Wandering subspaces of the Bergman space and the Dirichlet space over $\mathbb{D}^n$

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Abstract. Doubly commutativity of invariant subspaces of the Bergman space and the Dirichlet space over the unit polydisc $\mathbb{D}^n$ (with $n \geq 2$) is investigated. We show that for any non-empty subset $\alpha = \{\alpha_1, \ldots, \alpha_k\}$ of $\{1, \ldots, n\}$ and doubly commuting invariant subspace $\mathcal{S}$ of the Bergman space or the Dirichlet space over $\mathbb{D}^n$, the tuple consists of restrictions of co-ordinate multiplication operators $M_\alpha|_\mathcal{S} := (M_{z_{\alpha_1}}|_\mathcal{S}, \ldots, M_{z_{\alpha_k}}|_\mathcal{S})$ always possesses generating wandering subspace of the form

$$\bigcap_{i=1}^{k} (\mathcal{S} \ominus z_{\alpha_i}\mathcal{S}).$$

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1. Introduction

A closed subspace $\mathcal{W}$ of a Hilbert space $\mathcal{H}$ is said to be *generating wandering subspace* (following Halmos [5]) for an n-tuple $T = (T_1, \ldots, T_n)$ $(n \geq 1)$ of commuting bounded linear operators on $\mathcal{H}$ if

$$\mathcal{W} \perp T_1^{l_1} T_2^{l_2} \cdots T_n^{l_n} \mathcal{W}$$

for all $(l_1, \ldots, l_n) \in \mathbb{N}^n \setminus \{(0, \ldots, 0)\}$ and

$$\mathcal{H} = \overline{\text{span}}\{T_1^{l_1} T_2^{l_2} \cdots T_n^{l_n} h : h \in \mathcal{W}, l_1, \ldots, l_n \in \mathbb{N}\}.$$

In this case, the tuple T is said to have the *generating wandering subspace property.*

The main purpose of this paper is to investigate the following question:
Question: Let $(T_1, \ldots, T_n)$ be a commuting n-tuple of bounded linear operators on a Hilbert space $\mathcal{H}$. Does there exist a generating wandering subspace $\mathcal{W}$ for $(T_1, \ldots, T_n)$?

This question has an affirmative answer for the restriction of multiplication operator by the co-ordinate function M_z, to an invariant subspace of the Hardy space $H^2(\mathbb{D})$ (Beurlings theorem [3]) or the Bergman space $A^2(\mathbb{D})$ (Aleman, Richter and Sundberg [2]) or the Dirichlet space $\mathcal{D}(\mathbb{D})$ (Richter [8]) over the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. For $n \geq 2$, existence of generating wandering subspaces for general invariant subspaces of the Hardy space $H^2(\mathbb{D}^n)$ over the unit polydisc $\mathbb{D}^n$ rather fails spectacularly (cf. [9]).

Recall that the Hardy space over the unit polydisc $\mathbb{D}^n = \{\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| < 1, i = 1, \ldots, n\}$ is denoted by $H^2(\mathbb{D}^n)$ and defined by

$$H^2(\mathbb{D}^n) = \{f \in \mathcal{O}(\mathbb{D}^n) : \sup_{0 \leq r < 1} \int_{\mathbb{T}^n} |f(r\boldsymbol{z})|^2 d\boldsymbol{\theta} < \infty\},$$

where $d\boldsymbol{\theta}$ is the normalized Lebesgue measure on the torus $\mathbb{T}^n$, the distinguished boundary of $\mathbb{D}^n$, $r\boldsymbol{z} := (rz_1, \ldots, rz_n)$ and $\mathcal{O}(\mathbb{D}^n)$ denotes the set of all holomorphic functions on $\mathbb{D}^n$ (cf. [9]).

In [3], A. Beurling characterize all closed M_z-invariant subspaces of $H^2(\mathbb{D})$ in the following sense: Let $\mathcal{S} \neq \{0\}$ be a closed subspace of $H^2(\mathbb{D})$. Then $\mathcal{S}$ is M_z-invariant if and only if $\mathcal{S} = \theta H^2(\mathbb{D})$ for some inner function θ (that is, $\theta \in H^\infty(\mathbb{D})$ and $|\theta| = 1$ a.e. on the unit circle $\mathbb{T}$). In particular, Beurlings theorem yields the generating wandering subspace property for M_z-invariant subspaces of $H^2(\mathbb{D})$ as follows: if $\mathcal{S} = \theta H^2(\mathbb{D})$ is an M_z-invariant subspace of $H^2(\mathbb{D})$ then

$$\mathcal{S} = \sum_{m \geq 0} \oplus z^m \mathcal{W}, \tag{1.1}$$

where $\mathcal{W}$ is the generating wandering subspace for $M_z|_{\mathcal{S}}$ given by

$$\mathcal{W} = \mathcal{S} \ominus z\mathcal{S} = \theta H^2(\mathbb{D}) \ominus z\theta H^2(\mathbb{D}) = \theta\mathbb{C}.$$

In [9], W. Rudin showed that there are invariant subspaces $\mathcal{M}$ of $H^2(\mathbb{D}^2)$ which do not contain any bounded analytic function. In particular, the Beurling like characterization of $(M_{z_1}, \ldots, M_{z_n})$-invariant subspaces of $H^2(\mathbb{D}^n)$, in terms of inner functions on $\mathbb{D}^n$ is not possible.

On the other hand, the generating wandering subspace (and the Beurlings theorem) for the shift invariant subspaces of the Hardy space $H^2(\mathbb{D})$ follows directly from the classical Wold decomposition [12] theorem for isometries. Indeed, for a closed M_z-invariant subspace $\mathcal{S}(\neq \{0\})$ of $H^2(\mathbb{D})$,

$$\bigcap_{m=0}^{\infty} (M_z|_{\mathcal{S}})^m \mathcal{S} = \bigcap_{m=0}^{\infty} M_z^m \mathcal{S} \subseteq \bigcap_{m=0}^{\infty} M_z^m H^2(\mathbb{D}) = \{0\}.$$

Consequently, by Wold decomposition theorem for the isometry $M_z|_{\mathcal{S}}$ on $\mathcal{S}$ we have

$$\mathcal{S} = \sum_{m \geq 0} \oplus z^m \mathcal{W} \oplus (\bigcap_{m=0}^{\infty} (M_z|_{\mathcal{S}})^m \mathcal{S}) = \sum_{m \geq 0} \oplus z^m \mathcal{W},$$

where $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$ and hence, (1.1) follows. Therefore, the notion of generating wandering subspaces is stronger (as well as of independent interest) than the Beurlings characterization of M_z-invariant subspaces of $H^2(\mathbb{D})$.

To proceed, we now recall the following definition: a commuting n-tuple ($n \geq 2$) of bounded linear operators $(T_1, \ldots, T_n)$ on $\mathcal{H}$ is said to be *doubly commuting* if $T_i T_j^* = T_j^* T_i$ for all $1 \leq i < j \leq n$.

Natural examples of doubly commuting tuples of operators are the multiplication operators by the co-ordinate functions acting on the Hardy space or the Bergman space or the Dirichlet space over the unit polydisc $\mathbb{D}^n$ ($n \geq 2$).

Let $\mathcal{H} \subseteq \mathcal{O}(\mathbb{D}^n)$ be a reproducing kernel Hilbert space over $\mathbb{D}^n$ (see [1]) and multiplication operators $\{M_{z_1}, \ldots, M_{z_n}\}$ by co-ordinate functions are bounded. Then a closed $(M_{z_1}, \ldots, M_{z_n})$-invariant subspace $\mathcal{S}$ of $\mathcal{H}$ is said to be doubly commuting if the n-tuple $(M_{z_1}|_{\mathcal{S}}, \ldots, M_{z_n}|_{\mathcal{S}})$ is doubly commuting, that is, $R_i R_j^* = R_j^* R_i$ for all $1 \leq i < j \leq n$, where $R_i = M_{z_i}|_{\mathcal{S}}$.

In [11], third author and Sasane and Wick proved that any doubly commuting invariant subspace of $H^2(\mathbb{D}^n)$ (where $n \geq 2$) has the generating wandering subspace property (see [6] for $n = 2$). Also in [7], Redett and Tung obtained the same conclusion for doubly commuting invariant subspaces of the Bergman space $A^2(\mathbb{D}^2)$ over the bidisc $\mathbb{D}^2$.

In this paper we prove that doubly commuting invariant subspaces of the Bergman space $A^2(\mathbb{D}^n)$ and the Dirichlet space $\mathcal{D}(\mathbb{D}^n)$ have the generating wandering subspace property. Our result on the Bergman space over polydisc is a generalization of the base case $n = 2$ in [7]. Our analysis is based on the Wold-type decomposition result of S. Shimorin for operators closed to isometries [10].

The paper is organized as follows: in Section 2, we obtain some general results concerning the generating wandering subspaces of tuples of doubly commuting operators. We obtain generating wandering subspaces for doubly commuting shift invariant subspaces of the Bergman and Dirichlet spaces over $\mathbb{D}^n$ in Section 3.

2. Generating wandering subspace for tuple of doubly commuting operators

In this section we prove the multivariate version of the S. Shimorin's result for tuple of doubly commuting operators on a general Hilbert space. We show that for a tuple of doubly commuting operators $T = (T_1, \ldots, T_n)$ on a Hilbert space $\mathcal{H}$, if for any reducing subspace $\mathcal{S}_i$ of T_i the subspace $\mathcal{S}_i \ominus T_i \mathcal{S}_i$ is a generating wandering subspace for $T_i|_{\mathcal{S}_i}$, $i = 1, \ldots, n$, then $\bigcap\limits_{i=1}^{n} (\mathcal{H} \ominus T_i \mathcal{H})$ is a generating wandering subspace for T. We *fix for the rest of the paper* a natural number $n \geq 2$ and set $\Lambda_n := \{1, \ldots, n\}$.

For a closed subset $\mathcal{K}$ of a Hilbert space $\mathcal{H}$, an n-tuple of commuting operators $T = (T_1, \ldots, T_n)$ on $\mathcal{H}$ and a non-empty subset $\alpha = \{\alpha_1, \ldots, \alpha_k\} \subseteq \Lambda_n$, we write $[\mathcal{K}]_{T_\alpha}$ to denote the smallest closed joint $T_\alpha := (T_{\alpha_1}, \ldots, T_{\alpha_k})$-invariant subspace of $\mathcal{H}$ containing $\mathcal{K}$. In other words

$$[\mathcal{K}]_{T_\alpha} = \bigvee_{l_1,l_2,\ldots,l_k=0}^{\infty} T_{\alpha_1}^{l_1} T_{\alpha_2}^{l_2} \cdots T_{\alpha_k}^{l_k}(\mathcal{K}). \tag{2.1}$$

If α is a singleton set $\{i\}$ then we simply write $[\mathcal{K}]_{T_i}$.

For a non-empty subset $\alpha = \{\alpha_1, \ldots, \alpha_k\} \subseteq \Lambda_n$, we also denote by $\mathcal{W}_\alpha$ the following subspace of $\mathcal{H}$,

$$\mathcal{W}_\alpha = \bigcap_{i=1}^{k} (\mathcal{H} \ominus T_{\alpha_i}\mathcal{H}). \tag{2.2}$$

Again if α is a singleton set $\{i\}$ then we simply write $\mathcal{W}_i$. Thus with the above notation $\mathcal{W}_\alpha = \bigcap\limits_{\alpha_i \in \alpha} \mathcal{W}_{\alpha_i}$.

A bounded linear operator T on a Hilbert space $\mathcal{H}$ is *analytic* if $\bigcap\limits_{n=0}^{\infty} T^n\mathcal{H} = \{0\}$ and it is *concave* if it satisfies the following inequality

$$\|T^2x\|^2 + \|x\|^2 \leq 2\|Tx\|^2 \quad (x \in \mathcal{H}).$$

Multiplication by co-ordinate functions on the Dirichlet space over the unit polydisc are concave operators as we show in the next section.

For a single bounded operator T on a Hilbert space $\mathcal{H}$, the following result ensures the existence of the generating wandering subspace under certain conditions (see [8], [10]).

Theorem 2.1. *(Richter, Shimorin). Let T be an analytic operator on a Hilbert space $\mathcal{H}$ which satisfies one of the following properties:*
(i) $\| Tx + y \|^2 \leq 2(\| x \|^2 + \| Ty \|^2)$ $(x, y \in \mathcal{H})$,
(ii) T *is concave.*
Then $\mathcal{H} \ominus T\mathcal{H}$ is a generating wandering subspace for T, that is,

$$\mathcal{H} = [\mathcal{H} \ominus T\mathcal{H}]_T.$$

The following proposition is essential in order to generalize the above result for certain tuples of commuting operators.

Proposition 2.2. *Let $T = (T_1, \ldots, T_n)$ be a doubly commuting tuple of operators on $\mathcal{H}$. Then $\mathcal{W}_\alpha$ is T_j-reducing subspace for all non-empty subset $\alpha \subseteq \Lambda_n$ and $j \notin \alpha$, where $\mathcal{W}_\alpha$ is as in* (2.2).

Proof. Let $\alpha = \{\alpha_1, \ldots, \alpha_k\}$ be a non-empty subset of Λ_n and $j \notin \alpha$. First note that $\mathcal{W}_l = \operatorname{Ker} T_l^*$ for all $1 \leq l \leq n$ and therefore $\mathcal{W}_\alpha = \bigcap\limits_{i=1}^{k} \operatorname{Ker} T_{\alpha_i}^*$. Let $x \in \mathcal{W}_\alpha$, $\alpha_i \in \alpha$ and $y \in \mathcal{H}$. By doubly commutativity of T we have,

$$\langle T_j x, T_{\alpha_i} y\rangle = \langle T_{\alpha_i}^* T_j x, y\rangle = \langle T_j T_{\alpha_i}^* x, y\rangle = 0.$$

Therefore, $T_j\mathcal{W}_\alpha \perp T_{\alpha_i}\mathcal{H}$ and hence $T_j\mathcal{W}_\alpha \subseteq \mathcal{W}_{\alpha_i}$ for all $\alpha_i \in \alpha$. Thus $\mathcal{W}_\alpha$ is an invariant subspace for T_j. Also, by commutativity of T we have

$$\langle T_j^* x, T_{\alpha_i} y\rangle = \langle T_{\alpha_i}^* T_j^* x, y\rangle = \langle T_j^* T_{\alpha_i}^* x, y\rangle = 0,$$

for all $\alpha_i \in \alpha$ and $y \in \mathcal{H}$. This implies $T_j^* \mathcal{W}_\alpha \perp T_{\alpha_i}\mathcal{H}$ and then $T_j^* \mathcal{W}_\alpha \subseteq \mathcal{W}_{\alpha_i}$ for all $\alpha_i \in \alpha$. This completes the proof. □

Now we prove the main theorem in the general Hilbert space operator setting. Below for a set α, we denote by $\#\alpha$ the cardinality of α.

Theorem 2.3. *Let $T = (T_1, \dots, T_n)$ be a doubly commuting tuple of operators on $\mathcal{H}$ such that for any reducing subspace $\mathcal{S}_i$ of T_i, the subspace*

$$\mathcal{S}_i \ominus T_i \mathcal{S}_i$$

is a generating wandering subspace for $T_i|_{\mathcal{S}_i}$, $i = 1, \dots, n$. Then for each non-empty subset $\alpha = \{\alpha_1, \dots, \alpha_k\} \subseteq \Lambda_n$, the tuple $T_\alpha = (T_{\alpha_1}, \dots, T_{\alpha_k})$ has the generating wandering subspace property. Moreover, the corresponding generating wandering subspace is given by

$$\mathcal{W}_\alpha = \bigcap_{i=1}^{k} (\mathcal{H} \ominus T_{\alpha_i}\mathcal{H}).$$

Proof. First note that we only need to show $\mathcal{H} = [\mathcal{W}_\alpha]_{T_\alpha}$ for any non-empty subset α of Λ_n as the orthogonal property for wandering subspace is immediate. Now for a singleton set α the result follows form the assumption that $\mathcal{W}_i$ is a generating wandering subspace for T_i, $i = 1, \dots, n$. Now for $\#\alpha \geq 2$, it suffices to show that $[\mathcal{W}_\alpha]_{T_{\alpha_i}} = \mathcal{W}_{\alpha \setminus \{\alpha_i\}}$ for any $\alpha_i \in \alpha$. Because for $\alpha_i, \alpha_j \in \alpha$, one can repeat the procedure to get

$$[\mathcal{W}_\alpha]_{T_{\{\alpha_i, \alpha_j\}}} = [[\mathcal{W}_\alpha]_{T_{\alpha_i}}]_{T_{\alpha_j}} = \mathcal{W}_{\alpha \setminus \{\alpha_i, \alpha_j\}},$$

and continue this process until the set $\alpha \setminus \{\alpha_i, \alpha_j\}$ becomes a singleton set and finally apply the assumption for singleton set.

To this end, let $\alpha \subseteq \Lambda_n, \#\alpha \geq 2$ and $\alpha_i \in \alpha$. Consider the set $F = \mathcal{W}_{\alpha \setminus \{\alpha_i\}} \ominus T_{\alpha_i}(\mathcal{W}_{\alpha \setminus \{\alpha_i\}})$. Now by Proposition 2.2, $\mathcal{W}_{\alpha \setminus \{\alpha_i\}}$ is a reducing subspace for T_{α_i} and therefore

$$\begin{aligned} F &= \{x \in \mathcal{W}_{\alpha \setminus \{\alpha_i\}} : x \in \operatorname{Ker} T_{\alpha_i}^*\} \\ &= \mathcal{W}_{\alpha \setminus \{\alpha_i\}} \cap \mathcal{W}_{\alpha_i} \\ &= \mathcal{W}_\alpha. \end{aligned}$$

On the other hand, since $\mathcal{W}_{\alpha \setminus \{\alpha_i\}}$ is a reducing subspace for T_{α_i} then by assumption $F = \mathcal{W}_\alpha$ is a wandering subspace for T_{α_i}. Thus

$$[\mathcal{W}_\alpha]_{T_{\alpha_i}} = \mathcal{W}_{\alpha \setminus \{\alpha_i\}}.$$

This completes the proof. □

Combining Theorem 2.1 with the above theorem we have the following result, which is the case of our main interest.

Corollary 2.4. *Let $T = (T_1, \dots, T_n)$ be a commuting tuple of analytic operators on $\mathcal{H}$ such that T is doubly commuting and satisfies one of the following properties:*
(a) *T_i is concave for each $i = 1, \dots, n$,*
(b) $\|T_i x + y\|^2 \leq 2(\|x\|^2 + \|T_i y\|^2)$ $(x, y \in \mathcal{H}, i = 1, 2, \dots, n)$.

Then for any non-empty subset $\alpha = \{\alpha_1, \ldots, \alpha_k\} \subseteq \Lambda_n$, $\mathcal{W}_\alpha$ *is a generating wandering subspace for* $T_\alpha = (T_{\alpha_1}, \ldots, T_{\alpha_k})$, *where* $\mathcal{W}_\alpha$ *is as in* (2.2).

Proof. Note that if T satisfies condition (a) or (b) then for any $1 \le i \le n$ and reducing subspace $\mathcal{S}_i$ of T_i, $T_i|_{\mathcal{S}_i}$ also satisfies condition (a) or (b) respectively. Thus by Theorem 2.1, T satisfies the hypothesis of the above theorem and the proof follows. $\square$

We end this section by proving kind of converse of the above result and part of which is a generalization of [7], Theorem 3.

Theorem 2.5. *Let* $T = (T_1, \ldots, T_n)$ *be an* n*-tuple of commuting operators on* $\mathcal{H}$ *with the property*

$$\|T_i x + y\|^2 \le 2(\|x\|^2 + \|T_i y\|^2) \ (x, y \in \mathcal{H}, i = 1, 2, \ldots, n)$$

or T_i *is concave for all* $i = 1, \ldots, n$. *Then,*
(i) T *is doubly commuting on* $\mathcal{H}$, *and*
(ii) T_i *is analytic for all* $i = 1, 2, \ldots, n$
if and only if
(a) *for any non-empty subset* $\alpha = \{\alpha_1, \ldots, \alpha_k\} \subseteq \Lambda_n$, $\mathcal{W}_\alpha$ *is a generating wandering subspace for* $T_\alpha = (T_{\alpha_1}, \ldots, T_{\alpha_k})$ *and for* $k \ge 2$, $[\mathcal{W}_\alpha]_{T_{\alpha_i}} = \mathcal{W}_{\alpha \setminus \{\alpha_i\}}$ *for all* $\alpha_i \in \alpha$,
(b) T_i *commutes with* $T_j^* T_j$ *for all* $1 \le i < j \le n$.

Proof. The forward direction follows from Theorem 2.3.

For the converse, suppose that (a) and (b) hold. To show (i), let $1 \le i \le n$ be fixed. By assumption (a), $\mathcal{W}_i = [\mathcal{W}_{\{i,j\}}]_{T_j}$ for all $1 \le i \ne j \le n$. This shows that $\mathcal{W}_i$ is T_j invariant subspace for all $1 \le j \ne i \le n$. Let $x \in \mathcal{H}$. Since $[\mathcal{W}_i]_{T_i} = \mathcal{H}$ then there exists a sequence x_k converging to x such that

$$x_k = \sum_{m=0}^{N_k} T_i^m x_{m,k},$$

where each N_k is a nonnegative integer and $x_{m,k}$ is a member of $\mathcal{W}_i$. Now for any $1 \le j \ne i \le n$ we have,

$$T_i^* T_j x_k = \sum_{m=0}^{N_k} T_i^* T_j T_i^m x_{m,k} = \sum_{m=1}^{N_k} T_i^* T_j T_i^m x_{m,k}.$$

On the other hand,

$$T_j T_i^* x_k = \sum_{m=0}^{N_k} T_j T_i^* T_i^m x_{m,k} = \sum_{m=1}^{N_k} T_j T_i^* T_i^m x_{m,k} = \sum_{m=1}^{N_k} T_i^* T_j T_i^m x_{m,k},$$

where the last equality follows from (b). So $T_i^* T_j x_k = T_j T_i^* x_k$ and by taking limit $T_i^* T_j x = T_j T_i^* x$. Thus we have (i).
Finally, by Theorem 3.6 of [10] we have that

$$\mathcal{H} = [\mathcal{W}_i]_{T_i} \oplus \bigcap_{m=1}^{\infty} T_i^m \mathcal{H},$$

for all $1 \leq i \leq n$. By part (a), $[\mathcal{W}_i]_{T_i} = \mathcal{H}$ for all $1 \leq i \leq n$. Thus $\bigcap\limits_{m=1}^{\infty} T_i^m \mathcal{H} = \{0\}$ for all $1 \leq i \leq n$ and this completes the proof. $\square$

3. Generating wandering subspaces for invariant subspaces of $A^2(\mathbb{D}^n)$ and $\mathcal{D}(\mathbb{D}^n)$

In this section we apply the general theorem proved in the previous section to obtain generating wandering subspaces for invariant subspaces of the Bergman space and the Dirichlet space over polydisc.

The Bergman space over $\mathbb{D}^n$ is denoted by $A^2(\mathbb{D}^n)$ and defined by

$$A^2(\mathbb{D}^n) = \{f \in \mathcal{O}(\mathbb{D}^n) : \int_{\mathbb{D}^n} |f(\mathbf{z})|^2 d\mathbf{A}(\mathbf{z}) < \infty\},$$

where $d\mathbf{A}$ is the product area measure on $\mathbb{D}^n$.

Below for any invariant subspace $\mathcal{S}$ of $A^2(\mathbb{D}^n)$ and non-empty set $\alpha = \{\alpha_1, \ldots, \alpha_k\} \subseteq \Lambda_n$, we denote by $\mathcal{W}_\alpha^{\mathcal{S}}$ the following set:

$$\mathcal{W}_\alpha^{\mathcal{S}} := \bigcap_{i=1}^{k} (\mathcal{S} \ominus z_{\alpha_i} \mathcal{S}).$$

We denote by $M = (M_{z_1}, \ldots, M_{z_n})$ the n-tuple of co-ordinate multiplication operators on $A^2(\mathbb{D}^n)$.

Theorem 3.1. *Suppose $\mathcal{S}$ is a closed joint M-invariant subspace of $A^2(\mathbb{D}^n)$. If $\mathcal{S}$ is doubly commuting then for any non-empty subset $\alpha = \{\alpha_1, \ldots, \alpha_k\}$ of Λ_n, $\mathcal{W}_\alpha^{\mathcal{S}}$ is a generating wandering subspace for $M_\alpha|_{\mathcal{S}} = (M_{z_{\alpha_1}}|_{\mathcal{S}}, \ldots, M_{z_{\alpha_k}}|_{\mathcal{S}})$.*

The proof of the above theorem follows if we show the tuple of operators

$$(M_{z_1}|_{\mathcal{S}}, \ldots, M_{z_n}|_{\mathcal{S}})$$

on $\mathcal{S}$ satisfies all the hypothesis of Corollary 2.4. First note that by analyticity of $A^2(\mathbb{D}^n)$, all the co-ordinate multiplication operators $M_{z_i}, i = 1, \ldots, n$, on $A^2(\mathbb{D}^n)$ are analytic. Then $M_{z_i}|_{\mathcal{S}}$ is also analytic for all $i = 1, \ldots, n$. Thus the only hypothesis which remains to verify is either condition (a) or (b) of Corollary 2.4. In the next lemma we show that the tuple $(M_{z_1}, \ldots, M_{z_n})$ satisfies condition (b) and therefore so does $(M_{z_1}|_{\mathcal{S}}, \ldots, M_{z_n}|_{\mathcal{S}})$.

Lemma 3.2. *For $1 \leq i \leq n$, the operator*

$$M_{z_i} : A^2(\mathbb{D}^n) \to A^2(\mathbb{D}^n), \quad f \mapsto z_i f,$$

satisfies

$$\|M_{z_i} f + g\|^2_{A^2(\mathbb{D}^n)} \leq 2 \left(\|f\|^2_{A^2(\mathbb{D}^n)} + \|M_{z_i} g\|^2_{A^2(\mathbb{D}^n)} \right),$$

for all $f, g \in A^2(\mathbb{D}^n)$.

Before we prove this lemma, we recall a well known fact regarding the norm of a function in $A^2(\mathbb{D}^n)$ and prove an inequality. If f is in $A^2(\mathbb{D}^n)$ with the following power series expansion corresponding to i-th variable:

$$f(z_1, \cdots, z_n) = \sum_{k=0}^{\infty} \mathbf{a}_k z_i^k,$$

where $\mathbf{a}_k \in A^2(\mathbb{D}^{n-1})$ for all $k \in \mathbb{N}$, then

$$\|f\|^2_{A^2(\mathbb{D}^n)} = \sum_{k=0}^{\infty} \frac{\|\mathbf{a}_k\|^2_{A^2(\mathbb{D}^{n-1})}}{(k+1)}.$$

Next we prove the following inequality (can be found in [4], page 277, we include the proof for completeness) for any $z, w \in \mathbb{C}$ and $k \in \mathbb{N} \setminus \{0\}$,

$$\frac{|z+w|^2}{k+1} \leq 2\left(\frac{|z|^2}{k} + \frac{|w|^2}{k+2}\right). \tag{3.1}$$

To this end, note that $2k(k+2)\operatorname{Re}(z\bar{w}) \leq (k+2)^2|z|^2 + k^2|w|^2$, which follows from the inequality $|(k+2)z - kw|^2 \geq 0$. Now

$$\begin{aligned}
\frac{|z+w|^2}{k+1} &= \frac{|z|^2 + |w|^2 + 2\operatorname{Re}(z\bar{w})}{k+1} \\
&\leq \frac{|z|^2 + |w|^2}{k+1} + \frac{|z|^2(k+2)/k + |w|^2 k/(k+2)}{k+1} \\
&= 2\left(\frac{|z|^2}{k} + \frac{|w|^2}{k+2}\right).
\end{aligned}$$

Now we prove the lemma.

Proof. Let $1 \leq i \leq n$ be fixed. Let $f(z_1, \cdots, z_n) = \sum\limits_{k=0}^{\infty} \mathbf{a}_k z_i^k$ and $g(z_1, \cdots, z_n) = \sum\limits_{k=0}^{\infty} \mathbf{b}_k z_i^k$ be the power series expansions of two functions $f, g \in A^2(\mathbb{D}^n)$ with respect to z_i-th variable, where $\mathbf{a}_k, \mathbf{b}_k \in A^2(\mathbb{D}^{n-1})$ for all $k \in \mathbb{N}$. Then

$$(M_{z_i} f + g) = \sum_{k=0}^{\infty} \mathbf{a}_k z_i^{k+1} + \sum_{k=0}^{\infty} \mathbf{b}_k z_i^k = \mathbf{b}_0 + \sum_{k=1}^{\infty} (\mathbf{a}_{k-1} + \mathbf{b}_k) z_i^k.$$

Now

$$\begin{aligned}
\|M_{z_i} f + g\|^2_{A^2(\mathbb{D}^n)} &= \|\mathbf{b}_0\|^2_{A^2(\mathbb{D}^{n-1})} + \sum_{k=1}^{\infty} \frac{\|\mathbf{a}_{k-1} + \mathbf{b}_k\|^2_{A^2(\mathbb{D}^{n-1})}}{(k+1)} \\
&\leq \|\mathbf{b}_0\|^2_{A^2(\mathbb{D}^{n-1})} + 2\sum_{k=1}^{\infty} \left(\frac{\|\mathbf{a}_{k-1}\|^2_{A^2(\mathbb{D}^{n-1})}}{k} + \frac{\|\mathbf{b}_k\|^2_{A^2(\mathbb{D}^{n-1})}}{k+2}\right) \quad (\text{ by } (3.1)) \\
&= 2\left(\sum_{k=0}^{\infty} \frac{\|\mathbf{a}_k\|^2_{A^2(\mathbb{D}^{n-1})}}{k+1} + \sum_{k=1}^{\infty} \frac{\|\mathbf{b}_{k-1}\|^2_{A^2(\mathbb{D}^{n-1})}}{k+1}\right) \\
&= 2\left(\|f\|^2_{A^2(\mathbb{D}^n)} + \|M_{z_i} g\|^2_{A^2(\mathbb{D}^n)}\right).
\end{aligned}$$

The proof follows. $\square$

Now we turn our attention to the Dirichlet space over polydisc. The Dirichlet space over $\mathbb{D}$ is denoted by $\mathcal{D}(\mathbb{D})$ and defined by

$$\mathcal{D}(\mathbb{D}) := \{f \in \mathcal{O}(\mathbb{D}) : f' \in A^2(\mathbb{D})\}.$$

For any $f = \sum\limits_{k=0}^{\infty} a_k z^k \in \mathcal{D}(\mathbb{D})$, $\|f\|_{\mathcal{D}}^2 = \sum\limits_{k=0}^{\infty}(k+1)|a_k|^2$. The Dirichlet space over $\mathbb{D}^n$ is denoted by $\mathcal{D}(\mathbb{D}^n)$ and defined by

$$\mathcal{D}(\mathbb{D}^n) := \underbrace{\mathcal{D}(\mathbb{D}) \otimes \cdots \otimes \mathcal{D}(\mathbb{D})}_{\text{n-times}}.$$

Another way of expressing $\mathcal{D}(\mathbb{D}^n)$ is the following

$$\mathcal{D}(\mathbb{D}^n) := \{f \in \mathcal{O}\big(\mathbb{D}; \mathcal{D}(\mathbb{D}^{n-1})\big) : f = \sum_{k=0}^{\infty} \mathbf{a}_k z^k, \sum_{k=0}^{\infty}(k+1)\|\mathbf{a}_k\|^2_{\mathcal{D}(\mathbb{D}^{n-1})} < \infty\}.$$

In the above, $\mathbf{a}_k \in \mathcal{D}(\mathbb{D}^{n-1})$ for all $k \in \mathbb{N}$. The co-ordinate multiplication operators on $\mathcal{D}(\mathbb{D}^n)$ are also denoted by M_{z_i}, $i = 1, \ldots, n$. Since $\mathcal{D}(\mathbb{D}^n)$ contains holomorphic functions on $\mathbb{D}^n$ then M_{z_i} is analytic for all $i = 1, \ldots, n$. Let $1 \leq i \leq n$ be fixed. Now for $f \in \mathcal{D}(\mathbb{D}^n)$, let $f = \sum\limits_{k=0}^{\infty} \mathbf{a}_k z_i^k$ be the Taylor expansion of f corresponding to z_i-th variable, where $\mathbf{a}_k \in \mathcal{D}(\mathbb{D}^{n-1})$ for all $k \in \mathbb{N}$. Then $\|f\|^2 = \sum\limits_{k=0}^{\infty}(k+1)\|\mathbf{a}_k\|^2_{\mathcal{D}(\mathbb{D}^{n-1})}$ and

$$\begin{aligned}
\|M_{z_i}^2 f\|^2 + \|f\|^2 &= \sum_{k=2}^{\infty}(k+1)\|\mathbf{a}_{k-2}\|^2_{\mathcal{D}(\mathbb{D}^{n-1})} + \sum_{k=0}^{\infty}(k+1)\|\mathbf{a}_k\|^2_{\mathcal{D}(\mathbb{D}^{n-1})} \\
&= \sum_{k=0}^{\infty}(k-1)\|\mathbf{a}_k\|^2_{\mathcal{D}(\mathbb{D}^{n-1})} + \sum_{k=0}^{\infty}(k+1)\|\mathbf{a}_k\|^2_{\mathcal{D}(\mathbb{D}^{n-1})} \\
&= 2\sum_{k=1}^{\infty} k\|\mathbf{a}_k\|^2_{\mathcal{D}(\mathbb{D}^{n-1})} \\
&= 2\|M_{z_i} f\|^2.
\end{aligned}$$

Therefore M_{z_i} is concave for all $i = 1, \ldots, n$. Thus again by Corollary 2.4, we have proved the following result of generating wandering subspaces for invariant subspaces of Dirichlet space over polydisc.

Theorem 3.3. *Suppose $\mathcal{S}$ is a closed joint $(M_{z_1}, \ldots, M_{z_n})$-invariant subspace of $\mathcal{D}(\mathbb{D}^n)$. If $\mathcal{S}$ is doubly commuting then for any non-empty subset $\alpha = \{\alpha_1, \ldots, \alpha_k\}$ of Λ_n, $\mathcal{W}_\alpha^{\mathcal{S}}$ is a generating wandering subspace for $M_\alpha|_{\mathcal{S}} = (M_{z_{\alpha_1}}|_{\mathcal{S}}, \ldots, M_{z_{\alpha_k}}|_{\mathcal{S}})$, where*

$$\mathcal{W}_\alpha^{\mathcal{S}} = \bigcap_{i=1}^{k}(\mathcal{S} \ominus z_{\alpha_i}\mathcal{S}).$$

We conclude the paper with the remark that since the tuple of coordinate multiplication operators $(M_{z_1}, \dots, M_{z_n})$ on the Bergman space or the Dirichlet space over polydisc satisfies the hypothesis of Theorem 2.5, the same conclusion as in Theorem 2.5 holds for invariant subspaces of Bergman space or Dirichlet space over polydisc.

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