(\mathbb{B}^n)$, and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a nonconstant module map. Then

$$\|X\| = \|X_{\mathcal{Q}}\| \text{ if } n = 1,$$

and

$$\|X\| < \|X_{\mathcal{Q}}\| \text{ if } n > 1.$$

10. Extremal problem and its solution

This section provides characterizations of the extremal problem. See Section 2 for the definition and other basic details regarding the extremal problem.

Recall that, for $n = 1$, an interpolation problem is extremal if and only if the corresponding Pick matrix is positive semi-definite but not positive definite [3, Chapter 6]. Moreover, in this case, the extremal solution is unique and is a finite Blaschke product [3, Theorem 6.4]. In the case of $\mathbb{B}^n$, we give concrete characterizations of this problem using the quantitative and qualitative criteria studied earlier.

Fix interpolation data $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subseteq \mathbb{D}$. As in Section 7, or more specifically in Theorem 7.2, we define

$$\mathcal{Q}_{\mathcal{Z}} = \operatorname{span}\{K_n(\cdot, z_i) : i = 1, \ldots, m\},$$

and

$$\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}} = \mathcal{Q}_{\mathcal{Z}}^{conj} \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H_0^2(\mathbb{S}^n)].$$

Recall that $X_{\mathcal{Z},\mathcal{W}} \in \mathcal{B}_1(\mathcal{Q}_{\mathcal{Z}})$ is a module map satisfying

$$X_{\mathcal{Z},\mathcal{W}}^* K_n(\cdot, z_j) = \overline{w_j} K_n(\cdot, z_j),$$

for all $j = 1, \ldots, m$. Moreover, we have the functional $\Pi_{\mathcal{Z},\mathcal{W}} : (\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}, \|\cdot\|_1) \to \mathbb{C}$ defined by

$$\Pi_{\mathcal{Z},\mathcal{W}} f = \int_{\mathbb{S}^n} \psi_{\mathcal{Z},\mathcal{W}} f \, d\sigma,$$

for $f \in \mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}$, where (see Theorem 7.2)

$$\psi_{\mathcal{Z},\mathcal{W}} = \sum_{j=1}^m c_j K_n(\cdot, z_j),$$

and

$$\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} K_n(z_i, z_j) \end{bmatrix}_{m \times m}^{-1} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}.$$

Finally, recall from Corollary 7.3 that

$$\widetilde{\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}} = \ker \Pi_{\mathcal{Z},\mathcal{W}} = [\mathcal{Q}_{\mathcal{Z}}^{conj} \ominus \{\overline{\psi_{\mathcal{Z},\mathcal{W}}}\}] \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H_0^2(\mathbb{S}^n)].$$

Theorem 10.1. *Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subseteq \mathbb{D}$ be interpolation data. Then the following are equivalent:*

(1) $\mathcal{Z}$ and $\mathcal{W}$ solve the extremal problem.

(2) $\|\Pi_{\mathcal{Z},\mathcal{W}}\| = 1$.

(3) $dist_{L^1(\mathbb{S}^n)}\left(\frac{\overline{\psi_{\mathcal{Z},\mathcal{W}}}}{\|\psi_{\mathcal{Z},\mathcal{W}}\|_2^2}, \widetilde{\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}}\right) = 1$.

Proof. First, we prove that (1) implies (2). Suppose $\mathcal{Z}$ and $\mathcal{W}$ solve the extremal problem. This means, in particular, that there exists an interpolant. By Theorem 7.2, we have

$$\|\Pi_{\mathcal{Z},\mathcal{W}}\| \leq 1.$$

Assume, for contradiction, that $\|\Pi_{\mathcal{Z},\mathcal{W}}\| < 1$. Then, by Theorem 4.3 and Theorem 4.5, there exists $\theta \in \mathcal{S}(\mathbb{B}^n)$ such that $X_{\mathcal{Z},\mathcal{W}} = S_\theta$ and

$$\|\theta\|_\infty = \|X_{\mathcal{Z},\mathcal{W}}\| < 1.$$

Since $X_{\mathcal{Z},\mathcal{W}} = S_\theta$, by (7.4), we know that θ interpolates $\mathcal{Z}$ and $\mathcal{W}$. However, since the interpolation problem is extremal, this contradicts the definition. Therefore, we must have $\|\Pi_{\mathcal{Z},\mathcal{W}}\| = 1$.
To prove that (2) implies (1), assume that $\|\Pi_{\mathcal{Z},\mathcal{W}}\| = 1$. Since $\Pi_{\mathcal{Z},\mathcal{W}}$ is a contraction, by Theorem 4.3, Theorem 4.5, and Theorem 7.2, the interpolation problem has a solution $\theta \in \mathcal{S}(\mathbb{B}^n)$ with $\|\theta\|_\infty = 1$. Now, suppose $\varphi \in \mathcal{S}(\mathbb{B}^n)$ interpoles $\mathcal{Z}$ and $\mathcal{W}$ and satisfies

$$\|\varphi\|_\infty < 1.$$

By (7.4), we have $X_{\mathcal{Z},\mathcal{W}} = S_\varphi$. Now, in view of Theorem 4.5, we have

$$\|\Pi_{\mathcal{Z},\mathcal{W}}\| = \inf\{\|\theta\| : \theta \in \mathcal{S}(\mathbb{B}^n), S_\theta = X_{\mathcal{Z},\mathcal{W}}\}.$$

Thus,

$$\|\Pi_{\mathcal{Z},\mathcal{W}}\| \leq \|\varphi\|_\infty < 1,$$

which contradicts the assumption that $\|\Pi_{\mathcal{Z},\mathcal{W}}\| = 1$. Therefore, the interpolation problem is extremal. The equivalence of (2) and (3) follows from Lemma 5.1 in exactly the same way as in the proof of Theorem 5.3. □

Next, we formulate an extremal problem based on a given solvable interpolation problem. This approach will ultimately lead to a solution for the original solvable interpolation problem.

Theorem 10.2. *Suppose $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{B}^n$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subset \mathbb{D}$ are interpolation data, with $\mathcal{W} \neq \{0\}$. If $\mathcal{Z}$ and $\mathcal{W}$ solve the interpolation problem, then $\mathcal{Z}$ and $\widehat{\mathcal{W}}$ solve the extremal interpolation problem, where*

$$\widehat{\mathcal{W}} = \{\widehat{w}_1, \ldots, \widehat{w}_m\} \subset \mathbb{D},$$

and

$$\widehat{w}_j = \frac{1}{\|\Pi_{\mathcal{Z},\mathcal{W}}\|} w_j,$$

for all $j = 1, \ldots, m$.

Proof. Note that $\Pi_{\mathcal{Z},\mathcal{W}}$ is a nonzero bounded linear functional on $\mathcal{M}_{Q_{\mathcal{Z}}}$. Since the interpolation problem is solvable, Theorem 7.2 implies that

$$\|\Pi_{\mathcal{Z},\mathcal{W}}\| \leq 1.$$

For each $j = 1, \ldots, m$, define

$$\widehat{w}_j = \frac{1}{\|\Pi_{\mathcal{Z},\mathcal{W}}\|} w_j.$$

We claim that

$$\{\widehat{w}_j\}_{j=1}^m \subseteq \mathbb{D},$$

for $j = 1, \ldots, m$. Indeed, for a fixed $j \in \{1, \ldots, m\}$, using the definition of $X_{\mathcal{Z},\mathcal{W}}$ (recall that $X^*_{\mathcal{Z},\mathcal{W}} K_n(\cdot, z_j) = \overline{w_j} K_n(\cdot, z_j)$), we first note that

$$\frac{\|X^*_{\mathcal{Z},\mathcal{W}} K_n(\cdot, z_j)\|}{\|K_n(\cdot, z_j)\|} = |w_j|.$$

Since

$$\|X_{\mathcal{Z},\mathcal{W}}\| = \|X^*_{\mathcal{Z},\mathcal{W}}\| \geq \frac{\|X^*_{\mathcal{Z},\mathcal{W}} K_n(\cdot, z_j)\|}{\|K_n(\cdot, z_j)\|},$$

it follows that

$$|w_j| \leq \|X_{\mathcal{Z},\mathcal{W}}\|.$$

On the other hand, since $\mathcal{Q}_{\mathcal{Z}}$ is finite-dimensional, Lemma 9.2 implies (with the functional $X_{\mathcal{Q}}$ denoted by $\Pi_{\mathcal{Z},\mathcal{W}}$ in this context)

$$\|X_{\mathcal{Z},\mathcal{W}}\| \leq \|\Pi_{\mathcal{Z},\mathcal{W}}\|.$$

As $\|\Pi_{\mathcal{Z},\mathcal{W}}\| \leq 1$, we conclude that

$$\|X_{\mathcal{Z},\mathcal{W}}\| \leq 1,$$

that is, $X_{\mathcal{Z},\mathcal{W}}$ is a contraction. Now, we consider two cases.

Case I ($n > 1$): By Corollary 9.6, since the module map $X_{\mathcal{Z},\mathcal{W}}$ is a nonconstant contraction, we have

$$\|X_{\mathcal{Z},\mathcal{W}}\| < \|\Pi_{\mathcal{Z},\mathcal{W}}\| \leq 1,$$

and therefore,

$$|w_j| \leq \|X_{\mathcal{Z},\mathcal{W}}\|,$$

which proves the claim that $\hat{w}_j \in \mathbb{D}$.

Case II ($n = 1$): In this case, $\mathcal{Q}_{\mathcal{Z}} \subseteq H^2(\mathbb{D})$. By Corollary 9.7, we have

$$\|\Pi_{\mathcal{Z},\mathcal{W}}\| = \|X_{\mathcal{Z},\mathcal{W}}\|.$$

To obtain a contradiction, suppose that there exists $j \in \{1, \ldots, m\}$ such that

$$|w_j| = \|X_{\mathcal{Z},\mathcal{W}}\|.$$

By Sarason's commutant lifting theorem [45] (or see Theorem 9.1) applied to $X_{\mathcal{Z},\mathcal{W}}$ on $\mathcal{Q}_{\mathcal{Z}}$, there exists a function $\varphi \in \mathcal{S}(\mathbb{D})$ such that

$$X_{\mathcal{Z},\mathcal{W}} = S_\varphi \text{ and } \|X_{\mathcal{Z},\mathcal{W}}\| = \|\varphi\|_\infty.$$

Since $X^*_{\mathcal{Z},\mathcal{W}} = S^*_\varphi = T^*_\varphi|_{\mathcal{Q}}$, equation (7.4) gives

$$\varphi(z_j) = w_j,$$

and hence

$$|\varphi(z_j)| = |w_j| = \|X_{\mathcal{Z},\mathcal{W}}\| = \|\varphi\|_\infty.$$

By the maximum modulus principle, φ must be constant (note that $z_j \in \mathbb{D}$). Consequently, $X_{\mathcal{Z},\mathcal{W}} = \alpha I_{\mathcal{Q}}$, contradicting the assumption that $m \geq 2$. This proves the claim that $|\hat{w}_j| < 1$ for all $j = 1, \ldots, m$.

Therefore,

$$\widehat{\mathcal{W}} := \{\widehat{w}_1, \ldots, \widehat{w}_m\} \subseteq \mathbb{D}.$$

We claim that $\mathcal{Z}$ and $\widehat{\mathcal{W}}$ solve the extremal problem. Following the notation of Theorem 7.2, we proceed as follows. Define module map $X_{\mathcal{Z},\widehat{\mathcal{W}}} \in \mathcal{B}(\mathcal{Q}_{\mathcal{Z}})$ by

$$X^*_{\mathcal{Z},\widehat{\mathcal{W}}} K_n(\cdot, z_j) = \overline{\widehat{w}_j} K_n(\cdot, z_j).$$

that is,

$$X^*_{\mathcal{Z},\widehat{\mathcal{W}}} K_n(\cdot, z_j) = \frac{1}{\|\Pi_{\mathcal{Z},\mathcal{W}}\|} X^*_{\mathcal{Z},\mathcal{W}} K_n(\cdot, z_j),$$

for all $j = 1, \ldots, m$. In other words,

$$X_{\mathcal{Z},\widehat{\mathcal{W}}} = \frac{1}{\|\Pi_{\mathcal{Z},\mathcal{W}}\|} X_{\mathcal{Z},\mathcal{W}}.$$

As usual, define

$$\widehat{\psi} = X_{\mathcal{Z},\widehat{\mathcal{W}}}(P_{\mathcal{Q}_{\mathcal{Z}}} 1).$$

Recall that

$$\psi = X_{\mathcal{Z},\mathcal{W}}(P_{\mathcal{Q}_{\mathcal{Z}}} 1),$$

and hence

$$\psi = \|\Pi_{\mathcal{Z},\mathcal{W}}\| X_{\mathcal{Z},\widehat{\mathcal{W}}}(P_{\mathcal{Q}_{\mathcal{Z}}} 1) = \|\Pi_{\mathcal{Z},\mathcal{W}}\| \widehat{\psi}.$$

Finally, define the functional $\Pi_{\mathcal{Z},\widehat{\mathcal{W}}} : (\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}, \|\cdot\|_1) \to \mathbb{C}$ by

$$\Pi_{\mathcal{Z},\widehat{\mathcal{W}}} f = \int_{\mathbb{S}^n} \widehat{\psi} f \, d\sigma,$$

for all $f \in \mathcal{M}_{\mathcal{Q}}$. Therefore,

$$\Pi_{\mathcal{Z},\widehat{\mathcal{W}}} f = \frac{1}{\|\Pi_{\mathcal{Z},\mathcal{W}}\|} \Pi_{\mathcal{Z},\mathcal{W}} f,$$

$f \in \mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}$, and hence

$$\|\Pi_{\mathcal{Z},\widehat{\mathcal{W}}}\| = 1.$$

Consequently, Theorem 10.1 implies that $\mathcal{Z}$ and $\widehat{\mathcal{W}}$ solve the extremal problem. □

This result has an important consequence. For each $m \geq 1$, define (see Section 2)

$$\mathcal{E}_m(\mathbb{B}^n) = \{\varphi \in \mathcal{S}(\mathbb{B}^n) : \varphi \text{ is a solution to some } m\text{-point extremal problem}\}.$$

Corollary 10.3. *Let $m > 1$. Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{B}^n$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subset \mathbb{D}$ be interpolation data, with $\mathcal{W} \neq \{0\}$, and suppose that $\mathcal{Z}$ and $\mathcal{W}$ solve the interpolation problem. Define*

$$\lambda_{\mathcal{Z},\mathcal{W}} = \inf\{\|\eta\|_\infty : \eta \in \mathcal{S}(\mathbb{B}^n) \text{ interpolates } \mathcal{Z} \text{ and } \mathcal{W}\}.$$

Then $\lambda_{\mathcal{Z},\mathcal{W}} \in (0, 1]$, and there exists an extremal function $\varphi \in \mathcal{E}_m(\mathbb{B}^n)$ such that

$$\lambda_{\mathcal{Z},\mathcal{W}} \varphi,$$

interpolates $\mathcal{Z}$ and $\mathcal{W}$. Furthermore, $\lambda_{\mathcal{Z},\mathcal{W}}\varphi$ is a minimum-norm interpolant for the data $\mathcal{Z}$ and $\mathcal{W}$.

Proof. Since $\mathcal{Z}$ and $\mathcal{W}$ solve the interpolation problem, Theorem 10.2 implies that $\mathcal{Z}$ and $\widehat{\mathcal{W}}$ solve the extremal problem. That is, there exists $\varphi \in \mathcal{E}_m(\mathbb{B}^n)$ such that

$$\varphi(z_i) = \widehat{w}_i,$$

for all $i = 1, \ldots, m$. Recall that

$$\widehat{w}_j = \frac{1}{\|\Pi_{\mathcal{Z},\mathcal{W}}\|} w_j,$$

for all $j = 1, \ldots, m$, where $\Pi_{\mathcal{Z},\mathcal{W}}$ is the functional associated with the interpolation problem for $\mathcal{Z}$ and $\mathcal{W}$. Recall, by Theorem 7.2, that

$$\|\Pi_{\mathcal{Z},\mathcal{W}}\| \leq 1.$$

Set
$$\lambda_{\mathcal{Z},\mathcal{W}} = \|\Pi_{\mathcal{Z},\mathcal{W}}\|,$$
and define
$$\widetilde{\varphi} = \lambda_{\mathcal{Z},\mathcal{W}}\varphi.$$
Clearly, $\widetilde{\varphi} \in \mathcal{S}(\mathbb{B}^n)$ and interpolates the data $\mathcal{Z}$ and $\mathcal{W}$. Finally, recall from Theorem 4.5 the norm expression of $\Pi_{\mathcal{Z},\mathcal{W}}$ as
$$\|\Pi_{\mathcal{Z},\mathcal{W}}\| = \inf\{\|\theta\|_\infty : \theta \in \mathcal{S}(\mathbb{B}^n), S_\theta = X_{\mathcal{Z},\mathcal{W}}\}.$$
On the other hand, by (7.4), for every $\theta \in \mathcal{S}(\mathbb{B}^n)$,
$$S_\theta = X_{\mathcal{Z},\mathcal{W}},$$
if and only if θ interpolates $\mathcal{Z}$ and $\mathcal{W}$. Consequently,
$$\|\Pi_{\mathcal{Z},\mathcal{W}}\| = \inf\{\|\eta\|_\infty : \eta \in \mathcal{S}(\mathbb{B}^n) \text{ interpolates } \mathcal{Z} \text{ and } \mathcal{W}\},$$
and hence
$$|\lambda_{\mathcal{Z},\mathcal{W}}| = \|\Pi_{\mathcal{Z},\mathcal{W}}\|.$$
Finally, we have
$$\|\widetilde{\varphi}\|_\infty = \|\lambda_{\mathcal{Z},\mathcal{W}}\varphi\|_\infty = |\lambda_{\mathcal{Z},\mathcal{W}}| = \|\Pi_{\mathcal{Z},\mathcal{W}}\|.$$
This completes the proof of the corollary. $\square$

This result is particularly beneficial if one has a description of $\mathcal{E}_m(\mathbb{B}^n)$. To this end, in Section 12, we will apply this result to a specific case, such as the three-point interpolation problem. Moreover, this result may be viewed as an analogue of the classical interpolation results for the one-variable case. In particular, compare it with Corollary 1.8 in [17, page 132].

11. Examples of lifting

The goal of this section is to investigate the lifting problem through concrete examples. In particular, we consider examples of contractive module maps that do or do not admit a lift. This allows us, in particular, to apply the preceding results to concrete situations. Recall that an analytic function $f : \mathbb{B}^n \to \mathbb{C}$ can be expressed as a power series
$$f(z) = \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha z^k,$$
The degree of a monomial z^k, $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$, is defined as
$$\deg\, z^k = |k| := \sum_{j=1}^n k_j.$$
The degree of a polynomial is defined as the highest degree among its monomial terms. Given $m \in \mathbb{N}$, define
$$\mathcal{Q}_m = \{p \in \mathbb{C}[z_1, \ldots, z_n] : \deg\, p \leq m\}.$$
Clearly, $\mathcal{Q}_m$ is a finite-dimensional quotient module of $H^2(\mathbb{B}^n)$. Fix a polynomial $p \in \mathbb{C}[z_1, \ldots, z_n]$, and consider the module map
$$S_p = P_{\mathcal{Q}_m} T_p|_{\mathcal{Q}_m}.$$
We are interested in the lifting of S_p under the assumption that $\|p\|_2 = 1$.

Theorem 11.1. *Assume $n > 1$. Let $p \in \mathbb{C}[z_1, \dots, z_n]$, and let deg $p \leq m$. Suppose*

$$\|p\|_2 = 1.$$

Then $S_p \in \mathcal{B}(\mathcal{Q}_m)$ admits a lift if and only if p is a unimodular constant.

Proof. Let S_p admits a lift. We claim that

$$\|S_p\| = 1.$$

Indeed, since S_p admits a lift, there exists $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that

$$S_p = S_\varphi = P_{\mathcal{Q}_m} T_\varphi|_{\mathcal{Q}_m},$$

and hence $\|S_p\| \leq 1$. On the other hand, since 1 and p are in $\mathcal{Q}_m$, it follows that

$$S_p 1 = P_{\mathcal{Q}_m} p = p,$$

which implies

$$\|S_p\| \geq \|S_p 1\|_2 = \|p\|_2 = 1.$$

This proves the claim that $\|S_p\| = 1$. Since $\mathcal{Q}_m$ is finite-dimensional, Theorem 9.3 applies, and therefore

$$S_p = \alpha I_{\mathcal{Q}_m},$$

for some unimodular constant α. The above identity then implies that $p = \alpha$. The converse is obvious. $\square$

For each $m \geq 1$, define

$$J_m = \{f \in H^2(\mathbb{B}^n) : f(z) = \sum_{k \in \mathbb{Z}_+^n, |k| \geq m} a_k z^k\}.$$

Equivalently, J_m consists of analytic functions on $\mathbb{B}^n$ whose Taylor series contain no terms of degree less than m. The following lemma says, in particular, that J_m is a submodule.

Lemma 11.2. *$J_{m+1} = \mathcal{Q}_m^\perp$ for all $m \geq 1$.*

Proof. This is a simple consequence of the fact that $\{z^k : k \in \mathbb{Z}_+^n\}$ forms an orthogonal basis for $H^2(\mathbb{B}^n)$. $\square$

We now recall a perturbation theorem of Rudin [42, Theorem 4.3] concerning inner functions on $\mathbb{B}^n$, specialized here to the case of the ball algebra. This result will be applied to construct two nontrivial examples.

Theorem 11.3 (Rudin)**.** *Let g be a nonzero function in $A(\mathbb{B}^n)$ and let $m \in \mathbb{N}$. Assume that*

$$\|g\|_\infty < 1.$$

Then there exists an inner function $u \in H^\infty(\mathbb{B}^n)$ such that u and g have the same zeros in $\mathbb{B}^n$ and

$$u - g \in J_m.$$

The following result provides the first application of Rudin's theorem. In particular, it yields inner liftings of certain contractive module maps.

Example 11.4. Fix $m \in \mathbb{N}$, and set

$$\Lambda = \{k \in \mathbb{Z}_+^n : |k| \leq m\}.$$

Choose a finite set of scalars

$$\{a_k : k \in \Lambda\} \subseteq \mathbb{C}.$$

Assume that

$$0 < \sum_{k \in \Lambda} |a_k| \leq 1.$$

Then

$$p(z) := \sum_{|k| \leq m} a_k z^k,$$

is a nonzero polynomial in $\mathcal{Q}_m$. Consider the module map $S_p \in \mathcal{B}(\mathcal{Q}_m)$. As $p \in \mathcal{Q}_m$, we have

$$S_p 1 = P_{\mathcal{Q}_m} T_p 1 = p.$$

On the other hand, for $z \in \mathbb{B}^n$, since

$$|p(z)| \leq \sum_{|\alpha| \leq m} |a_\alpha z^\alpha| \leq \sum_{|\alpha| \leq m} |a_\alpha| = \sum_{\alpha \in \Lambda} |a_\alpha|,$$

we conclude that

$$\|p\|_{H^\infty(\mathbb{B}^n)} \leq \sum_{\alpha \in \Lambda} |a_\alpha| < 1.$$

By Rudin, Theorem 11.3, there exists an inner function $u \in H^\infty(\mathbb{B}^n)$ such that

$$u - p \in J_{m+1}.$$

In view of Lemma 11.2, we know that $J_{m+1} = \mathcal{Q}_m^\perp$. In other words, we have an inner function $u \in H^\infty(\mathbb{B}^n)$ such that

$$u - p \in \mathcal{Q}_m^\perp.$$

Corollary 3.2 now implies that S_p lifts to S_u.

It is curious to observe how two apparently independent results fit together, namely, the perturbation result of Rudin (Theorem 11.3) and the characterization of lifting along the lines of perturbations in Corollary 3.2.

Let $u \in H^\infty(\mathbb{B}^n)$ be an inner function. In this case, we know that $uH^2(\mathbb{B}^n)$ is a submodule. Therefore,

$$\mathcal{Q}_u := H^2(\mathbb{B}^n) \ominus uH^2(\mathbb{B}^n),$$

is a quotient module. This quotient module is traditionally referred to as a model space [31]. In the following, we give a criterion of lifting to ball algebra. We remark that this result holds specifically for $n > 1$.

Theorem 11.5. *Let $n > 1$, $u \in H^\infty(\mathbb{B}^n)$ be a nonconstant inner function, and let $X \in \mathcal{B}_1(\mathcal{Q}_u)$ be a module map. Assume that*

$$\psi := X(P_{\mathcal{Q}_u} 1) \in A(\mathbb{B}^n).$$

Then X admits a lift to $A(\mathbb{B}^n)$ if and only if

$$\|\psi\|_\infty \leq 1.$$

Moreover, in this case, the lifting is unique.

Proof. Suppose X lifts to $A(\mathbb{B}^n)$. By Corollary 3.3, there exists $\varphi \in A(\mathbb{B}^n) \cap \mathcal{S}(\mathbb{B}^n)$ such that

$$\psi - \varphi \in \mathcal{Q}_u^\perp = uH^2(\mathbb{B}^n).$$

Since φ and ψ are in $A(\mathbb{B}^n)$, it follows that $\psi - \varphi \in A(\mathbb{B}^n)$, and hence

$$\psi - \varphi \in A(\mathbb{B}^n) \cap uH^2(\mathbb{B}^n).$$

Assume, for contradiction, that $\psi \neq \varphi$. From the above, there exists a nonzero $g \in H^2(\mathbb{B}^n)$ such that

$$\psi - \varphi = ug.$$

However, this is impossible, in view of (9.2) (which is again a result of Rudin; see [42, Theorem 4.6]). This contradiction shows that $\psi = \varphi$. Consequently, $\|\psi\|_\infty = \|\varphi\|_\infty \leq 1$. The converse is immediate by Corollary 3.3. □

To better illustrate the theorem, we now construct a concrete example satisfying its hypotheses.

Example 11.6. Fix $m \in \mathbb{N}$ and multi-index $k \in \mathbb{Z}_+^n$ such that

$$|k| = m + 1.$$

Consider the polynomial

$$p(z) = \frac{1}{2} z^k \in H^2(\mathbb{B}^n).$$

By Rudin, Theorem 11.3, there exists a non-constant inner function $u \in H^\infty(\mathbb{B}^n)$ such that

$$u - p \in J_{m+1} = \mathcal{Q}_m^\perp.$$

By our choice, we have $p \in \mathcal{Q}_m^\perp$, and hence

$$u = p + (u - p) \in \mathcal{Q}_m^\perp.$$

This implies $uH^2(\mathbb{B}^n) \subseteq \mathcal{Q}_m^\perp$, equivalently, $\mathcal{Q}_m \subseteq \mathcal{Q}_u$. Let $q \in \mathbb{C}[z_1, \ldots, z_n]$ with deg $q \leq m$, and consider the module map $X = S_q \in \mathcal{B}(\mathcal{Q}_u)$ defined by

$$S_q f = P_{\mathcal{Q}_u} q f$$

Then,

$$X P_{\mathcal{Q}_u} 1 = S_{\mathcal{Q}_u} P_{\mathcal{Q}_u} 1 = P_{\mathcal{Q}_u} q P_{\mathcal{Q}_u} 1 = P_{\mathcal{Q}_u} q = q \in A(\mathbb{B}^n).$$

By Theorem 11.5, X admits a lift to $A(\mathbb{B}^n)$ if and only if

$$\|q\|_\infty \leq 1.$$

The existence and structural properties of inner functions have played a fundamental role throughout this paper. We conclude this section with yet another application of this existence.

As pointed out, quotient modules of $H^2(\mathbb{B}^n)$ for $n > 1$ are, by far, intricate objects. In what follows, we focus on a particular class of module maps that admit liftings. Remarkably, these liftings exhibit a rigidity phenomenon, which we discuss after the theorem.

We remark that any nonconstant inner function is contained in some nontrivial quotient module. To see this, fix a nonconstant inner function $\varphi \in H^\infty(\mathbb{B}^n)$. Consider the submodule

$$\mathcal{S} = \varphi H_0^2(\mathbb{B}^n).$$

Let $\mathcal{Q} = \mathcal{S}^\perp$. Then $\mathcal{Q}$ is a quotient module that contains φ. Indeed, for $f \in H_0^2(\mathbb{B}^n)$, we have

$$\langle \varphi, \varphi f \rangle = \langle 1, f \rangle = \overline{f(0)} = 0.$$

The above argument ensures that the hypothesis in the following theorem is non-vacuous.

Theorem 11.7. *Let $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ be a quotient module containing a nonconstant inner function $\varphi \in \mathcal{S}(\mathbb{B}^n)$, and let $\lambda \in \mathbb{C}$. Then the module map*

$$S_{\lambda\varphi} = P_{\mathcal{Q}} T_{\lambda\varphi}|_{\mathcal{Q}},$$

on $\mathcal{Q}$ admits a lift if and only if

$$|\lambda| \leq 1.$$

Proof. Following Theorem 4.3, we write $X = S_{\lambda\varphi}$ and $\psi = X(P_{\mathcal{Q}}1)$. We know that $\lambda\varphi \in \mathcal{Q}$, and hence

$$\psi = S_{\lambda\varphi} P_{\mathcal{Q}} 1 = P_{\mathcal{Q}} T_{\lambda\varphi} 1 = \lambda\varphi.$$

Then the functional $X_{\mathcal{Q}} : \mathcal{M}_{\mathcal{Q}} \to \mathbb{C}$ in Theorem 4.3 simplifies to

$$X_{\mathcal{Q}} f = \int_{\mathbb{S}^n} \psi f d\sigma = \int_{\mathbb{S}^n} \lambda\varphi f d\sigma = \lambda \int_{\mathbb{S}^n} \varphi f d\sigma,$$

and hence

$$|X_{\mathcal{Q}} f| = |\lambda| \Big| \int_{\mathbb{S}^n} \varphi f d\sigma \Big| \leq |\lambda| \int_{\mathbb{S}^n} |f| d\sigma = |\lambda| \|f\|_{L^1(\mathbb{S}^n)},$$

for all $f \in \mathcal{M}_{\mathcal{Q}}$. On the other hand, since φ is inner, we have

$$|X_{\mathcal{Q}} \overline{\varphi}| = |\lambda| \int_{\mathbb{S}^n} \varphi \bar{\varphi} d\sigma = |\lambda| \int_{\mathbb{S}^n} |\varphi|^2 d\sigma = |\lambda|,$$

and therefore

$$\|X_{\mathcal{Q}}\| = |\lambda|.$$

Hence, by Theorem 4.3, X admits a lift if and only if $|\lambda| \leq 1$. ☐

For a quotient module $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ and a function $\varphi \in \mathcal{S}(\mathbb{B}^n)$, the module map

$$X := S_\varphi,$$

always admits a lift, namely T_φ on $H^2(\mathbb{B}^n)$. However, it may happen that φ is not a Schur function, but there exists $\psi \in \mathcal{S}(\mathbb{B}^n)$ with

$$\varphi \neq \psi,$$

such that

$$X = S_\varphi = S_\psi.$$

Theorem 11.7 shows that, under the stated assumptions, $X = S_{\lambda\varphi}$ does not admit the aforementioned property. This rigidity phenomenon is a rather distinctive feature of the lifting theorem of the above class of module maps.

At the end of the proof of Corollary 9.4, we remarked that module maps on infinite-dimensional quotient modules may admit norm-preserving lifts. This stands in contrast to the finite-dimensional case, where Corollary 9.4 shows that a module map does not admit a norm-preserving lift whenever $n > 1$.

We clarify the infinite-dimensional quotient module aspect. For instance, fix a nonconstant inner function φ, and choose a quotient module $\mathcal{Q}$ containing φ. Consider the module map $X = S_\varphi$ on $\mathcal{Q}$. Clearly X admits a lift, namely T_φ. Moreover, we have

$$\|X\| = 1 = \|\varphi\|_\infty.$$

On the other hand, since φ is inner, it follows from (9.2) that

$$\varphi \notin A(\mathbb{B}^n).$$

Since $\varphi \in \mathcal{Q}$, it follows that

$$\mathcal{Q} \nsubseteq A(\mathbb{B}^n).$$

Consequently, $\mathcal{Q}$ is infinite-dimensional (see the discussion preceding (9.2); see also [19, Corollary 3.4] and [11, Chapter 2, Remark 2.5.5]).

12. Three-point interpolation

This section is devoted to some concrete examples of interpolation problems. An interpolation problem is said to be a *three-point interpolation problem* if the interpolation data set $\mathcal{Z}$ and $\mathcal{W}$ consist of three points: $\mathcal{Z} = \{z_1, z_2, z_3\} \subset \mathbb{B}^n$ and $\mathcal{W} = \{w_1, w_2, w_3\} \subset \mathbb{D}$. We begin by illustrating a solution to a three-point interpolation problem.

Define

$$\mathcal{F}_3(\mathbb{B}^n) = \mathcal{F}_D \cup \mathcal{F}_{ND},$$

where

$$\mathcal{F}_D = \{x = (x_1, \dots, x_n) \mapsto \frac{2x_1(1-\tau x_1) - \overline{\tau}\omega^2 x_2^2}{2(1-\tau x_1) - \omega^2 x_2^2}, |\tau| = 1, |\omega| \leq 1\}$$

and

$$\mathcal{F}_{ND} = \{x = (x_1, \dots, x_n) \mapsto \frac{x_1^2}{2-a^2} + \frac{2\sqrt{1-a^2}x_2}{2-a^2} : a \in [0,1)\}.$$

In [27, Theorem 2], Kosiński and Zwonek proved the following result: If the three-point Pick interpolation problem is extremal, then, upto a composition with automorphisms of $\mathbb{B}^n$ and $\mathbb{D}$, it is interpolated by a function belonging to $\mathcal{F}_3(\mathbb{B}^n)$.

The result of Kosiński and Zwonek, together with Corollary 10.3, yields a concrete representation of at least one solution to a given three-point interpolation problem:

Theorem 12.1. *Suppose $\mathcal{Z} = \{z_1, z_2, z_3\} \subset \mathbb{B}^n$ and $\mathcal{W} = \{w_1, w_2, w_3\} \subset \mathbb{D}$ are interpolation data, with $\mathcal{W} \neq \{0\}$. If $\mathcal{Z}$ and $\mathcal{W}$ solve the interpolation problem, then there exist*

$$\varphi \in \mathcal{F}_3(\mathbb{B}^n),$$

and $\lambda \in (0,1]$ such that

$$\lambda\varphi,$$

solve the interpolation problem for $\mathcal{Z}$ and $\mathcal{W}$. Moreover, λ can be chosen to satisfy the minimality condition

$$\lambda = \inf\{\|\eta\|_\infty : \eta \in \mathcal{S}(\mathbb{B}^n) \text{ interpolates } \mathcal{Z} \text{ and } \mathcal{W}\}.$$

The final part of the above result follows the same lines of proof of Corollary 10.3. Now we consider a definite 3-point interpolation problem.

Example 12.2. Let $n > 1$. Denote by $\{e_j\}_{j=1}^n$ the standard orthonormal basis of $\mathbb{C}^n$, and define

$$\lambda = \sqrt{1 - \left(\frac{3}{8}\right)^{\frac{1}{n}}},$$

and

$$M = \max_{1 \leq j \leq 3} \|K_n(\cdot, z_j)\|_\infty.$$

Note that $M > 1$. For each $j = 1, 2, 3$, define

$$z_j = \lambda e_j.$$

Finally, define

$$\begin{cases} w_1 = \frac{1}{2M}, \\ w_2 = \frac{1}{3M}, \\ w_3 = \frac{1}{4M}. \end{cases}$$

We now consider the three-point interpolation problem corresponding to the data $\mathcal{Z} = \{z_1, z_2, z_3\}$ and $\mathcal{W} = \{w_1, w_2, w_3\}$. As in Theorem 7.2, we let

$$\psi_{\mathcal{Z},\mathcal{W}} = c_1 K_n(\cdot, z_1) + c_2 K_n(\cdot, z_2) + c_3 K_n(\cdot, z_3),$$

where the scalars c_1, c_2, and c_3 are given by

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} K_n(z_i, z_j) \end{bmatrix}_{3\times 3}^{-1} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

Since $\langle z_j, z_j\rangle = \lambda^2 \delta_{ij}$, and $K_n(z, w) = \frac{1}{(1-\langle z,w\rangle)^n}$ for all $z, w \in \mathbb{B}^n$, it follows that

$$K_n(z_t, z_t) = \frac{1}{(1 - \langle z_t, z_t\rangle)^n} = \frac{1}{(1-\lambda^2)^n},$$

for all $t = 1, 2, 3$, and

$$K_n(z_i, z_j) = 1,$$

for all $i \neq j$. Therefore,

$$[K_n(z_i, z_j)]_{3\times 3} = \begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix},$$

where

$$a = \frac{1}{(1-\lambda^2)^n}.$$

Computing the inverse of the matrix $[K_n(z_i, z_j)]_{3\times 3}$, we obtain

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{1}{(a-1)(a+2)} \begin{bmatrix} (a+1)w_1 - w_2 - w_3 \\ -w_1 + w_2(a+1) - w_3 \\ -w_1 - w_2 + (a+1)w_3 \end{bmatrix}.$$

This further implies that $c_1, c_2, c_3 > 0$. Now

$$\begin{aligned} \|\psi_{\mathcal{Z},\mathcal{W}}\|_\infty &\leq c_1 \|K_n(\cdot, z_1)\|_\infty + c_2 \|K_n(\cdot, z_2)\|_\infty + c_3 \|K_n(\cdot, z_3)\|_\infty \\ &\leq M(c_1 + c_2 + c_3) \\ &\leq \|\psi\|_\infty \leq M \frac{1}{(a+2)}(w_1 + w_2 + w_3) \\ &= \frac{13}{12(a+2)} \\ &< 1, \end{aligned}$$

that is, $\|\psi_{\mathcal{Z},\mathcal{W}}\|_\infty < 1$. By Theorem 7.2, the interpolation problem has a solution.

Remark 12.3. It is easy to see that the function ψ in the above example itself interpolates the given data $\mathcal{Z}$ and $\mathcal{W}$.

13. Concluding remarks

We begin with some general remarks. Approximation, interpolation and lifting are central themes in the theory of Hilbert function spaces and are closely connected to several applied areas of mathematics, including electrical engineering (cf. Helton [23, 23]). Results in these directions contribute to the understanding of a wide range of problems, such as the structure of submodules and quotient modules; the theory of linear operators; complex variables; and techniques involving reproducing kernel Hilbert spaces, including recent developments in machine learning. It is also important to note that for $n = 1$, Schur functions admit useful fractional linear-type representations, also known as realization formulas [3]. However, various complications arise when attempting to represent Schur functions in higher dimensions. Another fundamental problem concerns the structure of submodules and quotient modules of $H^2(\mathbb{B}^n)$. We hope that the results presented in this paper will contribute to the further development and understanding of these areas.

In the passage from one-variable to multivariable Hilbert function spaces, the two most natural domains are the polydisc and the unit ball. In a recent paper, [13] resolved the lifting and interpolation problems on $\mathbb{D}^n$. Some of the qualitative and quantitative features established there also arise in our study of $\mathbb{B}^n$. With this comparison in mind, we make the following general observation:

(1) The construction of the function $\psi_{\mathcal{Z},\mathcal{W}}$, as well as the qualitative and quantitative criteria established in this paper, was also obtained in the polydisc setting in [13]. In the present work, we prove these implications using a combination of techniques; some new and others carefully adapted, with significant modifications, from $\mathbb{D}^n$ setting. A fundamental difference in the ball setting is the absence of a convenient orthonormal basis for $L^2(\mathbb{S}^n)$; in contrast, for $L^2(\mathbb{T}^n)$, the monomials form an orthonormal basis. This feature played a central role in the interpolation and lifting results for $\mathbb{D}^n$ in [13], and its absence highlights a key technical challenge in the present context.
(2) While working on $\mathbb{B}^n$ (as well as on $\mathbb{D}^n$ [13]), we apply substantially finer techniques that allow us to avoid many of the subtleties inherent in several-variable function theory. On the one hand, such refined tools are essential for eliminating case-by-case analyses and obtaining a general result. On the other hand, the results also suggest that, when focusing on specific settings, such as $\mathbb{B}^n$ or $\mathbb{D}^n$, there remains significant scope for further sharpening and improving both the results presented here and those in the earlier work [13].
(3) While working on $\mathbb{B}^n$, we found that results concerning Schur functions and Hilbert function spaces are often considerably more involved than in other standard domains, such as $\mathbb{D}^n$. This may be partly due to the lack of explicit examples of inner functions on the ball. Indeed, as pointed out earlier, many of the results, examples, and counterexamples in this paper rely crucially on the structure of inner functions on the ball.

Before closing, we outline some simple yet useful general observations. First, we note that every module map on a finite-dimensional quotient module admits a lift to a function in $H^\infty(\mathbb{B}^n)$. The crucial point underlying the proof is that these quotient modules are naturally realized inside the ball algebra.

Proposition 13.1. *Let $\mathcal{Q}$ be a finite-dimensional quotient module of $H^2(\mathbb{B}^n)$, and let $X \in \mathcal{B}(\mathcal{Q})$ be a module map. Then there exists $\varphi \in H^\infty(\mathbb{B}^n)$ such that*

$$X = S_\varphi.$$

Moreover, φ can be chosen as

$$\varphi = X(P_{\mathcal{Q}}1),$$

and therefore,

$$X = S_{X(P_{\mathcal{Q}}1)}.$$

Proof. As $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ is finite-dimensional, it follows from (9.1) (or see [19, Corollary 3.4] and [11, Chapter 2, Remark 2.5.5]) that

$$\mathcal{Q} \subset A(\mathbb{B}^n) \subset H^\infty(\mathbb{B}^n).$$

In particular, we have

$$\psi := X(P_{\mathcal{Q}}1) \in \mathcal{Q} \subseteq A(\mathbb{B}^n),$$

that is, $\psi \in H^\infty(\mathbb{B}^n)$. If we write 0 as

$$\psi - \psi = 0 \in \mathcal{Q}^\perp,$$

then Theorem 3.1 implies that $X = S_\psi$ and completes the proof of the lemma. □

Throughout the paper, the space of mixed functions $\mathcal{M}(\mathbb{S}^n)$ has played an important role. We now discuss a decomposition of this space that aligns with the homogeneous decomposition studied by Rudin [43]. Recall that

$$\mathcal{M}(\mathbb{S}^n) = L^2(\mathbb{S}^n) \ominus [H^2(\mathbb{S}^n) \dot{+} H^2(\mathbb{S}^n)^{conj}].$$

For each $p, q \in \mathbb{Z}_+$, define $H(p,q)$ as the vector space of all harmonic homogeneous polynomials on $\mathbb{C}^n$ that have total degree p in the variables $z_1, \ldots, z_n$, and total degree q in the variables $\overline{z_1}, \ldots, \overline{z_n}$. This notion is related to spherical harmonic functions, as discussed in [43, Section 12.1]. In view of the identification

$$f \longleftrightarrow f|_{\mathbb{S}^n},$$

one can represent $H(p,q)$ as a vector subspaces of $L^2(\mathbb{S}^n)$ for all $p, q \in \mathbb{Z}_+$. Moreover, by [43, Theorem 12.2.3], it follows that $L^2(\mathbb{S}^n)$ is the direct sum of the pairwise orthogonal spaces $H(p,q)$, $p, q \in \mathbb{Z}_+$:

$$L^2(\mathbb{S}^n) = \bigoplus_{p,q \in \mathbb{Z}_+} H(p,q).$$

Observe $H(p,0)$ consists of holomorphic homogeneous polynomials of degree p for all $p \in \mathbb{Z}_+$. Moreover

$$H^2(\mathbb{S}^n) = \bigoplus_{p \in \mathbb{Z}_+} H(p,0)$$

Now, by taking conjugates, it follows that

$$H^2(\mathbb{S}^n)^{conj} = \bigoplus_{q \in \mathbb{Z}_+} H(0,q)$$

Using these direct sum decompositions, along with that of $L^2(\mathbb{S}^n)$, we finally conclude that

$$\mathcal{M}(\mathbb{S}^n) = \bigoplus_{p,q \geq 1} H(p,q)$$

Remark 13.2. The above description of $\mathcal{M}(\mathbb{S}^n)$ can be used to provide a slightly different proof of Theorem 4.3. However, the proof presented there also has the potential to work in more general cases, such as weighted Bergman spaces.

Alongside inner functions, extremal functions constitute one of the central themes of this paper. We conclude by emphasizing some compelling reasons why the class of extremal functions deserves special attention. Recall that $\mathcal{E}(\mathbb{B}^n)$ denotes the class of extremal functions.

(1) Unlike the one-variable setting, there is no concrete description or example of inner functions on $\mathbb{B}^n$, and in particular there is no satisfactory analogue or example of finite Blaschke products whenever $n > 1$.
(2) The class $\mathcal{E}(\mathbb{B}^n)$ contains many explicit examples that can be constructed via extremal interpolation problems (cf. [27, Theorem 2]).
(3) The Carathéodory approximation theorem and Pick's theorem on $\mathbb{B}^n$, established in this paper, along with the striking parallels between inner and extremal functions also outlined here, suggest that extremal functions occupy a central place in the theory of Hilbert function spaces.

These observations suggest that extremal functions merit a systematic investigation. One fundamental question is to understand their structure and obtain explicit representations, at least for rational extremal functions. An equally intriguing question is whether there exist non-rational extremal functions on $\mathbb{B}^n$ when $n > 1$.

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