COMMUTANT LIFTING AND INTERPOLATION ON THE UNIT BALL

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Abstract. We solve the commutant lifting and interpolation problems in the setting of the Hardy space and Schur functions on the open unit ball of $\mathbb{C}^n$. Our solutions also signify the role of inner functions on the unit ball, objects whose existence was once in doubt and to which Aleksandrov, Rudin, and others made fundamental contributions. Some of our results, particularly those related to inner functions, are new even in the classical, one-variable case.

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1. Introduction

Denote by $\mathbb{B}^n$ the open unit ball in $\mathbb{C}^n$, and let $H^\infty(\mathbb{B}^n)$ denote the space of all scalar-valued bounded analytic functions on $\mathbb{B}^n$. Equipped with the supremum norm $\|\cdot\|_\infty$, this forms a commutative Banach algebra. The closed unit ball of $H^\infty(\mathbb{B}^n)$ is denoted by $\mathcal{S}(\mathbb{B}^n)$, that is,

$$\mathcal{S}(\mathbb{B}^n) = \{\varphi \in H^\infty(\mathbb{B}^n) : \|\varphi\|_\infty \leq 1\}.$$

The elements of $\mathcal{S}(\mathbb{B}^n)$ are called *Schur functions*. For $n = 1$, Schur functions admit fractional linear-type representations, also known as realization formulas [1]. However, and this is relevant to the content of this paper, various complications arise when one attempts to represent Schur functions in higher variables.

The following interpolation problem is classical and is referred to as the *Nevanlinna–Pick interpolation problem*:

Problem 1.1. Given m distinct points $\{z_i\} \subseteq \mathbb{B}^n$ and a set of m values $\{w_i\}_{i=1}^m$ in $\mathbb{D}$, determine when there exists $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that

$$\varphi(z_i) = w_i,$$

for all $i = 1, \ldots, m$.

2010 *Mathematics Subject Classification.* 30E05, 46J15, 47A57, 32A40, 32A65, 15B05, 30H10, 30J15, 42B15.

Key words and phrases. Commutant lifting, Hardy space, unit ball, analytic functions, inner functions, perturbations, interpolation, L^p-spaces.

When $n = 1$ (in which case we write $\mathbb{B}^1 = \mathbb{D}$), the above interpolation problem was solved independently by Nevanlinna [14] and Pick [16] in the 1910s, and the resulting criterion is now known as the Nevanlinna–Pick interpolation theorem on the unit disc: There exists $\varphi \in \mathcal{S}(\mathbb{D})$ such that $\varphi(z_i) = w_i$ for all $i = 1, \ldots, m$, if and only if the $m \times m$ *Pick matrix*

$$\left[\frac{1 - z_i \bar{z}_j}{1 - w_i \bar{w}_j}\right]_{m \times m},$$

is positive semi-definite.

The Nevanlinna–Pick interpolation theorem, along with the problem of its multivariable counterparts, has long occupied a central place in the study of Hilbert function spaces. In fact, obtaining a criterion for interpolation in several variables has remained an open problem for many years, and it has only recently been resolved in the case of the polydisc [7]. It is important to emphasize that a characterization based on the positivity of a Pick matrix is generally not expected in higher dimensions, as such criteria are intrinsically tied to the theory of complete Nevanlinna–Pick kernels. The kernel of primary importance on $\mathbb{B}^n$, namely the Szegö kernel, does not belong to this class except for the case of $n = 1$. Recall that the Szegö kernel on $\mathbb{B}^n$ is the reproducing kernel $\mathbb{S}_n : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$ defined by

$$\mathbb{S}_n(z, w) = \frac{1}{(1 - \langle z, w \rangle)^n},$$

where $\langle z, w \rangle = \sum_{i=1}^n z_i \bar{w}_i$ for all $z, w \in \mathbb{B}^n$. Also, for each $w \in \mathbb{B}^n$, define the *Szegö kernel function* $\mathbb{S}_n(\cdot, w) : \mathbb{B}^n \to \mathbb{C}$ by

$$(\mathbb{S}_n(\cdot, w))(z) = \mathbb{S}_n(z, w),$$

for all $z \in \mathbb{B}^n$.

The aim of this paper is to provide a solution to the interpolation problem on $\mathbb{B}^n$. Following in the footsteps of Sarason [20], our approach first establishes a commutant lifting theorem for $H^2(\mathbb{B}^n)$, the Hardy space over $\mathbb{B}^n$, and then uses this result to solve the interpolation problem. It should be noted that the commutant lifting theorem for $H^2(\mathbb{B}^n)$, in the case $n > 1$, is itself an important and challenging problem. Thus, in this paper we resolve two problems of independent interest in function theory and the theory of function Hilbert spaces.

One notable aspect of the solutions obtained in this paper is their connection to inner functions on $\mathbb{B}^n$, functions whose very existence was once in doubt for $n > 1$. A function $\varphi \in H^\infty(\mathbb{B}^n)$ is said to be *inner* if

$$|\varphi(\zeta)| = 1,$$

for almost every $\zeta \in \mathbb{S}^n$ (in the sense of K-limits; see Section 3). Clearly, if φ is inner, then

$$\varphi \in \mathcal{S}(\mathbb{B}^n).$$

It was only in 1982 that Aleksandrov [3] constructed highly pathological examples of inner functions on $\mathbb{B}^n$, $n > 1$. In the following year, Rudin [17] observed that, despite their pathological nature, such functions exist in enormous abundance. Our results offer a concrete link to these observations of Aleksandrov and Rudin, a connection that we now elaborate on before proceeding further.

Given m distinct points $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$ and a set of m values $\{w_i\}_{i=1}^m \subseteq \mathbb{D}$, define

$$\psi_{\mathcal{Z},\mathcal{W}} = \sum_{j=1}^m c_j \mathbb{S}_n(\cdot, z_j).$$

where the scalars $\{c_j\}_{j=1}^m$ are given by

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} \mathbb{S}_n(z_1,z_1) & \mathbb{S}_n(z_1,z_2) & \dots & \mathbb{S}_n(z_1,z_m) \\ \mathbb{S}_n(z_2,z_1) & \mathbb{S}_n(z_2,z_2) & \dots & \mathbb{S}_n(z_2,z_m) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{S}_n(z_m,z_1) & \mathbb{S}_n(z_m,z_2) & \dots & \mathbb{S}_n(z_m,z_m) \end{bmatrix}^{-1} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}.$$

Note that the inverse of the above matrix is well defined, since the matrix in question is a Gram matrix. Moreover, observe that

$$\psi_{\mathcal{Z},\mathcal{W}} \in H^\infty(\mathbb{B}^n).$$

In fact, it is in the ball algebra $A(\mathbb{B}^n)$ (to be defined later). We have the following as one of the solutions to the interpolation problem on the ball (see Theorem 6.4):

Theorem 1.2. *There exists $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that*

$$\varphi(z_j) = w_j,$$

for all $j = 1, \dots, m$, if and only if there exists a sequence of inner functions $\{u_i\} \subseteq \mathcal{S}(\mathbb{B}^n)$ such that

$$\lim_i u_i(z_j) = \psi_{\mathcal{Z},\mathcal{W}}(z_j),$$

for all $j = 1, \dots, m$.

In addition to the above, we present several other characterizations of the interpolation problem, which are collected in Theorem 1.5. The significance of the preceding result lies in the way it highlights the inherent complexity of the interpolation problem and its deep connection to the subtleties of function theory on $\mathbb{B}^n$, $n > 1$.

We now proceed to explain the remaining results, beginning with commutant lifting on $H^2(\mathbb{B}^m)$, the Hardy space over $\mathbb{B}^m$, which plays a key role in the solution of the interpolation problem. Recall that $H^2(\mathbb{B}^n)$ is the reproducing kernel Hilbert space corresponding to the kernel function $\mathbb{S}_n$. It is also known that $H^2(\mathbb{B}^n)$ consists of all analytic functions $f : \mathbb{B}^n \to \mathbb{C}$ satisfying

$$\|f\| := \Big(\sup_{0<r<1} \int_{\mathbb{S}^n} |f(r\zeta)|^2 \, d\sigma(\zeta) \Big)^{\frac{1}{2}},$$

where $d\sigma$ denotes the normalized surface (Lebesgue) measure on the sphere $\mathbb{S}^n = \partial \mathbb{B}^n$. Given a function $\varphi : \mathbb{B}^n \to \mathbb{C}$, we have the property that $\varphi H^2(\mathbb{B}^n) \subseteq H^2(\mathbb{B}^n)$ (in which case such a φ is called a *multiplier*) if and only if

$$\varphi \in H^\infty(\mathbb{B}^n).$$

Such a multiplier φ defines a multiplication operator T_φ on $H^2(\mathbb{B}^n)$ by

$$T_\varphi f = \varphi f,$$

for all $f \in H^2(\mathbb{B}^n)$. Moreover, we have the norm identity $\|T_\varphi\| = \|\varphi\|_\infty$. The multiplication operator T_φ is also known as the *analytic Toeplitz operator* with symbol φ. The coordinate functions $\varphi = z_i$ give rise to the special multiplication or Toeplitz operators T_{z_i}, $i = 1, \dots, n$. Moreover, we have the fundamental commutation property: the commutant of the coordinate multipliers is precisely the algebra $H^\infty(\mathbb{B}^n)$:

$$\{T_{z_1}, \dots, T_{z_n}\}' = \{T_\varphi : \varphi \in H^\infty(\mathbb{B}^n)\}.$$

We also need to consider backward shift invariant subspaces (known as quotient modules) of $H^2(\mathbb{B}^n)$. A closed subspace $\mathcal{Q}$ of $H^2(\mathbb{B}^n)$ is called a *quotient module* if

$$T_{z_i}^* \mathcal{Q} \subseteq \mathcal{Q},$$

for all $i = 1, \dots, n$ [5, 6, 10]. For each $\varphi \in H^\infty(\mathbb{B}^n)$, we define the *module map* S_φ on $\mathcal{Q}$ by

$$S_\varphi = P_{\mathcal{Q}} T_\varphi|_{\mathcal{Q}},$$

where $P_{\mathcal{Q}}$ denotes the orthogonal projection of $H^2(\mathbb{B}^n)$ onto $\mathcal{Q}$ [6]. We say that an operator $X \in \mathcal{B}(\mathcal{Q})$ is a *module map* if

$$X S_{z_i} = S_{z_i} X,$$

for all $i = 1, \dots, n$. In other words, a module map is a bounded linear operator on a quotient module that commutes with the coordinate multiplication operators compressed to the quotient module. Given a Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(\mathcal{H})$ the space of all bounded linear operators on $\mathcal{H}$, and we set

$$\mathcal{B}_1(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \|T\| \leq 1\}.$$

With this terminology and structure in place, we can now formulate the commutant lifting problem as follows:

Problem 1.3. Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{B}^n)$, and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. When does there exist a Schur function $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that

$$X = S_\varphi?$$

If such an operator $X \in \mathcal{B}_1(\mathcal{Q})$ admits a representation $X = S_\varphi$ for some $\varphi \in \mathcal{S}(\mathbb{B}^n)$, then we say that X admits a *lift*, or that X admits a *lift to* T_φ, or simply that X is *liftable*.

Equivalently, this is a question about the commutativity of the following diagram, together with the additional requirement that $\varphi \in \mathcal{S}(\mathbb{B}^n)$:

$$\begin{tikzcd}
H^2(\mathbb{B}^n) \arrow[r, "T_\varphi"] & H^2(\mathbb{B}^n) \arrow[d, "i_{\mathcal{Q}}^*"] \\
\mathcal{Q} \arrow[u, "i_{\mathcal{Q}}"] \arrow[r, "X"'] & \mathcal{Q}
\end{tikzcd}$$

where $i_{\mathcal{Q}} : \mathcal{Q} \longrightarrow H^2(\mathbb{B}^n)$ denotes the inclusion map (here we note that $S_\varphi = i_{\mathcal{Q}}^* T_\varphi i_{\mathcal{Q}}$).

We now focus on the case where $n = 1$. In this particular situation, Sarason's seminal work asserts that every such module map $X \in \mathcal{B}_1(\mathcal{Q})$ admits a lift [20]. And even more, the lift preserves the sup norm: there exists $\varphi \in \mathcal{S}(\mathbb{D})$ such that $X = S_\varphi$ and

$$\|X\| = \|\varphi\|_\infty.$$

Sarason further applied his lifting theorem to kernel function-based finite-dimensional quotient modules, thereby recovering the positivity condition of the Pick matrix in the context of Nevanlinna–Pick interpolation problem. Now we turn to several variables. In what follows, unless otherwise stated, the case of $n = 1$ will also be included in the discussion.

First we establish the essential tools that would be needed for the lifting theorem. For a function $f \in L^2(\mathbb{S}^n)$, we denote its conjugate by $\overline{f}$. For a subset S of $L^2(\mathbb{S}^n)$, we write S^{conj} for the collection of all conjugate functions of elements of S; that is,

$$S^{conj} = \{\overline{f} : f \in S\}.$$

Because K-limits naturally allow boundary identification, we will, whenever needed, regard $H^2(\mathbb{B}^n)$ as canonically identified with $H^2(\S^n)$ without additional explanation (see Section 3 for further details), where

$$H^2(\mathbb{S}^n) = \overline{\mathbb{C}[z_1, \dots, z_n]}^{L^2(\mathbb{S}^n)}.$$

Next, we define the space of "mixed functions" (also see [7]):

$$\mathcal{M}(\mathbb{S}^n) := L^2(\mathbb{S}^n) \ominus [H^2(\mathbb{S}^n) + H^2(\mathbb{S}^n)^{conj}].$$

Note that $\mathcal{M}(\mathbb{S}) = \{0\}$. Define the space of Hardy functions "vanishing at 0" as

$$H_0^2(\mathbb{S}^n) = \{f \in H^2(\mathbb{S}^n) : \langle f, 1\rangle_{H^2(\mathbb{S}^n)} = 0\}.$$

Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{B}^n)$. We consider $\mathcal{Q}^{conj}$, $\mathcal{M}(\mathbb{S}^n)$, and $H_0^2(\mathbb{S}^n)$ as subspaces of $L^1(\mathbb{S}^n)$ and define

$$\mathcal{M}_{\mathcal{Q}} = \mathcal{Q}^{conj} \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H_0^2(\mathbb{S}^n)].$$

where $\dot{+}$ denotes the skew sum in the Banach space $L^1(\mathbb{S}^n)$. This is in line with the corresponding constructions in the polydisc case, as introduced in [7].

Another essential component is the notion of inner functions on $\mathbb{B}^n$. Note that for a long time, it was widely conjectured that no nonconstant inner functions exist on $\mathbb{B}^n$, $n > 1$. This belief persisted until 1981, when Aleksandrov constructed highly pathological examples of such functions; shortly thereafter, Löw [12] independently established their existence (also see Hakim and Shibony [11]; and see Rudin's historical comments in [17]). In what follows, we also highlight the subtle nature of inner functions on the unit ball and their connection to the lifting problem. More specifically, motivated by Rudin's result on the w^*-density of inner functions in $\mathcal{S}(\mathbb{B}^n)$ (see Theorem 5.1 or [17, Theorem 5.3(b)]), we introduce the following notion: Let $\mathcal{S} \subseteq L^1(\mathbb{S}^n)$, $\varphi \in H^2(\mathbb{B}^n)$, and let $\{u_i\} \subseteq \mathcal{S}(\mathbb{B}^n)$ be a sequence of inner functions. We say that

$$w^* - \lim_i u_i = \varphi \text{ on } \mathcal{S},$$

if for every $f \in \mathcal{S}$, we haave

$$\lim_i \int_{\mathbb{S}^n} u_i f \, d\sigma = \int_{\mathbb{S}^n} \psi f \, d\sigma.$$

The following is a summary of all the commutant lifting theorems obtained in this paper. It is important to note that these (and all others except Theorem 7.5) apply also to the case $n = 1$, and hence many of them are new even in the single-variable setting.

Theorem 1.4. *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{B}^n)$, and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. Set*

$$\psi = X(P_{\mathcal{Q}} 1).$$

The following conditions are equivalent:

(1) (Commutant lifting) X admits a lift.

(2) *(Perturbations) There exists $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that*

$$\psi - \varphi \in \mathcal{Q}^{\perp}.$$

Moreover, in this case X lifts to S_φ.

(3) *(Qualitative property) $X_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|.\|_1) \longrightarrow \mathbb{C}$ is a contraction, where*

$$X_{\mathcal{Q}} f = \int_{\mathbb{S}^n} \psi f \, d\sigma,$$

for all $f \in \mathcal{M}_{\mathcal{Q}}$.

(4) *(Quantitative property) $dist_{L^1(\mathbb{S}^n)}\left(\frac{\overline{\psi}}{\|\psi\|_2^2}, \tilde{\mathcal{M}_{\mathcal{Q}}}\right) \geq 1$, where*

$$\tilde{\mathcal{M}_{\mathcal{Q}}} =: \ker X_{\mathcal{Q}} = [\mathcal{Q}^{conj} \ominus \{\overline{\psi}\}] \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H_0^2(\mathbb{S}^n)].$$

(5) *(Inner functions - I) There exists a sequence of inner functions $\{u_i\} \subseteq \mathcal{S}(\mathbb{B}^n)$ such that*

$$w^* - \lim_i u_i = \psi \text{ on } \mathcal{M}_{\mathcal{Q}}.$$

(6) *(Inner functions - II) There exists a sequence of inner functions $\{u_i\} \subseteq \mathcal{S}(\mathbb{B}^n)$ such that*

$$w^* - \lim_i u_i = \psi \text{ on } \mathcal{Q}^{conj}.$$

In the context of the above theorem, we further point out that Lemma 4.2 yields the following identity:

$$\ker X_{\mathcal{Q}} = \tilde{\mathcal{M}_{\mathcal{Q}}}.$$

The equivalence of (1) and (2) is less involved, yet it plays a central role in establishing the remaining equivalence conditions as well as several other results in this paper. The perturbation property in (2) is connected, perhaps coincidentally, with Rudin's perturbation theorem for inner functions, a connection that we will explore further in Section 7.

The equivalence of conditions (1), (3), and (4) was established earlier in the setting of the Hardy space over the unit polydisc [7]. In the present work, in the setting of the unit ball, we revisit these implications using a combination of techniques: some new, and some adapted, carefully and with significant modifications, from the polydisc setting. In fact, in the present case, condition (2) plays a crucial role in deriving the remaining implications. Overall, the techniques used both here and in [7] are substantially finer, enabling us to avoid many of the subtleties inherent in several-variable function theory. On the one hand, such refined tools are essential for eliminating case-by-case analyses and obtaining a general result. On the other hand, the theorem also indicates that, when one focuses on specific settings, either the polydisc or the unit ball, there remains considerable room for sharpening and improving the results presented here as well as those in the earlier work [7].

Now we turn to a detailed presentation of our solution to the interpolation problem:

Theorem 1.5. *Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$ be a set of m distinct points, and let $\mathcal{W} = \{w_i\}_{i=1}^m \subseteq \mathbb{D}$ be a set of m values. Define*

$$\psi_{\mathcal{Z},\mathcal{W}} = \sum_{j=1}^m c_j \mathbb{S}_n(\cdot, z_j),$$

where the scalars $\{c_1, \ldots, c_m\}$ *are given by*

$$\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \left[\mathbb{S}_n(z_i, z_j)\right]^{-1}_{m\times m} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}.$$

The following conditions are equivalent:

(1) *There exists* $\varphi \in \mathcal{S}(\mathbb{B}^n)$ *such that*
$$\varphi(z_i) = w_i,$$
for all $i = 1, \ldots, m$.

(2) $\mathcal{J} : (\mathcal{M}_{\mathcal{Q}_\mathcal{Z}}, \|\cdot\|_1) \to \mathbb{C}$ *is a contraction , where*
$$\mathcal{J}_{\mathcal{Z},\mathcal{W}} f = \int_{\mathbb{S}^n} \psi_{\mathcal{Z},\mathcal{W}} f \, d\sigma,$$
and
$$\mathcal{M}_{\mathcal{Q}_\mathcal{Z}} = \mathcal{Q}_\mathcal{Z}^{conj} \dot{+} [\mathcal{M}(\mathbb{S}^n) + H_0^2(\mathbb{S}^n)].$$

(3) $d_{L^1(\mathbb{S}^n)}\left(\frac{\overline{\psi_{\mathcal{Z},\mathcal{W}}}}{\|\psi_{\mathcal{Z},\mathcal{W}}\|_2^2}, \widetilde{\mathcal{M}_{\mathcal{Q}_\mathcal{Z}}}\right) \geq 1$, *where*
$$\widetilde{\mathcal{M}_{\mathcal{Q}_\mathcal{Z}}} = [\mathcal{Q}_\mathcal{Z}^{conj} \ominus \{\overline{\psi_{\mathcal{Z},\mathcal{W}}}\}] \dot{+} [\mathcal{M}_n \dot{+} H_0^2(\mathbb{S}^n)].$$

(4) *There exists a sequence of inner functions* $\{u_i\} \subseteq \mathcal{S}(\mathbb{B}^n)$ *such that*
$$\lim_i u_i(z_j) = \psi_{\mathcal{Z},\mathcal{W}}(z_j),$$
for all $j = 1, \ldots, m$.

Similar equivalence conditions (1)–(3) were previously observed in the polydisc setting in [7]. Condition (4), however, is new even in the classical one-variable case.

The remainder of the paper is organized as follows. In Section 2, we present characterizations of lifting in terms of perturbations of Schur functions and quotient modules. Section 3 provides characterizations of lifting based on the contractivity of certain linear functions. In the following section, we utilize these results to introduce a characterization using a distance function that bears some resemblance to the Nehari theorem. In Section 5, we relate lifting to inner functions and present some more characterizations of lifting via sequences of inner functions. All the lifting theorems obtained earlier are then applied in Section 6 to provide characterizations of interpolation for Schur functions. Finally, Section 7 is devoted to examples.

2. Characterizations by perturbations

Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{B}^n)$. In the context of Problem 1.3, observe that if $X := S_\varphi$ for some $\varphi \in \mathcal{S}(\mathbb{B}^n)$, then it is evident that $X \in \mathcal{B}_1(\mathcal{Q})$ and $XS_{z_i} = S_{z_i}X$ for all $i = 1, \ldots, n$. That is, X is a contractive module map on $\mathcal{Q}$. Moreover,
$$\|X\| = \|S_\varphi\| \leq \|\varphi\|_\infty.$$
Therefore, the commutant lifting problem is concerned with establishing the converse: whether a contractive module map on $\mathcal{Q}$ necessarily arises as a compressed multiplication operator by a Schur class function.

In this section, we present a solution to this problem that is technically less involved, yet it will play an important role in addressing the relatively more involved results that follow. At the same time, the solution provided here reveals a perturbation property that is of independent interest.

Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{B}^n)$. We give a special attention to the function $P_{\mathcal{Q}}1 \in \mathcal{Q}$. This function is of particular interest, as it is a joint cyclic vector in the sense that

$$\mathcal{Q} = \overline{\text{span}}\{S_z^k(P_{\mathcal{Q}}1) : k \in \mathbb{Z}_+^n\}. \tag{2.1}$$

To see this, for each $k \in \mathbb{Z}_+^n$, we compute

$$S_z^k P_{\mathcal{Q}}1 = P_{\mathcal{Q}} T_z^k|_{\mathcal{Q}} P_{\mathcal{Q}}1 = P_{\mathcal{Q}} T_z^k P_{\mathcal{Q}}1 = P_{\mathcal{Q}} T_z^k 1 = P_{\mathcal{Q}} T_z^k 1 = P_{\mathcal{Q}} z^k.$$

The claim now follows from the fact that

$$H^2(\mathbb{B}^n) = \overline{\text{span}}\{z^k : k \in \mathbb{Z}_+^n\}.$$

A closed subspace $\mathcal{S} \subseteq H^2(\mathbb{B}^n)$ is called a *submodule* if $\mathcal{S}^\perp$ is a quotient module of $H^2(\mathbb{B}^n)$. Equivalently, $\mathcal{S}$ is a joint invariant subspace:

$$z_i \mathcal{S} \subseteq \mathcal{S},$$

for all $i = 1, \ldots, n$. It is well known that the structure of submodules and quotient modules of $H^2(\mathbb{B}^n)$ is complicated, and unlike the case $n = 1$ (which is covered by Beurling's theorem and the classical Sz.-Nagy–Foiaș model theory [13]), no satisfactory classification or understanding is available. This complexity presents a major obstacle in understanding the several-variable commutant lifting theorem as well as the interpolation problem (and vice versa as well). Our first characterization of commutant lifting for contractive module maps relies on perturbations of certain natural functions and establishes a connection with submodules of $H^2(\mathbb{B}^n)$. We now briefly summarize the essential ingredients needed for the statement.

We now introduce a more general framework for the lifting theorem. We consider subalgebras of $H^\infty(\mathbb{B}^n)$ for lifting maps; more specifically, we consider the ball algebra, since this closed subalgebra of $H^\infty(\mathbb{B}^n)$ will be used in subsequent results. Recall that the ball algebra $A(\mathbb{B}^n)$ is the Banach algebra consisting of all analytic functions on $\mathbb{B}^n$ that admit a continuous extension to $\overline{\mathbb{B}^n}$. We set

$$\mathcal{A} = H^\infty(\mathbb{B}^n) \text{ or } A(\mathbb{B}^n).$$

In the definition of lifting,

Note that the statement that a contractive module map X acting on a quotient module admits a lift means, as usual, that there exists a Schur function $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that $X = S_\varphi$. If φ lies merely in $\mathcal{A}$ (and is not necessarily a Schur function), then we say that X admits a lift to $\mathcal{A}$ (in this case, X is not necessarily contractive).

Let $X \in \mathcal{B}(\mathcal{Q})$ be a module map. In what follows, we will always write (suppressing X, as it will be clear from the context)

$$\psi = X(P_{\mathcal{Q}}1).$$

This function

$$\psi \in \mathcal{Q},$$

will play a central role throughout the developments that follow. For instance, in the following, we show that perturbations of ψ by functions from $\mathcal{A}$ are a key to solving the commutant lifting problem.

Theorem 2.1. *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{B}^n)$, and let $X \in \mathcal{B}(\mathcal{Q})$ be a module map. Then X admits a lift to $\mathcal{A}$ if and only if there exists $\varphi \in \mathcal{A}$ such that*

$$\psi - \varphi \in \mathcal{Q}^\perp.$$

Moreover, in this case X lifts to S_φ.

Proof. Suppose X admits a lift. That is, there exists $\varphi \in \mathcal{A}$ such that

$$X = S_\varphi.$$

Then

$$\psi = XP_{\mathcal{Q}}1 = S_\varphi P_{\mathcal{Q}}1 = P_{\mathcal{Q}} T_\varphi P_{\mathcal{Q}}1.$$

Since $\mathcal{Q}^\perp$ is a submodule, it follows that

$$\varphi \mathcal{Q}^\perp \subseteq \mathcal{Q}^\perp,$$

and hence $P_{\mathcal{Q}} T_\varphi P_{\mathcal{Q}} = P_{\mathcal{Q}} T_\varphi$. This implies

$$\psi = P_{\mathcal{Q}} \varphi,$$

so we conclude that

$$\psi - \varphi = P_{\mathcal{Q}}\varphi - \varphi = -P_{\mathcal{Q}^\perp}\varphi \in \mathcal{Q}^\perp.$$

Conversely, assume that $\psi - \varphi \in \mathcal{Q}^\perp$. Since $\psi \in \mathcal{Q}$, we must have $P_{\mathcal{Q}}\psi = \psi$, and hence

$$\psi = P_{\mathcal{Q}}\varphi.$$

Pick $k \in \mathbb{Z}_+^n$. As X is a module map, $XS_{z_i} = S_{z_i}X$ for all $i = 1, \ldots, n$, and hence

$$XS_z^k = S_z^k X.$$

Therefore

$$XS_z^k P_{\mathcal{Q}}1 = S_{\boldsymbol{z}}^k XP_{\mathcal{Q}}1 = S_z^k \psi.$$

Moreover, we note that T_φ is an analytic Toeplitz operator. Therefore, $T_z^k T_\varphi = T_z^k T_\varphi$, which implies $S_z^k S_\varphi = S_z^k S_\varphi$, and hence

$$\begin{aligned} X(S_z^k P_{\mathcal{Q}}1) &= S_z^k \psi \\ &= S_z^k P_{\mathcal{Q}}\varphi \\ &= S_z^k \varphi \\ &= S_z^k S_\varphi P_{\mathcal{Q}}1 \\ &= S_\varphi (S_z^k P_{\mathcal{Q}}1). \end{aligned}$$

In view of (2.1), it follows that $X = S_\varphi$, which completes the proof of the theorem. ■

We now address the commutant lifting problem in its standard form. The following result holds, with a proof analogous to that of Theorem 2.1 (simply replace $\mathcal{A}$ with $\mathcal{S}(\mathbb{B}^n)$):

Theorem 2.2. *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{B}^n)$, and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. Then X admits a lift if and only if there exists $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that*

$$\psi - \varphi \in \mathcal{Q}^\perp.$$

Moreover, in this case X lifts to S_φ.

Define

$$\mathcal{SB}(\mathbb{B}^n) = \mathcal{S}(\mathbb{B}^n) \cap A(\mathbb{B}^n).$$

In the setting of the above theorem, if, in addition, the lifting symbol φ is also in the ball algebra (that is, if $\varphi \in \mathcal{SB}(\mathbb{B}^n)$), then we say that X admits a lift to $\mathcal{SB}(\mathbb{B}^n)$. Again, the following holds with a proof similar to that of Theorem 2.1:

Corollary 2.3. *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{B}^n)$, and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. Then X admits a lift to $\mathcal{SB}(\mathbb{B}^n)$ if and only if there exists $\varphi \in \mathcal{SB}(\mathbb{B}^n)$ such that*

$$\psi - \varphi \in \mathcal{Q}^{\perp}.$$

Moreover, in this case X lifts to S_φ.

We remark that the problem of classifying liftings in higher dimensions is a nontrivial and meaningful problem. As we will soon see (cf. Section 7), in contrast to the case $n = 1$, where every contractive module map admits a lift (by Sarason's theorem [20]), in higher dimensions there exist contractive module maps that may or may not admit liftings.

Before concluding this section, we note that the perturbation problem above is, perhaps coincidentally, connected with another perturbation phenomenon for inner functions studied by Rudin, a connection that we will discuss further in Section 7.

3. Qualitative characterizations

In this section, we go deeper into the structure of the Hardy space $H^2(\mathbb{B}^n)$. All notations and facts presented here are standard for $H^2(\mathbb{B}^n)$. For further details, we refer the reader to the classic [18].

We denote by $L^2(\mathbb{S}^n)$ the Hilbert space of all square-integrable functions on $\mathbb{S}^n$ with respect to σ. Let $\zeta \in \mathbb{S}^n$ and let $\alpha > 1$. In analogy with the notion of an angular region in one variable, define $\mathcal{D}_\alpha(\zeta)$ by (see Rudin [18, 5.4.1])

$$\mathcal{D}_\alpha(\zeta) = \Big\{ z \in \mathbb{B}^n : |1 - \langle z, \zeta \rangle| < \frac{\alpha}{2}(1 - \|z\|^2) \Big\}.$$

A function $f : \mathbb{B}^n \to \mathbb{C}$ is said to have $K-$limit c at $\zeta \in \mathbb{S}^n$ if, for every $\alpha > 1$,

$$\lim_{\substack{z \to \zeta \\ z \in \mathcal{D}_\alpha(\zeta)}} f(z) = c.$$

If this limit exists, we write (see [18, 5.4.6])

$$(K - \lim f)(\zeta) = c.$$

The notion of K-limits gives boundary values of the Hardy functions as follows (see [18, 5.6.6]): For each $f \in H^2(\mathbb{B}^n)$, the function $f^* : \mathbb{S}^n \to \mathbb{C}$ exists a.e. where

$$f^*(\zeta) = (K - \lim f)(\zeta). \tag{3.1}$$

Moreover, we have the following:

$$\lim_{r \to 1} \int_{\mathbb{S}^n} |f^* - f_r|^2 d\sigma = 0,$$

where, f_r, $0 < r < 1$, is defined by

$$f_r(\zeta) = f(r\zeta),$$

for $\zeta \in \mathbb{S}^n$ a.e. Define

$$H^2(\mathbb{S}^n) = \overline{\{f|_{\mathbb{S}^n} : f \in A(\mathbb{B}^n)\}}^{L^2(\mathbb{S}^n)},$$

and

$$H^\infty(\mathbb{S}^n) = H^2(\mathbb{S}^n) \cap L^\infty(\mathbb{S}^n).$$

By [18, 5.6.9], we know that $f \mapsto f^*$ is linear isometry from $H^2(\mathbb{B}^n)$ onto $H^2(\mathbb{S}^n)$. In view of this, we will often identify $H^2(\mathbb{B}^n)$ with $H^2(\mathbb{S}^n)$ via the unitary operator $U : H^2(\mathbb{B}^n) \to H^2(\mathbb{S}^n)$, where

$$Uf = f^*,$$

for all $f \in H^2(\mathbb{B}^n)$. Note that the set of polynomials

$$\{p^* : p \in \mathbb{C}[z_1, \dots z_n]\},$$

is dense in $H^2(\mathbb{S}^n)$. We also remark that if $K - \lim f$ exists for a function f, then

$$K - \lim \bar{f} = \overline{K - \lim f}.$$

Now we set up the notations needed for the second lifting theorem. Although some of these have already appeared in the introduction, we present them again here with the necessary elaboration. Recall that the space of mixed functions is defined by

$$\mathcal{M}(\mathbb{S}^n) = L^2(\mathbb{S}^n) \ominus [H^2(\mathbb{S}^n) + H^2(\mathbb{S}^n)^{conj}].$$

From the definition of $\mathcal{M}(\mathbb{S}^n)$ it is clear that $\mathcal{M}(\mathbb{S}^n)$ is self-adjoint, that is,

$$f \in \mathcal{M}(\mathbb{S}^n),$$

if and only if

$$\overline{f} \in \mathcal{M}(\mathbb{S}^n).$$

Moreover, we note that if $\mathcal{S}$ is a closed subspace of $L^2(\mathbb{S}^n)$ then $\mathcal{S}^{conj}$ is also a closed subspace of $L^2(\mathbb{S}^n)$. Recall that $H^2_0(\mathbb{S}^n)$ is the closed subspace of $H^2(\mathbb{S}^n)$ consisting of functions vanishing at 0, that is,

$$H^2_0(\mathbb{S}^n) = \{f \in H^2(\mathbb{S}^n) : f(0) = 0\}$$

Note that here we are using the identification of $H^2(\mathbb{B}^n)$ with $H^2(\mathbb{S}^n)$ via the $K - \lim$ of the Hardy functions (via the unitary operator U defined as above). More specifically, we representing $H^2_0(\mathbb{S}^n)$ as

$$H^2_0(\mathbb{S}^n) = \{f^* \in H^2(\mathbb{S}^n) : f(0) = 0, f \in H^2(\mathbb{B}^n)\}.$$

Equivalently, we have

$$H^2_0(\mathbb{S}^n) = \{f \in H^2(\mathbb{S}^n) : \langle f, 1\rangle_{H^2(\mathbb{S}^n)} = 0\}.$$

Similarly, if S is a subset of $H^2(\mathbb{S}^n)$, then we represent S^{conj} as

$$S^{conj} = \{\overline{f} : f \in S\}.$$

Fix a quotient module $\mathcal{Q}$ of $H^2(\mathbb{S}^n)$ and define

$$\mathcal{M}_{\mathcal{Q}} = \mathcal{Q}^{conj} \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H^2_0(\mathbb{S}^n)].$$

where $\dot{+}$ denotes the skew sum in the Banach space $L^1(\mathbb{S}^n)$. In other words, we consider $\mathcal{M}_{\mathcal{Q}}$ as a subspace of $L^1(\mathbb{S}^n)$:

$$(\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1) \subset L^1(\mathbb{S}^n).$$

Now if we consider $\mathcal{M}_{\mathcal{Q}}$ to be a subspace of the Hilbert space $L^2(\mathbb{S}^n)$, then the skew sum mentioned above becomes an orthogonal sum. We keep this observation as a lemma for future reference:

Lemma 3.1. *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{B}^n)$. Then $\mathcal{M}(\mathbb{S}^n)$, $H^2_0(\mathbb{S}^n)$, and $\mathcal{Q}^{conj}$ are pairwise orthogonal closed subspaces of $L^2(\mathbb{S}^n)$. Moreover,*

$$H^2_0(\mathbb{S}^n) \perp H^2(\mathbb{S}^n)^{conj}.$$

However, $\mathcal{M}_{\mathcal{Q}}$ need not be closed when considered as a subspace of $L^1(\mathbb{S}^n)$.

Remark 3.2. Using the identification of $H^2(\mathbb{B}^n)$ and $H^2(\mathbb{S}^n)$, we have

$$H^2(\mathbb{S}^n) + H^2(\mathbb{S}^n)^{conj} = H^2_0(\mathbb{S}^n) \oplus H^2(\mathbb{S}^n)^{conj},$$

which readily implies that $H^2(\mathbb{S}^n) + H^2(\mathbb{S}^n)^{conj}$ is a closed subspace.

We also recall the duality of the L^p-spaces in our present context. For each $\varphi \in L^\infty(\mathbb{S}^n)$, define $\chi_\varphi \in (L^1(\mathbb{S}^n))^*$ by

$$\chi_\varphi f = \int_{\mathbb{S}^n} f\varphi \, d\sigma,$$

for all $f \in L^\infty(\mathbb{S}^n)$. This yields the duality (an isometric and bijection map)

$$(L^1(\mathbb{S}^n))^* \cong L^\infty(\mathbb{S}^n). \tag{3.2}$$

This duality will play a key role in the next result, which we refer to as a quantitative lifting theorem:

Theorem 3.3. *Let $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ be a quotient module and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. Define $X_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|.\|_1) \to \mathbb{C}$ by*

$$X_{\mathcal{Q}} f = \int_{\mathbb{S}^n} \psi f d\sigma,$$

for all $f \in \mathcal{M}_{\mathcal{Q}}$, where

$$\psi = X(P_{\mathcal{Q}}1).$$

Then X admits a lift if and only if $X_{\mathcal{Q}}$ is a contraction on $(\mathcal{M}_{\mathcal{Q}}, \|.\|_1)$.

Proof. First, we assume that $X \in \mathcal{B}(\mathcal{Q})$ admits a lift. By Theorem 2.2, there exists a Schur function $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that

$$\psi - \varphi \in \mathcal{Q}^\perp.$$

Since φ is a Schur function, by taking K-limit, we obtain

$$\|\varphi\|_{H^\infty(\mathbb{S}^n)} \leq 1.$$

By the duality (3.2), we know that $\chi_\varphi \in (L^1(\mathbb{S}^n))^*$, where

$$\chi_\varphi f = \int_{\mathbb{S}^n} \varphi f \, d\sigma,$$

for all $f \in L^1(\mathbb{S}^n)$. We also know that

$$\|\chi_\varphi\| = \|\varphi\|_{H^\infty(\mathbb{S}^n)} \leq 1.$$

We claim that

$$\chi_\varphi|_{\mathcal{M}_{\mathcal{Q}}} = X_{\mathcal{Q}}. \tag{3.3}$$

To prove this, first we pick $f \in \mathcal{Q}^{conj}$. We have

$$\begin{aligned} X_{\mathcal{Q}} f &= \int_{\mathbb{S}^n} \psi f d\sigma \\ &= \int_{\mathbb{S}^n} \psi \overline{\overline{f}} d\sigma \\ &= \langle \psi, \overline{f} \rangle_{H^2(\mathbb{S}^n)} \\ &= \langle \psi, \overline{f} \rangle_{H^2(\mathbb{B}^n)}. \end{aligned}$$

Since $\psi - \varphi \in \mathcal{Q}^{\perp}$, it follows that $\langle \psi - \varphi, \overline{f} \rangle_{H^2(\mathbb{B}^n)} = 0$, and hence

$$\langle \psi, \overline{f} \rangle_{H^2(\mathbb{B}^n)} = \langle \varphi, \overline{f} \rangle_{H^2(\mathbb{B}^n)}.$$

This implies

$$\begin{aligned} X_{\mathcal{Q}} f &= \langle \varphi, \overline{f} \rangle_{H^2(\mathbb{B}^n)} \\ &= \langle \varphi, \overline{f} \rangle_{H^2(\mathbb{S}^n)} \\ &= \int_{\mathbb{S}^n} \varphi f d\sigma \\ &= \chi_{\varphi} f, \end{aligned}$$

that is, $X_{\mathcal{Q}}|_{\mathcal{Q}^{conj}} = \chi_{\varphi}|_{\mathcal{Q}^{conj}}$. For $f \in \mathcal{M}(\mathbb{S}^n)$, we have

$$X_{\mathcal{Q}} f = \int_{\mathbb{S}^n} \psi f d\sigma = \langle f, \overline{\psi} \rangle_{H^2(\mathbb{S}^n)} = 0 = \chi_{\varphi} f.$$

This shows that $X_{\mathcal{Q}}|_{\mathcal{M}(\mathbb{S}^n)} = \chi_{\varphi}|_{\mathcal{M}(\mathbb{S}^n)}$. Finally, if $f \in H^2(\mathbb{S}^n)_0$, then, as $\mathcal{Q}^{conj} \perp H^2_0(\mathbb{S}^n)$, we have

$$X_{\mathcal{Q}} f = \int_{\mathbb{S}^n} \psi f d\sigma = 0,$$

and on the other hand, as $\varphi \in H^2(\mathbb{S}^n)$ and $f \in H^2_0(\mathbb{S}^n)$, we have

$$\chi_{\varphi} f = \int_{\mathbb{S}^n} \varphi f d\sigma = 0 = X_{\mathcal{Q}} f.$$

Therefore, $X_{\mathcal{Q}}|_{H^2(\mathbb{S}^n)_0} = \chi_{\varphi}|_{H^2(\mathbb{S}^n)_0} = 0$, and proves the claim that $X_{\mathcal{Q}} = \chi_{\varphi}|_{\mathcal{M}_{\mathcal{Q}}}$. By using this, we have

$$\|X_{\mathcal{Q}}\| \leq \|\chi_{\varphi}\| = \|\varphi\|_{\infty} \leq 1,$$

which proves that $X_{\mathcal{Q}}$ is a contractive functional on $\mathcal{M}_{\mathcal{Q}}$.
For the converse direction, assume that $X_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|.\|_1) \longrightarrow \mathbb{C}$ is a contractive functional. By the Hahn-Banach extension theorem, there exists $\theta \in L^{\infty}(\mathbb{S}^n)$ such that

$$\chi_{\theta}|_{\mathcal{M}_{\mathcal{Q}}} = X_{\mathcal{Q}},$$

and

$$\|\theta\|_{\infty} = \|X_{\mathcal{Q}}\| \leq 1.$$

For $f \in \mathcal{Q}^{conj}$, as $\chi_{\theta} f = X_{\mathcal{Q}} f$, we have

$$\int_{\mathbb{S}^n} \theta f d\sigma = \int_{\mathbb{S}^n} \psi f d\sigma,$$

and hence

$$\int_{\mathbb{S}^n} (\theta - \psi) f d\sigma = 0. \tag{3.4}$$

We claim that $\theta \in H^2(\mathbb{S}^n)$. To this end, we first note that $\theta \in L^2(\mathbb{S}^n)$. Let $f \in \mathcal{M}(\mathbb{S}^n)$. As $\mathcal{M}(\mathbb{S}^n)$ is self-adjoint, we have $\overline{f} \in \mathcal{M}(\mathbb{S}^n)$, and hence

$$\begin{aligned}\langle \theta, f\rangle_{L^2(\mathbb{S}^n)} &= \int_{\mathbb{S}^n} \theta \overline{f} d\sigma \\ &= \chi_\theta \overline{f} \\ &= X_{\mathcal{Q}} \overline{f} \\ &= \int_{\mathbb{S}^n} \psi \overline{f} \\ &= \langle \psi, f\rangle_{L^2(\mathbb{S}^n)} \\ &= 0.\end{aligned}$$

This implies

$$\theta \in L^2(\mathbb{S}^n) \ominus \mathcal{M}(\mathbb{S}^n).$$

In view of Remark 3.2, we know that the subspace $H^2(\mathbb{S}^n) + H^2(\mathbb{S}^n)^{conj}$ is closed in $L^2(\mathbb{S}^n)$, which yields

$$\begin{aligned}L^2(\mathbb{S}^n) \ominus \mathcal{M}(\mathbb{S}^n) &= L^2(\mathbb{S}^n) \ominus \Big(L^2(\mathbb{S}^n) \ominus [H^2(\mathbb{S}^n) + H^2(\mathbb{S}^n)^{conj}]\Big) \\ &= H^2(\mathbb{S}^n) + H^2(\mathbb{S}^n)^{conj},\end{aligned}$$

and consequently

$$\theta \in H^2(\mathbb{S}^n) + H^2(\mathbb{S}^n)^{conj}.$$

There exists $\eta_1 \in H^2(\mathbb{B}^n)^{conj}$ and $\eta_2 \in H^2(\mathbb{B}^n)$ such that

$$\theta = \eta_1^* + \eta_2^*.$$

Define

$$\varphi_1 = \eta_1 - \eta_1(0),$$

and

$$\varphi_2 = \eta_2 + \eta_1(0).$$

Then $\varphi_1^* \in H^2_0(\mathbb{S}^n)^{conj}$ and $\varphi_2^* \in H^2(\mathbb{S}^n)$, and clearly

$$\theta = \varphi_1^* + \varphi_2^*.$$

As $\varphi_1^* \in H^2_0(\mathbb{S}^n)^{conj}$, we have $\overline{\varphi_1^*} \in H^2_0(\mathbb{S}^n)$, and hence

$$\begin{aligned}\langle \theta, \varphi_1^*\rangle_{L^2(\mathbb{S}^n)} &= \int_{\mathbb{S}^n} \theta \overline{\varphi_1^*} d\sigma \\ &= \chi_\theta(\overline{\varphi_1^*}) \\ &= X_{\mathcal{Q}}(\overline{\varphi_1^*}) \\ &= \int_{\mathbb{S}^n} \psi^* \overline{\varphi_1^*} d\sigma \\ &= \langle \overline{\varphi_1^*}, \overline{\psi^*}\rangle_{L^2(\mathbb{S}^n)} \\ &= 0,\end{aligned}$$

which implies

$$\langle \theta, \varphi_1^*\rangle_{L^2(\mathbb{S}^n)} = 0.$$

Now as $\theta = \varphi_1^* + \varphi_2^*$ and $\langle \varphi_1^*, \varphi_2^* \rangle_{L^2(\mathbb{S}^n)} = 0$, we have

$$0 = \langle \theta, \varphi_1^* \rangle_{L^2(\mathbb{S}^n)} = \langle \varphi_1^* + \varphi_2^*, \varphi_1^* \rangle_{L^2(\mathbb{S}^n)} = \|\varphi_1^*\|^2_{L^2(\mathbb{S}^n)},$$

that is, $\varphi_1^* = 0$, and hence

$$\theta = \varphi_2^* \in H^2(\mathbb{S}^n).$$

We also have $\theta \in L^\infty(\mathbb{S}^n)$, so that

$$\varphi_2^* \in L^\infty(\mathbb{S}^n) \cap H^2(\mathbb{S}^n).$$

As $\varphi_2 \in H^2(\mathbb{B}^n)$, we have

$$\varphi_2 = P[\varphi_2^*],$$

where P denotes the Poisson integral (see [18, page 41]. In other words, for any $z \in \mathbb{B}^n$, we have

$$\varphi_2(z) = P[\varphi_2^*](z) = \int_{\mathbb{S}^n} P(z, \zeta)\varphi_2^*(\zeta) d\sigma(\zeta),$$

where $P(z, \zeta)$ is the Poisson kernel:

$$P(z, \zeta) = \frac{(1 - \|z\|^2)^n}{|1 - \langle z, \zeta \rangle|^{2n}}.$$

for all $z \in \mathbb{B}^n$ and $\zeta \in \mathbb{S}^n$. For $z \in \mathbb{B}^n$, we compute

$$\begin{aligned} |\varphi_2(z)| &= \Big| \int_{\mathbb{S}^n} P(z, \zeta)\varphi_2^*(\zeta) d\sigma(\zeta) \Big| \\ &\leq \int_{\mathbb{S}^n} P(z, \zeta)|\varphi_2^*(\zeta)| d\sigma(\zeta) \\ &\leq \|\varphi_2^*\|_{L^\infty(\mathbb{S}^n)}, \end{aligned}$$

which proves that $\varphi_2 \in H^\infty(\mathbb{B}^n)$, and

$$\|\varphi_2\|_{H^\infty(\mathbb{B}^n)} \leq \|\varphi_2^*\|_{L^\infty(\mathbb{S}^n)} = \|\theta\|_{L^\infty(\mathbb{S}^n)} \leq 1.$$

We recall from (3.4) that

$$\int_{\mathbb{S}^n} (\theta - \psi) f d\sigma = 0,$$

for all $f \in \mathcal{Q}^{conj}$. Now $\theta = \varphi_2^*$, for $\varphi \in H^\infty(\mathbb{B}^n)$ implies

$$\varphi_2 - \psi \in \mathcal{Q}^\perp.$$

Finally, since $\|\varphi\|_\infty \leq 1$, by Theorem 2.2, we conclude that X admits a lift. This completes the proof of the theorem. ∎

The kernel space of the functional $X_{\mathcal{Q}}$ will play an important role in obtaining a quantitative characterization of possible liftings.

The proof of the above result differs from the approach used in the polydisc case. Specifically, we rely more heavily on the perturbation characterization of the lifting problem. Another point of departure is that the Fourier series techniques used in the polydisc case are not as effective here; instead, our argument relies primarily on the construction of the space $\mathcal{M}(\mathbb{S}^n)$.

4. Quantitative characterizations

We use all the characterizations obtained thus far in this paper to propose a quantitative characterization of lifting. Here, "quantitative" refers to a distance formula that bears a resemblance to the classical Nehari theorem.

Given a normed linear space X, a nonempty set $S \subset X$ and $x \in X$, the distance from x to S is defined by

$$\mathrm{dist}_X(x, S) = \inf\{\|x - y\| : y \in S\}.$$

The following lemma is elementary, and we do not claim it as a new observation.

Lemma 4.1. *Let X be normed linear space, f be a nonzero functional in X^*, and let $x \in X$. Then*

$$dist_X(x, \ker f) = \frac{|f(x)|}{\|f\|}.$$

Proof. Let $y \in \ker f$. Then

$$|f(x)| = |f(x - y)| \leq \|f\|\|x - y\|.$$

and hence

$$\frac{|f(x)|}{\|f\|} \leq \|x - y\|,$$

so that

$$\mathrm{dist}_X(x, \ker f) \geq \frac{|f(x)|}{\|f\|}.$$

For the reverse inequality, we recall that $\|f\| = \sup_{y \neq 0} \frac{|f(y)|}{\|y\|}$. Therefore, there exists a sequence $\{y_n\} \subseteq X$ such that

$$f(y_n) \neq 0,$$

for all n, and

$$\frac{|f y_n|}{\|y_n\|} \to \|f\|.$$

Since

$$x - \frac{f(x)}{f(y_n)} y_n \in \ker f,$$

we have

$$\begin{aligned}
\mathrm{dist}_X(x, \ker f) &\leq \left\| x - x - \frac{f(x)}{f(y_n)} y_n \right\| \\
&= \left\| \frac{f(x)}{f(y_n)} y_n \right\| \\
&= \frac{|f(x)|}{|f(y_n)|} \|y_n\| \\
&\longrightarrow \frac{|f(x)|}{\|f\|},
\end{aligned}$$

which implies $\mathrm{dist}_X(x, \ker f) \leq \frac{|f(x)|}{\|f\|}$. ∎

Given a quotient module $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ and a module map $X \in \mathcal{B}_1(\mathcal{Q})$, we have defined $\psi = X(P_{\mathcal{Q}}1)$, and

$$\mathcal{M}_{\mathcal{Q}} = \mathcal{Q}^{conj} \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H_0^2(\mathbb{S}^n)].$$

Under the given assumptions, also recall that $X_{\mathcal{Q}} : \mathcal{M}_{\mathcal{Q}} \to \mathbb{C}$ is defined by

$$X_{\mathcal{Q}} f = \int_{\mathbb{S}^n} \psi f \, d\sigma,$$

for all $f \in \mathcal{M}_{\mathcal{Q}}$. Define

$$\widetilde{\mathcal{M}_{\mathcal{Q}}} = [\mathcal{Q}^{conj} \ominus \{\overline{\psi}\}] \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H_0^2(\mathbb{S}^n)].$$

Lemma 4.2. $\ker X_{\mathcal{Q}} = \widetilde{\mathcal{M}_{\mathcal{Q}}}$.

Proof. We write $X_{\mathcal{Q}} f = \int_{\mathbb{S}^n} \psi f d\sigma$ as

$$X_{\mathcal{Q}} f = \langle f, \overline{\psi} \rangle_{L^2(\mathbb{S}^n)},$$

for all $f \in \mathcal{M}_{\mathcal{Q}}$, and consequently

$$\ker X_{\mathcal{Q}} = [\mathcal{Q}^{conj} \ominus \{\overline{\psi}\}] \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H_0^2(\mathbb{S}^n)],$$

which completes the proof of the lemma. ∎

In view of the above and the distance function obtained in Lemma 4.1, we have the following characterization of liftable operators on quotient modules:

Theorem 4.3. *Let $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ be a quotient module, and let $X \in \mathcal{B}_1(\mathcal{Q})$. Then X admits a lift if and only if*

$$dist_{L^1(\mathbb{S}^n)}\Big(\frac{\overline{\psi}}{\|\psi\|_2^2}, \widetilde{\mathcal{M}_{\mathcal{Q}}}\Big) \geq 1.$$

Proof. Lemma 4.2 yields

$$\ker X_{\mathcal{Q}} = \widetilde{\mathcal{M}_{\mathcal{Q}}}.$$

Thus

$$\mathrm{dist}_{L^1(\mathbb{S}^n)}\Big(\frac{\overline{\psi}}{\|\psi\|_2^2}, \widetilde{\mathcal{M}_{\mathcal{Q}}}\Big) = \mathrm{dist}_{L^1(\mathbb{S}^n)}\Big(\frac{\overline{\psi}}{\|\psi\|_2^2}, \ker X_{\mathcal{Q}}\Big).$$

Observe that

$$X_{\mathcal{Q}}\Big(\frac{\overline{\psi}}{\|\psi\|_2^2}\Big) = \int_{\mathbb{S}^n} \psi \frac{\overline{\psi}}{\|\psi\|_2^2} d\sigma = 1,$$

We then have, by Lemma 4.1, that

$$\mathrm{dist}_{L^1(\mathbb{S}^n)}\Big(\frac{\overline{\psi}}{\|\psi\|_2^2}, \widetilde{\mathcal{M}_{\mathcal{Q}}}\Big) = \frac{1}{\|X_{\mathcal{Q}}\|}.$$

Now, we turn to Theorem 3.3, which states that X admits a lift if and only if

$$\|X_{\mathcal{Q}}\| \leq 1.$$

In view of the above, this condition is equivalent to requiring that

$$\mathrm{dist}_{L^1(\mathbb{S}^n)}\Big(\frac{\overline{\psi}}{\|\psi\|_2^2}, \widetilde{\mathcal{M}_{\mathcal{Q}}}\Big) \geq 1.$$

This completes the proof of the theorem. ∎

The above proof differs from the polydisc version as obtained in [7, Theorem 4.3]. Here, in addition to applying Lemma 4.1, we have highlighted the key identity (see Lemma 4.2)

$$\ker X_{\mathcal{Q}} = [\mathcal{Q}^{conj} \ominus \{\overline{\psi}\}] \dot{+} [\mathcal{M}(\mathbb{S}^n) \dot{+} H_0^2(\mathbb{S}^n)],$$

which holds for a general quotient module $\mathcal{Q}$ and module map $X \in \mathcal{B}_1(\mathcal{Q})$. This same identity also holds in the polydisc setting. In a sense, the above identity provides deeper insight into the structure of the functional $X_{\mathcal{Q}}$.

5. Characterizations by inner functions

In this section, we connect the lifting of contractive module maps with inner functions. We begin by recalling the following key result due to Rudin [17, Theorem 5.3(b)]:

Theorem 5.1 (Rudin). *Let $f \in \mathcal{S}(\mathbb{B}^n)$. Then there is a sequence of inner functions $\{u_i\}$ in $\mathcal{S}(\mathbb{B}^n)$ such that*

$$w^* - \lim_i u_i = f \text{ on } L^1(\mathbb{S}^n).$$

In the above, by $w^* - \lim_i u_i = f$ on a set $S \subseteq L^1(\mathbb{S}^n)$, we mean that

$$\lim_i \int_{\mathbb{S}^n} u_i g d\sigma = \int_{\mathbb{S}^n} f g d\sigma,$$

for all $g \in S$. In other words, we have the following:

$$\overline{\{u \in H^\infty(\mathbb{B}^n) : u \text{ is inner}\}}^{w^*} = \mathcal{S}(\mathbb{B}^n).$$

In view of this, we introduce the following definition:

With this, we continue in the setting of Theorem 3.3 and present yet another characterization of lifting in terms of inner functions. Given a quotient module $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ and a module map $X \in \mathcal{B}_1(\mathcal{Q})$, we recall that

$$X_{\mathcal{Q}} f = \int_{\mathbb{S}^n} \psi f d\sigma \qquad (f \in \mathcal{M}_{\mathcal{Q}}),$$

defines a functional $X_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1) \longrightarrow \mathbb{C}$, where

$$\psi = X(P_{\mathcal{Q}} 1),$$

and

$$\mathcal{M}_{\mathcal{Q}} = \mathcal{Q}^{conj} \dot{+} [\mathcal{M}_n \dot{+} H_0^2(\mathbb{S}^n)].$$

Theorem 5.2. *Let $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$, and $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map, and let $\psi = X(P_{\mathcal{Q}} 1)$. Then X admits a lift if and only if there is a sequence of inner functions $\{u_i\} \subseteq \mathcal{S}(\mathbb{B}^n)$ such that*

$$w^* - \lim_i u_i = \psi \text{ on } \mathcal{M}_{\mathcal{Q}}.$$

Proof. Suppose X admits a lift. From the proof of Theorem 3.3, and specifically by equation (3.3), there exists $\theta \in \mathcal{S}(\mathbb{B}^n)$ such that for $f \in \mathcal{M}_{\mathcal{Q}}$, we have

$$X_{\mathcal{Q}} f = \int_{\mathbb{S}^n} \theta f d\sigma.$$

By Theorem 5.1, there exists a sequence of inner functions $\{u_i\} \subseteq \mathcal{S}(\mathbb{B}^n)$ such that

$$w^* - \lim_i u_i = \theta \text{ on } L^1(\mathbb{S}^n).$$

In particular, $w^* - \lim_i u_i = \theta$ on $\mathcal{M}_{\mathcal{Q}}$. Moreover, for each $f \in \mathcal{M}_{\mathcal{Q}}$, since

$$X_{\mathcal{Q}} f = \int_{\mathbb{S}^n} \psi f d\sigma = \int_{\mathbb{S}^n} \theta f d\sigma,$$

it follows that

$$\lim_i \int_{\mathbb{S}^n} u_i f d\sigma = \int_{\mathbb{S}^n} \psi f d\sigma,$$

for all $f \in \mathcal{M}_{\mathcal{Q}}$, that is, $w^* - \lim_i u_i = \psi$ on $\mathcal{M}_{\mathcal{Q}}$.
Conversely, assume that $w^* - \lim_i u_i = \psi$ on $\mathcal{M}_{\mathcal{Q}}$. For $f \in \mathcal{M}_{\mathcal{Q}}$, we have

$$\begin{aligned} |X_{\mathcal{Q}} f| &= \Big| \int_{\mathbb{S}^n} \psi f d\sigma \Big| \\ &= \Big| \lim_i \int_{\mathbb{S}^n} u_i f d\sigma \Big| \\ &\leq \lim_i \int_{\mathbb{S}^n} |u_i f| d\sigma \\ &= \|f\|_1. \end{aligned}$$

In other words, the functional $X_{\mathcal{Q}}$ on $(\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1)$ is a contraction, and therefore, by Theorem 3.3, it follows that X admits a lift. ∎

Now, we apply the construction of $\mathcal{M}_{\mathcal{Q}}$ in more detail to prove a more economical version of the above theorem:

Corollary 5.3. *Let $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$, and $X \in \mathcal{B}(\mathcal{Q})$ be a module map, and let $\psi = X(P_{\mathcal{Q}} 1)$. Then X admits a lift if and only if there is a sequence of inner functions $\{u_i\} \subseteq \mathcal{S}(\mathbb{B}^n)$ such that*

$$w^* - \lim_i u_i = \psi \text{ on } \mathcal{Q}^{conj}.$$

Proof. The necessary part is a special case of Theorem 5.2. For sufficiency, suppose there exists a sequence of inner functions $\{u_i\} \subseteq \mathcal{S}(\mathbb{B}^n)$ such that $\lim_i u_i = \psi$ on $\mathcal{Q}^{conj}$. If we can prove that

$$w^* - \lim_i u_i = \psi \text{ on } \mathcal{M}_{\mathcal{Q}},$$

then the sufficiency part of this corollary will follow from Theorem 5.2. To prove the claim, pick any $g \in \mathcal{M}_{\mathcal{Q}} = \mathcal{Q}^{conj} \dot{+} [\mathcal{M}_n \dot{+} H^2_0(\mathbb{S}^n)]$. Then, there exist $g_1 \in \mathcal{Q}^{conj}$, $g_2 \in \mathcal{M}(\mathbb{S}^n)$, and $g_3 \in H^2_0(\mathbb{S}^n)$ such that

$$g = g_1 + g_2 + g_3.$$

Fix an i. Since $g_2 \in \mathcal{M}(\mathbb{S}^n)$, we have

$$\int_{\mathbb{S}^n} u_i g_2 \, d\sigma = \langle u_i, \overline{g_2} \rangle_{L^2(\mathbb{S}^n)} = 0 = \langle \psi, \overline{g_2} \rangle = \int_{\mathbb{S}^n} \psi g_2 d\sigma,$$

and also

$$\int_{\mathbb{S}^n} u_i g_3 d\sigma = \langle u_i g_3, 1 \rangle = u_i(0) g_3(0) = 0,$$

as $g_3 \in H^2_0(\mathbb{S}^n)$. Also, since

$$\int_{\mathbb{S}^n} \psi g_3 d\sigma = \langle \psi, \overline{g_3} \rangle = 0,$$

we have

$$\int_{\mathbb{S}^n} u_i g d\sigma = \int_{\mathbb{S}^n} u_i(g_1 + g_2 + g_3) d\sigma = \int_{\mathbb{S}^n} u_i g_1 d\sigma.$$

In view of our assumption, as $g_1 \in \mathcal{Q}^{conj}$, we have

$$\int_{\mathbb{S}^n} u_i g_1 d\sigma \to \int_{\mathbb{S}^n} \psi g_1 d\sigma,$$

and consequently

$$\int_{\mathbb{S}^n} u_i g d\sigma = \int_{\mathbb{S}^n} u_i g_1 d\sigma \to \int_{\mathbb{S}^n} \psi g_1 d\sigma = \int_{\mathbb{S}^n} \psi g d\sigma,$$

which completes the proof of the corollary. ■

We now focus on lifting on finite-dimensional quotient modules. In order to accomplish a meaningful result in this setting, we specialize to a weak topology. Let $S \subseteq L^1(\mathbb{S}^n)$. For each $f \in S$, define $\hat{f} \in (L^\infty(\mathbb{S}^n))^*$ by

$$\hat{f}(\varphi) = \int_{\mathbb{S}^n} \phi f d\sigma,$$

for all $\varphi \in L^\infty(\mathbb{S}^n)$, and set

$$\hat{S} := \{\hat{f} : f \in S\} \subseteq (L^\infty(\mathbb{S}^n))^*.$$

We let τ_S denote the weak-topology on $L^\infty(\mathbb{S}^n)$ generated by the family of functionals $\hat{S}$. In other words,

$$(L^\infty(\mathbb{S}^n), \tau_S),$$

is the smallest topology on $L^\infty(\mathbb{S}^n)$ for which each functional $\hat{f} : L^\infty(\mathbb{S}^n) \to \mathbb{C}$, $f \in S$, is continuous. Therefore, we have the following: Let $\{\varphi_i\}$ be a sequence in $L^\infty(\mathbb{S}^n)$. Then

$$\{\varphi_i\} \longrightarrow \varphi \text{ in } (L^\infty(\mathbb{S}^n), \tau_{\mathcal{Q}}),$$

if and only if

$$\hat{f}(\varphi_i) \longrightarrow \hat{f}(\varphi),$$

for all $f \in S$. The following result establishes a connection between the weak topology introduced above and the lifting on finite-dimensional quotient modules:

Theorem 5.4. *Let $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ be a finite-dimensional quotient module, and $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. Then X admits a lift if and only if there is a sequence of inner functions $\{u_i\} \subseteq \mathcal{S}(\mathbb{B}^n)$ such that*

$$\{u_i\} \longrightarrow \psi \text{ in } (L^\infty(\mathbb{S}^n), \tau_{\mathcal{Q}^{conj}}),$$

where

$$\psi = X(P_{\mathcal{Q}} 1).$$

Proof. Since $\mathcal{Q}$ is finite-dimensional, it follows that

$$\mathcal{Q} \subseteq H^\infty(\mathbb{B}^n).$$

In particular, we have

$$\psi \in L^\infty(\mathbb{S}^n),$$

Now, convergence $\{u_i\} \longrightarrow \psi$ in $(L^\infty(\mathbb{S}^n), \tau_{\mathcal{Q}^{cconj}})$ is equivalent to the convergence of $\hat{f}(u_i) \longrightarrow \hat{f}(\psi)$ for all $f \in \mathcal{Q}^{cconj}$. Equivalently, we have

$$\int_{\mathbb{S}^n} u_i f d\sigma \longrightarrow \int_{\mathbb{S}^n} \psi f d\sigma,$$

for all $f \in \mathcal{Q}^{conj}$. The result now follows from Corollary 5.3. ∎

Consider the set of all inner functions as a subset of $L^\infty(\mathbb{S}^n)$. Let $\mathcal{Q}$ be a finite-dimensional quotient module and let $X \in \mathcal{B}_1(\mathcal{Q})$. Then the above result can be stated as follows: X admits a lift if and only if

$$X(P_{\mathcal{Q}}1) \in \overline{\{\varphi \in \mathcal{S}(\mathbb{B}^n) : \varphi \text{ is inner}\}}^{(L^\infty(\mathbb{S}^n),\tau_{\mathcal{Q}})}.$$

6. Interpolation

As noted earlier, in the one-variable setting, Sarason [20] showed that the solution to the classical Nevanlinna–Pick interpolation problem follows from his commutant lifting theorem. In a similar spirit, as we also pointed out earlier, we apply the characterization of lifting on the unit ball to derive a solution to the interpolation problem for Schur functions on $\mathbb{B}^n$.

Some results and proofs presented in this section parallel both Sarason's original arguments and the corresponding results for the polydisc interpolation problem [7], which is expected and also part of the plan of the paper. Although additional technical considerations arise due to the greater complexity of the multivariable setting. At the same time, there are substantial variations in both methodology and results. For example, we present new insights into interpolation through inner functions, which apply to the unit ball and yield new results even in the classical $n = 1$ case.

The following lemma serves as the first instance where our approach deviates from the classical and polydisc framework. Specifically, we apply the perturbation solution to the lifting problem as established in Corollary 2.1.

Lemma 6.1. *Let $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ be a finite-dimensional quotient module and let $X \in \mathcal{B}(\mathcal{Q})$ be a module map. Then there exists $\varphi \in H^\infty(\mathbb{D}^n)$ such that*

$$X = S_\varphi.$$

Moreover, φ can be chosen as

$$\varphi = X(P_{\mathcal{Q}}1),$$

and therefore,

$$X = S_{X(P_{\mathcal{Q}}1)}.$$

Proof. As $\mathcal{Q} \subseteq H^2(\mathbb{B}^n)$ is finite-dimensional, it follows that (cf. [10, Corollary 3.4] and [5, Chapter 2, Remark 2.5.5])

$$\mathcal{Q} \subset A(\mathbb{B}^n) \subset H^\infty(\mathbb{B}^n).$$

In particular, we have

$$\psi := X(P_{\mathcal{Q}}1) \in \mathcal{Q} \subseteq A(\mathbb{B}^n),$$

that is, $\psi \in H^\infty(\mathbb{B}^n)$. If we write 0 as

$$\psi - \psi = 0 \in \mathcal{Q}^\perp,$$

then Corollary 2.1 implies that $X = S_\psi$ and completes the proof of the lemma. ∎

Recall that $H^2(\mathbb{B}^n)$ is a reproducing kernel Hilbert space corresponding to the Szegö kernel on the ball $\mathbb{B}^n$:

$$\mathbb{S}_n(z, w) = \frac{1}{(1 - \langle z, w\rangle)^n}.$$

Given m distinct points $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$, define

$$\mathcal{Q}_{\mathcal{Z}} = \text{span } \{\mathbb{S}_n(\cdot, z_i) : 1 \leq i \leq m\}.$$

In view of the kernel property

$$T_{z_j}^* \mathbb{S}_n(\cdot, w) = \overline{w_j}\mathbb{S}_n(\cdot, w),$$

for all $j = 1, \ldots, n$, and $w \in \mathbb{B}^n$, it follows that $\mathcal{Q}_{\mathcal{Z}}$ is a finite-dimensional quotient module. Observe that

$$\mathcal{Q} \subset A(\mathbb{B}^n) \subset H^\infty(\mathbb{B}^n).$$

Moreover, since $\{z_i\}_{i=1}^m$ is a set of distinct points, the $m \times m$ Gram matrix

$$\begin{bmatrix}\mathbb{S}_n(z_i, z_j)\end{bmatrix}_{m\times m} = \begin{bmatrix} \mathbb{S}_n(z_1, z_1) & \mathbb{S}_n(z_1, z_2) & \ldots & \mathbb{S}_n(z_1, z_m) \\ \mathbb{S}_n(z_2, z_1) & \mathbb{S}_n(z_2, z_2) & \ldots & \mathbb{S}_n(z_2, z_m) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{S}_n(z_m, z_1) & \mathbb{S}_n(z_m, z_2) & \ldots & \mathbb{S}_n(z_m, z_m) \end{bmatrix},$$

is invertible. We are now ready for a solution to the interpolation problem:

Theorem 6.2. *Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$ be a set of m distinct points, and let $\{w_i\}_{i=1}^m \subseteq \mathbb{D}$ be a set of m values. Consider m scalars $\{c_i\}_{i=1}^m$ defined by*

$$\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix}\mathbb{S}_n(z_i, z_j)\end{bmatrix}_{m\times m}^{-1} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix},$$

and set

$$\psi_{\mathcal{Z},\mathcal{W}} = \sum_{j=1}^m c_j \mathbb{S}_n(\cdot, z_j).$$

Then there exists $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that

$$\varphi(z_i) = w_i,$$

for all $i = 1, \ldots, m$, if and only if

$$\mathcal{J}_{\mathcal{Z},\mathcal{W}} f = \int_{\mathbb{S}^n} \psi_{\mathcal{Z},\mathcal{W}} f \, d\sigma$$

defines a contraction $\mathcal{J} : (\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}, \|\cdot\|_1) \to \mathbb{C}$, where

$$\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}} = \mathcal{Q}_{\mathcal{Z}}^{conj} \dot{+} [\mathcal{M}(\mathbb{S}^n) + H_0^2(\mathbb{S}^n)].$$

Proof. Following Sarason, we define $X_{\mathcal{Z},\mathcal{W}} \in \mathcal{B}(\mathcal{Q}_{\mathcal{Z}})$ by

$$X_{\mathcal{Z},\mathcal{W}}^* \mathbb{S}_n(\cdot, z_i) = \overline{w_i}\mathbb{S}_n(\cdot, z_i).$$

for all $i = 1, 2 \ldots, m$. Clearly, $X_{\mathcal{Z},\mathcal{W}}$ is a module map. Moreover, there exists $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that $\varphi(z_i) = w_i$ for all $i = 1, \ldots, m$, if and only if

$$S_\varphi = X_{\mathcal{Z},\mathcal{W}}.$$

This follows from the fact that $\mathcal{Q}_{\mathcal{Z}} = \text{span}\{\mathbb{S}_n(\cdot, z_i) : i = 1, \ldots, m\}$ and

$$X_{\mathcal{Z},\mathcal{W}}^* \mathbb{S}_n(\cdot, z_i) = \overline{w_i}\mathbb{S}_n(\cdot, z_i) = S_\varphi^* \mathbb{S}_n(\cdot, z_i),$$

for all $i = 1, \ldots, m$ (also see the claim part in the proof of Theorem 6.6 of [7]). Therefore, by Lemma 6.1, there exists a Schur function $\varphi \in \mathcal{S}(\mathbb{B}^n)$ interpolating the given data if and only if $X_{\mathcal{Z},\mathcal{W}}$ admits a lift. Set

$$\psi = X_{\mathcal{Z},\mathcal{W}}(P_{\mathcal{Q}_{\mathcal{Z}}}1).$$

By Theorem 3.3, the operator $X_{\mathcal{Z},\mathcal{W}}$ on $\mathcal{Q}_{\mathcal{Z}}$ admits a lift if and only if the functional

$$X_{\mathcal{Q}_{\mathcal{Z}}} f = \int_{\mathbb{S}^n} \psi f \, d\sigma.$$

for all $f \in \mathcal{Q}_{\mathcal{Z}}$, defines a contraction on $(\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}, \|\cdot\|_1)$. Therefore, it remains to prove that

$$\psi_{\mathcal{Z},\mathcal{W}} = \psi.$$

To this end, we observe, since $\psi \in \mathcal{Q}_{\mathcal{Z}}$, there exist scalars $c_1, \ldots c_m$ such that

$$\psi = \sum_{j=1}^{m} c_j \mathbb{S}_n(\cdot, z_j).$$

Since $\mathcal{Q}_{\mathcal{Z}}$ is finite-dimensional, Lemma 6.1 implies $X = S_\psi$. Therefore,

$$X^* \mathbb{S}_n(\cdot, z_j) = S_\psi^* \mathbb{S}_n(\cdot, z_j),$$

for all $j = 1, \ldots, m$. As $X^* \mathbb{S}_n(\cdot, z_j) = \overline{w_j} \mathbb{S}_n(\cdot, z_j)$ and $S_\psi^* \mathbb{S}_n(\cdot, z_j) = \overline{\psi(z_j)} \mathbb{S}_n(\cdot, z_j)$, we conclude that

$$\overline{w_j} \mathbb{S}_n(\cdot, z_j) = \overline{\psi(z_j)} \mathbb{S}_n(\cdot, z_j),$$

and hence

$$\psi(z_j) = w_j,$$

for all $j = 1, \ldots, m$. In particular, for each $j = 1, \ldots, m$, we have

$$w_j = \psi(z_j) = \sum_{i=1}^{m} c_i \mathbb{S}_n(z_j, z_i),$$

and gives the desired identity $\psi_{\mathcal{Z},\mathcal{W}} = \psi$. ∎

Since $\psi_{\mathcal{Z},\mathcal{W}} = \psi = X_{\mathcal{Z},\mathcal{W}}(P_{\mathcal{Q}_{\mathcal{Z}}}1)$, Theorem 4.3 yields the following consequence:

Corollary 6.3. *Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$ be a set of m distinct points, and let $\{w_i\}_{i=1}^m \subseteq \mathbb{D}$ be a set of m values. Consider m scalars $\{c_i\}_{i=1}^m$ defined by*

$$\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \big[\mathbb{S}_n(z_i, z_j)\big]_{m\times m}^{-1} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}.$$

Let

$$\psi_{\mathcal{Z},\mathcal{W}} = \sum_{j=1}^{m} c_j \mathbb{S}_n(\cdot, z_j),$$

and let

$$\widetilde{\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}} = [\mathcal{Q}_{\mathcal{Z}}^{conj} \ominus \langle \overline{\psi_{\mathcal{Z},\mathcal{W}}} \rangle] \dot{+} [\mathcal{M}_n \dot{+} H_0^2(\mathbb{S}^n)].$$

Then there exists $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that

$$\varphi(z_i) = w_i,$$

for all $i = 1, \dots, m$, if and only if

$$d_{L^1(\mathbb{S}^n)}\Big(\frac{\overline{\psi_{\mathcal{Z},\mathcal{W}}}}{\|\psi_{\mathcal{Z},\mathcal{W}}\|_2^2}, \widetilde{\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}}\Big) \geq 1.$$

In the following, we establish a connection between interpolation and inner functions.

Theorem 6.4. *Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subseteq \mathbb{B}^n$ be a set of m distinct points, and let $\{w_i\}_{i=1}^m \subseteq \mathbb{D}$ be a set of m values. Consider m scalars $\{c_i\}_{i=1}^m$ defined by*

$$\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \big[\mathbb{S}_n(z_i, z_j)\big]_{m\times m}^{-1} \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}.$$

Then there exists $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that

$$\varphi(z_i) = w_i,$$

for all $i = 1, \dots, m$, if and only if there exists a sequence of inner functions $\{u_i\} \subseteq \mathcal{S}(\mathbb{B}^n)$ such that

$$\lim_i u_i(z_j) = \psi_{\mathcal{Z},\mathcal{W}}(z_j),$$

for all $j = 1, \dots, m$.

Proof. We follow the setting of Theorem 6.2. We know that $\psi_{\mathcal{Z},\mathcal{W}} = X_{\mathcal{Z},\mathcal{W}} P_{\mathcal{Q}_{\mathcal{Z}}} 1$. By Corollary 5.3, the operator $X_{\mathcal{Z},\mathcal{W}}$ on $\mathcal{Q}_{\mathcal{Z}}$ admits a lift if and only if there is a sequence of inner functions $\{u_i\}$ such that

$$w^* - \lim_i u_i = \psi_{\mathcal{Z},\mathcal{W}} \text{ on } \mathcal{Q}_{\mathcal{Z}}^{conj},$$

that is,

$$\int_{\mathbb{S}^n} u_i f \, d\sigma \longrightarrow \int_{\mathbb{S}^n} \psi_{\mathcal{Z},\mathcal{W}} f \, d\sigma,$$

for all $f \in \mathcal{Q}_{\mathcal{Z}}^{conj}$. This condition, therefore, is equivalent to the interpolation condition: There exists $\varphi \in \mathcal{S}(\mathbb{B}^n)$ such that

$$\varphi(z_i) = w_i,$$

for all $i = 1, \dots, m$. On the other hand, since $\mathcal{Q}_{\mathcal{Z}} = \operatorname{span}\{\mathbb{S}_n(\cdot, z_j) : 1 \leq j \leq m\}$ is finite-dimensional, the condition $\lim_i u_i = \psi_{\mathcal{Z},\mathcal{W}}$ on $\mathcal{Q}_{\mathcal{Z}}^{conj}$ is equivalent to the condition that

$$\int_{\mathbb{S}^n} u_i \overline{\mathbb{S}_n(\cdot, z_j)} d\sigma \longrightarrow \int_{\mathbb{S}^n} \psi_{\mathcal{Z},\mathcal{W}} \overline{\mathbb{S}_n(\cdot, z_j)} d\sigma,$$

for all $j = 1, \dots, m$. In view of this and the reproducing kernel property, we compute

$$\begin{aligned} u_i(z_j) &= \langle u_i, \mathbb{S}_n(\cdot, z_j)\rangle \\ &= \int_{\mathbb{S}^n} u_i \overline{\mathbb{S}_n(\cdot, z_j)} \\ &\longrightarrow \int_{\mathbb{S}^n} \psi_{\mathcal{Z},\mathcal{W}} \overline{\mathbb{S}_n(\cdot, z_j)} d\sigma \\ &= \langle \psi_{\mathcal{Z},\mathcal{W}}, \mathbb{S}_n(\cdot, z_j)\rangle \\ &= \psi_{\mathcal{Z},\mathcal{W}}(z_j), \end{aligned}$$

that is, $u_i(z_j) \longrightarrow \psi_{\mathcal{Z},\mathcal{W}}(z_j)$ for all $j = 1, \dots, m$. This completes the proof of the result. ∎

The above interpolation criterion is new even in the case $n = 1$. More specifically, when $n = 1$, one has a more effective matrix-positivity characterization, namely the positive semi-definiteness of the classical Pick matrix. In this setting, the solution to the interpolation problem is unique, and it is a finite Blaschke product of degree p whenever the Pick matrix has low rank:

$$p := \operatorname{rank}\left[\frac{1-w_i\overline{w_j}}{1-z_i\overline{z_j}}\right]_{m\times m} < m.$$

For the case in which the Pick matrix has full rank (the most interesting situation), the above theorem applies and yields perhaps the next best alternative: there exists a sequence of inner functions converging pointwise to a function $\psi_{\mathcal{Z},\mathcal{W}}$ on all the prescribed data $\{z_1, \dots, z_m\}$.

7. Examples

The goal of this section is to construct concrete examples illustrating situations in which lifting does and does not occur. This allows us, in particular, to apply the preceding results to explicit situations. Recall that a holomorphic function $f : \mathbb{B}^n \to \mathbb{C}$ can be expressed as a power series

$$f(z) = \sum_{\alpha\in\mathbb{Z}_+^n} a_\alpha z^k,$$

where $z^k = z_1^{k_1} \cdots z_n^{\mathbb{S}_n}$ for all $k = (k_1, \dots, \mathbb{S}_n) \in \mathbb{Z}_+^n$. The degree of a monomial z^k is defined as

$$\deg\, z^k = |k| := \sum_{j=1}^{n} k_j.$$

The degree of a polynomial is defined as the highest degree among its monomial terms. Given $m \in \mathbb{N}$, define

$$\mathcal{Q}_m = \{p \in \mathbb{C}[z_1, \dots, z_n] : \deg\, p \le m]\}.$$

Clearly, $\mathcal{Q}_m$ is a finite-dimensional quotient module of $H^2(\mathbb{B}^n)$. Fix a polynomial $p \in \mathbb{C}[z_1, \dots, \boldsymbol{z}_n]$, and consider the module map

$$S_p = P_{\mathcal{Q}_m} T_p|_{\mathcal{Q}_m}.$$

We are interested in the lifting of S_p under the assumption that $\|p\|_2 = 1$. The polydisc version [7, Corollary 3.3] is essentially the same, but the present proof deviates slightly from that of the polydisc case. We also aim to make the present proof self-contained for the sake of completeness.

Theorem 7.1. *Let $p \in \mathbb{C}[z_1, \dots, z_n]$ be a polynomial, and let deg $p \le m$. Suppose*

$$\|p\|_2 = 1.$$

Then $S_p \in \mathcal{B}(\mathcal{Q}_m)$ admits a lift if and only if p is a unimodular constant.

Proof. Clearly, S_p is a module map. Let $\psi = S_p(P_{\mathcal{Q}_m}1)$. As 1 and p are in $\mathcal{Q}_m$, we have $P_{\mathcal{Q}_m}1 = 1$ and $P_{\mathcal{Q}_m}p = p$. We have

$$\psi = S_p 1 = P_{\mathcal{Q}_m} T_p 1 = P_{\mathcal{Q}_m} p = p.$$

By Theorem 3.3, we know that S_p admits a lift if and only if $X_m : (\mathcal{M}_{\mathcal{Q}_m}, \|.\|_1) \to \mathbb{C}$ is a contraction, where

$$X_m f = \int_{\mathbb{S}^n} p f \, d\sigma,$$

for all $f \in \mathcal{M}_{\mathcal{Q}_m}$. Also note that

$$\|p\|_1 = \int_{\mathbb{S}^n} |p| d\sigma = \int_{\mathbb{S}^n} |p1| d\sigma \leq \Big(\int_{\mathbb{S}^n} |p|^2 d\sigma \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{S}^n} |1| d\sigma \Big)^{\frac{1}{2}} = 1. \tag{7.1}$$

Now assume that S_p admits a lift. As

$$X_m \bar{p} = \int_{\mathbb{S}^n} |p|^2 d\sigma = \|p\|_2^2 = 1,$$

and

$$|X_m \bar{p}| \leq \|\overline{p}\|_1 = \|p\|_1,$$

it follows that

$$1 = \|p\|_2^2 \leq \|p\|_1 \leq \|p\|_2 = 1,$$

and hence $\|p\|_2 = 1 = \|p\|_1$. Then (7.1) implies that equality holds in the Cauchy–Schwarz inequality:

$$\int_{\mathbb{S}^n} |p \times 1| d\sigma = \Big(\int_{\mathbb{S}^n} |p|^2 d\sigma \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{S}^n} |1| d\sigma \Big)^{\frac{1}{2}}$$

Therefore, there exists a constant λ such that

$$|p| = \lambda.$$

Since $\|p\|_1 = 1$, we have $|\lambda| = 1$, and hence

$$|p| = 1.$$

This implies that p is an inner function. Finally, by [17, Theorem 4.6], we conclude that p is a unimdular constant. The converse is obvious. ∎

Now we want to compute the submodule corresponding to the quotient module. To this end, for each $m \geq 1$, define

$$J_m = \{ f \in H^2(\mathbb{B}^n) : f(z) = \sum_{k \in \mathbb{Z}_+^n, |k| \geq m} a_\alpha z^k \}.$$

Equivalently, J_m consists of holomorphic functions on $\mathbb{B}^n$ whose Taylor series contain no terms of degree less than m. The following lemma is a simple consequence of the fact that $\{z^k : k \in \mathbb{Z}_+^n\}$ forms an orthogonal basis for $H^2(\mathbb{B}^n)$.

Lemma 7.2. *$J_{m+1} = \mathcal{Q}_m^\perp$ for all $m \geq 1$.*

We now recall a perturbation theorem of Rudin [17, Theorem 4.3] concerning inner functions on $\mathbb{B}^n$, specialized here to the case of the ball algebra. This result will be applied to construct two nontrivial examples.

Theorem 7.3 (Rudin)**.** *Let g be a nonzero function in $A(\mathbb{B}^n)$ and let $m \in \mathbb{N}$. Assume that*

$$\|g\|_\infty < 1.$$

Then there exists an inner function $u \in H^\infty(\mathbb{B}^n)$ such that u and g have the same zeros in $\mathbb{B}^n$ and

$$u - g \in J_m.$$

The following is the first application of Rudin's theorem. In particular, this yields the existence of lifting of contractive module maps:

Example 7.4. Fix $m \in \mathbb{N}$, and set

$$\Lambda = \{\alpha \in \mathbb{Z}_+^n : |\alpha| \leq m\}.$$

Choose a finite set of scalars

$$\{a_\alpha : \alpha \in \Lambda\} \subseteq \mathbb{C}.$$

Assume that

$$0 < \sum_{\alpha \in \Lambda} |a_\alpha| \leq 1.$$

Then

$$p(z) := \sum_{|\alpha| \leq m} a_k z^k,$$

is a nonzero polynomial in $\mathcal{Q}_m$. Consider the module map $S_p \in \mathcal{B}(\mathcal{Q}_m)$. As $p \in \mathcal{Q}_m$, we have

$$S_p 1 = P_{\mathcal{Q}[m]} T_p 1 = p.$$

For each $z \in \mathbb{B}^n$, we have

$$|p(z)| \leq \sum_{|\alpha| \leq m} |a_\alpha z^\alpha| \leq \sum_{|\alpha| \leq m} |a_\alpha| = \sum_{\alpha \in \Lambda} |a_\alpha|,$$

which implies

$$\|p\|_{H^\infty(\mathbb{B}^n)} \leq \sum_{\alpha \in \Lambda} |a_\alpha| < 1.$$

By Theorem 7.3, there exists an inner function $u \in H^\infty(\mathbb{B}^n)$ such that

$$u - p \in J_{m+1}.$$

In view of Lemma 7.2, we know that $J_{m+1} = \mathcal{Q}_m^\perp$. In other words, we have an inner function $u \in H^\infty(\mathbb{B}^n)$ such that

$$u - p \in \mathcal{Q}_m^\perp.$$

Theorem 2.2 now implies that S_p lifts to S_u.

It is curious to observe how two apparently independent results fit together, namely, the perturbation result of Rudin (Theorem 7.3) and the characterization of lifting along the lines of perturbations in Theorem 2.2.

Let $\varphi \in H^\infty(\mathbb{B}^n)$ be an inner function. In this case, we know that the Toeplitz operator T_φ is an isometry and hence $\varphi H^2(\mathbb{B}^n)$ is a submodule, and consequently

$$\mathcal{Q}_u := H^2(\mathbb{B}^n) \ominus u H^2(\mathbb{B}^n),$$

is a quotient module. This quotient module is traditionally referred to as a model space [13]. In the following, we give a criterion of lifting to ball algebra. We remark that this result holds specifically for $n > 1$.

Theorem 7.5. *Let $n > 1$, $u \in H^\infty(\mathbb{B}^n)$ be a non-constant inner function, and let $X \in \mathcal{B}_1(\mathcal{Q}_u)$ be a module map. Assume that*

$$\psi := X(P_{\mathcal{Q}_u} 1) \in A(\mathbb{B}^n).$$

Then X admits a lift to $A(\mathbb{B}^n)$ if and only if

$$\|\psi\|_\infty \leq 1.$$

Proof. Suppose X lifts to $A(\mathbb{B}^n)$. By Corollary 2.3, there exists $\varphi \in A(\mathbb{B}^n) \cap \mathcal{S}(\mathbb{B}^n)$ such that

$$\psi - \phi \in \mathcal{Q}_u^{\perp} = uH^2(\mathbb{B}^n).$$

Since $\varphi \in A(\mathbb{B}^n)$, it follows that $\psi - \varphi \in A(\mathbb{B}^n)$, and hence

$$\psi - \varphi \in A(\mathbb{B}^n) \cap uH^2(\mathbb{B}^n).$$

Assume, for contradiction, that $\psi \neq \phi$. From the above, there exists a nonzero $g \in H^2(\mathbb{B}^n)$ such that

$$\psi - \phi = ug.$$

However, this is impossible by Theorem 4.6 of [17]. This contradiction shows that $\psi = \phi$. Consequently, $\|\psi\|_\infty = \|\phi\|_\infty \leq 1$. The converse is immediate by Lemma 2.3. ∎

To clarify the conditions of the above theorem, it is necessary to consider a concrete example:

Example 7.6. Fix $m \in \mathbb{N}$ and multi-index $k \in \mathbb{Z}_+^n$ such that

$$|\alpha| = m + 1.$$

Consider the polynomial

$$p(z) = \frac{1}{2} z^k \in H^2(\mathbb{B}^n).$$

By Rudin, Theorem 7.3, there exists a non-constant inner function $u \in H^\infty(\mathbb{B}^n)$ such that

$$u - p \in J_{m+1} = \mathcal{Q}_m^{\perp}.$$

By our choice, we have $p \in \mathcal{Q}_m^{\perp}$, and hence

$$u = p + (u - p) \in \mathcal{Q}_m^{\perp}.$$

This implies $uH^2(\mathbb{B}^n) \subseteq \mathcal{Q}_m^{\perp}$, equivalently, $\mathcal{Q}_m \subseteq \mathcal{Q}_u$. Let $q \in \mathbb{C}[z_1, \ldots, z_n]$ with $\deg q \leq m$, and consider the module map $X = S_q \in \mathcal{B}(\mathcal{Q}_u)$ defined by

$$S_q f = P_{\mathcal{Q}_u} q f$$

Then,

$$X P_{\mathcal{Q}_u} 1 = S_{\mathcal{Q}_u} P_{\mathcal{Q}_u} 1 = P_{\mathcal{Q}_u} q P_{\mathcal{Q}_u} 1 = P_{\mathcal{Q}_u} q = q \in A(\mathbb{B}^n).$$

By Theorem 7.5, X admits a lift to $A(\mathbb{B}^n)$ if and only if

$$\|q\|_\infty \leq 1.$$

We remark that, by invoking the $w*$-density property of inner functions on the polydisc [19, Theorem 5.5.1], an analogous argument yields the corresponding polydisc version of the solution to the interpolation problem as stated in Theorem 6.4.

We conclude this paper by remarking that the lifting and interpolation problems in several variables are of different levels of intricacy on the unit ball than on the polydisc. While solutions to these problems in the polydisc setting have been developed in the recent paper [7], as well as in several earlier works that established results of considerable depth and influence [2, 4], the literature contains virtually no comparable attempts, let alone partial results, within the framework of the unit ball (however, see [8, 9] and the references there in). This disparity reflects the greater complexity of the function theory associated with Hilbert function spaces on the ball.

In particular, the connection between quotient modules of the corresponding Hardy spaces and inner functions highlights a fundamental difference between the ball and the polydisc.

A notable aspect of this paper is its exploration of the relationship between interpolation problems and inner functions on the ball, which exemplifies the challenges inherent in the Hilbert function space theory in this setting.

Acknowledgement: The research of the second named author is supported in part by TARE (TAR/2022/000063), ANRF, Department of Science & Technology (DST), Government of India.

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