

ORTHOGONALITY IN BANACH SPACES VIA PROJECTIVE TENSOR PRODUCT

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ABSTRACT. Let X be a complex Banach space and $x, y \in X$. By definition, we say that x is Birkhoff-James orthogonal to y if $\|x + \lambda y\|_X \geq \|x\|_X$ for all $\lambda \in \mathbb{C}$. We prove that x is Birkhoff-James orthogonal to y if and only if there exists a semi-inner product φ on X such that $\|\varphi\| = 1$, $\varphi(x, x) = \|x\|^2$ and $\varphi(x, y) = 0$. A similar result holds for C^* -algebras. A key point in our approach to orthogonality is the representations of bounded bilinear maps via projective tensor product spaces.

1. INTRODUCTION

This paper deals with the notion of Birkhoff-James orthogonality [2, 8, 9] for vectors in Banach spaces via projective tensor product of Banach spaces. Let X be a normed linear space over the scalar field \mathbb{K} , where \mathbb{K} is either \mathbb{R} or \mathbb{C} . Suppose $x, y \in X$. We say that x is *Birkhoff-James orthogonal* (or simply *orthogonal*) to y if

$$\|x + \lambda y\|_X \geq \|x\|_X,$$

for all $\lambda \in \mathbb{K}$. We denote this by $x \perp_B y$. If X is a Hilbert space, then this notion coincides with the usual orthogonality, that is, $x \perp_B y$ if and only if

$$\langle x, y \rangle_X = 0.$$

The objective of this note is to present an abstract characterization of orthogonality of vectors in Banach spaces over \mathbb{K} . We are motivated by Bhatia and Šemrl's investigation [3, Theorem 1.1 and Remark 3.1] of orthogonality in the setting of bounded linear operators on Hilbert spaces:

Theorem 1.1. (*Bhatia and Šemrl*) *Let S and T be bounded operators on a Hilbert space \mathcal{H} .*

(i) *Then $S \perp_B T$ if and only if there exists a sequence $\{h_n\}$ of unit vectors such that $\|Sh_n\| \rightarrow \|S\|$, and $\langle Sh_n, Th_n \rangle \rightarrow 0$ as $n \rightarrow \infty$.*

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(ii) If we additionally assume that \mathcal{H} is a finite dimensional Hilbert space, then $S \perp_B T$ if and only if there exists a unit vector $h \in \mathcal{H}$ such that

$$\|Sh\| = \|S\| \quad \text{and} \quad \langle Sh, Th \rangle = 0.$$

We treat the above Bhatia-Šemrl result (more specifically, part (ii) of Theorem 1.1) as a paradigm and examine orthogonality of vectors in Banach spaces in terms of semi-inner products. Our main result, in the setting of Banach spaces, is the following:

Theorem 1.2. *Let X be a Banach space, $x, y \in X$, and $x \neq 0$. Then $x \perp_B y$ if and only if there exists a semi-inner product $\varphi : X \oplus_\infty X \rightarrow \mathbb{K}$ such that $\|\varphi\| = 1$ and*

$$\varphi(x, x) = \|x\|^2 \quad \text{and} \quad \varphi(x, y) = 0.$$

Actually, this also holds for normed linear spaces, but we will return to it at the end of this paper.

As an immediate consequence of the above, we also prove an orthogonality result in the setting of C^* -algebras (see Corollary 2.2). Here, for a pair of Banach spaces X and Y , we define $X \oplus_\infty Y$ to be the Banach space

$$X \times Y = \{(x, y) : x \in X, y \in Y\},$$

with the norm

$$\|(x, y)\|_\infty = \max\{\|x\|_X, \|y\|_Y\},$$

for all $x \in X$ and $y \in Y$. Also recall that a semi-inner product [6, Page 1] on a vector space V is scalar-valued function $\varphi : V \times V \rightarrow \mathbb{K}$ such that for all $x, y, z \in V$ and $\alpha, \beta \in \mathbb{K}$, we have

- (a) $\varphi(\alpha x + \beta y, z) = \alpha\varphi(x, z) + \beta\varphi(y, z)$,
- (b) $\varphi(x, y) = \overline{\varphi(y, x)}$,
- (c) $\varphi(x, x) \geq 0$.

It is easy to check that, if φ is a semi-inner product, then φ is an inner product if and only if $\varphi(x, x) = 0$ implies $x = 0$. It is worth noting that the notion of a semi-inner product is defined somewhat differently by different researchers. In the orthogonality context, we point out the papers by Giles [7] and Lumer [11].

The main ingredients of our approach to the orthogonality problem are: (1) Bhatia and Šemrl's orthogonality of bounded linear operators on infinite-dimensional Hilbert spaces (part (i) of Theorem 1.1 above), (2) projective tensor product techniques (see Theorem 1.3 below), and (3) some standard Banach space techniques (like Banach-Alaoglu theorem).

Before we proceed with the main content of the paper, let us shortly review the existing literature on orthogonality. The notion of Birkhoff-James orthogonality is an active research area. In fact, Bhatia and Šemrl's paper [3] on orthogonality of operators

on Hilbert spaces has stimulated extensive research for the past two decades. For instance, orthogonality of a pair of compact operators acting on a reflexive Banach space to a normed linear space has been studied by Sain, Paul and Mal [16]. Their investigation involves finer geometric Banach space techniques. Also see [14] on orthogonality of linear operators on finite dimensional Banach spaces, [15] on norm attainment and orthogonality, and [12] on approximate orthogonality. More recent advances can be found, for instance, in the quickly growing literature [1, 4, 5, 10, 18, 19] (also see the references therein).

Given Banach spaces X and Y , we denote the Banach space of all bounded linear operators from X to Y by $\mathcal{B}(X, Y)$, and we let X_1 denote the closed unit ball in X . If $Y = X$, then we write $\mathcal{B}(X)$. A bilinear (sesquilinear) map $B : X \times Y \rightarrow Z$ is said to be bounded if there exists $M > 0$ such that

$$\|B(x, y)\| \leq M \quad (x \in X_1, y \in Y_1).$$

We denote the Banach space of all bounded bilinear (sesquilinear) maps from $X \times Y$ to Z by $\text{Bil}(X \times Y, Z)$ ($\text{Ses}(X \times Y, Z)$). Here

$$\|B\| = \sup\{\|B(x, y)\| : x \in X_1, y \in Y_1\},$$

for all $B \in \text{Bil}(X \times Y, Z)$ ($B \in \text{Ses}(X \times Y, Z)$). As a tool for the proof of the main result, we use the notion of projective tensor product. The projective tensor product $X \hat{\otimes}_\pi Y$ of Banach spaces X and Y is the completion of the algebraic tensor product $X \otimes Y$ under the projective norm

$$\|u\|_\pi = \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

Our key point is the following result [13, Theorem 2.9] concerning representations of bounded bilinear maps via projective tensor product spaces.

Theorem 1.3. *Let X, Y and Z be Banach spaces, and let $B \in \text{Bil}(X \times Y, Z)$. Then there exists a unique $\tilde{B} \in \mathcal{B}(X \hat{\otimes}_\pi Y, Z)$ such that*

$$\tilde{B}(x \otimes y) = B(x, y) \quad (x \in X, y \in Y).$$

Moreover, the correspondence $B \longleftrightarrow \tilde{B}$ is an isometric isomorphism between $\text{Bil}(X \times Y, Z)$ and $\mathcal{B}(X \hat{\otimes}_\pi Y, Z)$.

We fix some more notation that we will use from now on. Given a vector space X over \mathbb{K} , we denote by \overline{X} the complex conjugate vector space of X . That is, $\overline{X} = X$ with the same additive group structures, but with the scalar multiplication \star defined by

$$(1.1) \quad \alpha \star x = \overline{\alpha}x,$$

for all $\alpha \in \mathbb{K}$ and $x \in \overline{X}$. Clearly, if $(X, \|\cdot\|)$ is normed linear space over \mathbb{K} , then there is an anti-linear isometric isomorphism between $(X, \|\cdot\|)$ and $(\overline{X}, \|\cdot\|)$. If \mathcal{H} is a Hilbert space, then $\overline{\mathcal{H}}$ is identified with the dual (the space of continuous linear functionals) of \mathcal{H} by Riesz representation theorem.

Note that the space $\text{Bil}(X \oplus_\infty \overline{X}, \mathbb{K})$ is isometrically isomorphic to the space $\text{Ses}(X \oplus_\infty X, \mathbb{K})$. The correspondence is given by

$$(1.2) \quad \psi(x, y) = \varphi(x, y),$$

where $\psi \in \text{Bil}(X \oplus_\infty X, \mathbb{K})$, $\varphi \in \text{Ses}(X \oplus_\infty X, \mathbb{K})$, $x \in X$ and $y \in Y$.

Note added in proof: After completion of the present paper, Professor Kallol Paul kindly pointed out to us that our main result, Theorem 1.2 also follows (with a little more work) from James [9, Theorem 2.1]. Note, however, that our result uses the technique of projective tensor product and some basic geometric technique of Banach spaces. Our approach connects two subtle notions, namely, orthogonality in Banach spaces and projective tensor product of Banach spaces, which is different in the spirit of the ongoing Birkhoff-James orthogonality program. As we will soon see, the semi-inner product φ in Theorem 1.2 is naturally governed by projective tensor product of Banach spaces. We believe that the inherited structure of projective tensor product of Banach spaces in orthogonality will prove useful in future investigations.

2. MAIN RESULTS

We begin with the proof of Theorem 1.2. But before we do so, let us recall the classical Banach–Alaoglu theorem: Let X be a Banach space over \mathbb{K} . Then the closed unit ball $(X^*)_1$ of the dual X^* is compact with respect to the weak* topology on X^* .

Proof of Theorem 1.2: Suppose $x \perp_B y$. Note that the Banach space X is isometrically isomorphic to a closed subspace of $C((X^*)_1)$, where $(X^*)_1$ endowed with the weak* topology is a compact set. Here the correspondence is given by the formula $X \ni u \mapsto \hat{u}$, where

$$\hat{u}(f) = f(u) \quad (f \in (X^*)_1).$$

Next we note that the commutative Banach algebra $C((X^*)_1)$ is isometrically isomorphic to a closed subspace of $\mathcal{B}(L^2(\mu))$ for some σ -finite measure μ . Here the correspondence is given by the formula $C((X^*)_1) \ni g \mapsto M_g$, where $M_g : L^2(\mu) \rightarrow L^2(\mu)$ is the multiplication operator defined by

$$M_g(h) = gh \quad (h \in L^2(\mu)).$$

Using the above identifications, we have $M_{\hat{x}}, M_{\hat{y}} \in \mathcal{B}(L^2(\mu))$ and

$$M_{\hat{x}} \perp_B M_{\hat{y}}.$$

By the infinite dimensional part of Theorem 1.1, there exists a sequence $\{h_n\}$ of unit vectors in $L^2(\mu)$ such that

$$\|M_{\hat{x}}(h_n)\|_{L^2(\mu)} \rightarrow \|M_{\hat{x}}\|_{\mathcal{B}(L^2(\mu))},$$

and

$$\langle M_{\hat{x}}(h_n), M_{\hat{y}}(h_n) \rangle_{L^2(\mu)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

For each $n \geq 1$, define $\psi_n : X \oplus_\infty \overline{X} \rightarrow \mathbb{K}$ by

$$\psi_n(z, w) = \langle M_{\hat{z}}(h_n), M_{\hat{w}}(h_n) \rangle_{L^2(\mu)} \quad (z \in X, w \in \overline{X}).$$

Now we prove that ψ_n is bilinear. Clearly, ψ_n is linear in its first variable. Suppose $\alpha_1, \alpha_2 \in \mathbb{K}$ and $z, w_1, w_2 \in X$. Then

$$\begin{aligned} \psi_n(z, (\alpha_1 \star w_1 + \alpha_2 \star w_2)) &= \langle M_{\hat{z}}(h_n), M_{\widehat{\alpha_1 \star w_1 + \alpha_2 \star w_2}}(h_n) \rangle \\ &= \langle M_{\hat{z}}(h_n), M_{\widehat{\alpha_1 w_1 + \alpha_2 w_2}}(h_n) \rangle \\ &= \langle \hat{z}h_n, (\alpha_1 \hat{w}_1 + \alpha_2 \hat{w}_2)h_n \rangle \\ &= \alpha_1 \langle \hat{z}h_n, \hat{w}_1 h_n \rangle + \alpha_2 \langle \hat{z}h_n, \hat{w}_2 h_n \rangle, \end{aligned}$$

that is

$$\psi_n(z, (\alpha_1 \star w_1 + \alpha_2 \star w_2)) = \alpha_1 \psi_n(z, w_1) + \alpha_2 \psi_n(z, w_2),$$

and hence ψ_n is bilinear for all $n \geq 1$. To prove that ψ_n is bounded, suppose $(z, w) \in X \oplus_\infty \overline{X}$ and $\|(z, w)\|_\infty \leq 1$. Then

$$\begin{aligned} |\psi_n(z, w)| &\leq \|M_{\hat{z}}\|_{\mathcal{B}(L^2(\mu))} \|M_{\hat{w}}\|_{\mathcal{B}(L^2(\mu))} \|h_n\|_{L^2(\mu)}^2 \\ &= \|\hat{z}\|_{C((X^*)_1)} \|\hat{w}\|_{C((X^*)_1)} \\ &= \|z\|_X \|w\|_X, \end{aligned}$$

which implies that $\|\psi_n\| \leq 1$ for all n . Consequently $\{\psi_n\}_{n \geq 1} \subseteq \text{Bil}(X \oplus_\infty \overline{X}, \mathbb{K})$. Note, by Theorem 1.3, that

$$\text{Bil}(X \oplus_\infty \overline{X}, \mathbb{K}) \simeq \mathcal{B}(X \hat{\otimes}_\pi \overline{X}, \mathbb{K}).$$

Here the correspondence is given by $\text{Bil}(X \oplus_\infty \overline{X}, \mathbb{K}) \ni \psi \mapsto \tilde{\psi}$, where

$$\tilde{\psi}(z \otimes w) = \psi(z, w) \quad (z, w \in X).$$

Since $\|\psi_n\| \leq 1$, we have $\|\tilde{\psi}_n\| \leq 1$ for each n . By applying the Banach-Alaoglu theorem, one can find a subsequence $\{\tilde{\psi}_{n_k}\}$ of $\{\tilde{\psi}_n\}$ and a map $\tilde{\psi} \in \mathcal{B}(X \hat{\otimes}_\pi \overline{X}, \mathbb{K})$ with $\|\tilde{\psi}\| \leq 1$ such that

$$\tilde{\psi}_{n_k} \xrightarrow{w^*} \tilde{\psi}.$$

This yields $\psi_{n_k} \rightarrow \psi$ in the pointwise topology, that is

$$\psi_{n_k}(z, w) \rightarrow \psi(z, w) \quad (z, w \in X).$$

By isometry, $\|\psi\| \leq 1$. Observe that

$$\psi_{n_k}(x, x) = \langle M_{\hat{x}}(h_{n_k}), M_{\hat{x}}(h_{n_k}) \rangle_{L^2(\mu)} = \|M_{\hat{x}}(h_{n_k})\|_{L^2(\mu)}^2.$$

But since

$$\|M_{\hat{x}}(h_{n_k})\|_{L^2(\mu)}^2 \rightarrow \|M_{\hat{x}}\|_{\mathcal{B}(L^2(\mu))}^2 = \|\hat{x}\|^2,$$

it follows that

$$\psi_{n_k}(x, x) \rightarrow \|x\|^2,$$

for all $x \in X$. On the other hand, $\psi_{n_k}(x, x) \rightarrow \psi(x, x)$. Hence

$$\psi(x, x) = \|x\|^2 \quad (x \in X).$$

Since

$$\psi_{n_k}(x, y) = \langle M_{\hat{x}}(h_{n_k}), M_{\hat{x}}(h_{n_k}) \rangle_{L^2(\mu)} \rightarrow 0,$$

we have $\psi(x, y) = 0$. Note that if $\|\psi\| < 1$, then $\psi(x, x) < \|x\|^2$. Hence $\|\psi\| = 1$. Now observe that the identification in (1.2) provides a sesquilinear map $\varphi \in \text{Ses}(X \oplus_\infty X, \mathbb{K})$, with

$$\varphi(x, y) = \psi(x, y) \quad (x, y \in X).$$

This gives $\varphi(z, z) \geq 0$ for all $z \in X$, and $\varphi(z, w) = \overline{\varphi(w, z)}$ for all $z, w \in X$. Hence φ is a semi-inner product on X such that $\|\varphi\| = 1$, $\varphi(x, x) = \|x\|^2$ and $\varphi(x, y) = 0$.

Conversely, suppose that such a φ exists. Then for all $\lambda \in \mathbb{K}$,

$$\|x\|^2 = |\varphi(x, x + \lambda y)| \leq \|\varphi\| \|x\| \|x + \lambda y\|,$$

which implies that $\|x\| \leq \|x + \lambda y\|$, that is, $x \perp_B y$. This completes the proof of the theorem.

Remark 2.1. It is worth pointing out that besides Theorem 1.1 and Theorem 1.3, the inclusion

$$X \hookrightarrow C((X^*)_1) \hookrightarrow \mathcal{B}(L^2(\mu)),$$

also plays an important role in our proof.

Now we turn to orthogonality of elements in C^* -algebras. The following result follows from Theorem 1.2.

Corollary 2.2. *Let \mathcal{A} be a C^* -algebra and $a, b \in \mathcal{A}$ and $a \neq 0$. Then $a \perp_B b$ if and only if there exists a bilinear map $\psi : \mathcal{A} \oplus_\infty \mathcal{A} \rightarrow \mathbb{C}$ such that $\|\psi\| = 1$, $\psi(a, a^*) = \|a\|^2$ and $\psi(a, b^*) = 0$.*

Proof. Suppose $a \perp_B b$. Then by Theorem 1.2, there exists a semi-inner product $\varphi : \mathcal{A} \oplus_\infty \mathcal{A} \rightarrow \mathbb{C}$ such that $\|\varphi\| = 1$, $\varphi(a, a) = \|a\|^2$ and $\varphi(a, b) = 0$. Define $\psi : \mathcal{A} \oplus_\infty \mathcal{A} \rightarrow \mathbb{C}$ by

$$\psi(a, b) = \varphi(a, b^*).$$

Then ψ is the required map. To prove the converse, suppose such a ψ exists. Then

$$\begin{aligned} \|a\|^2 &= |\psi(a, (a + \lambda b)^*)| \\ &\leq \|\psi\| \|a\| \|(a + \lambda b)^*\| \\ &= \|a\| \|a + \lambda b\|, \end{aligned}$$

that is, $\|a + \lambda b\| \geq \|a\|$ for all $\lambda \in \mathbb{C}$. □

We refer the reader to [4, Proposition 4.1] for a more natural version of orthogonality in the setting of C^* -algebras.

It is worth pointing out that Theorem 1.2 is also applicable to finite dimensional Banach spaces. Clearly, Theorem 1.2 and Corollary 2.2 are analogous to Bhatia and Šemrl classifications (see part (ii) of Theorem 1.1) of orthogonality of matrices on finite dimensional Hilbert spaces. On the other hand, our results uses the Bhatia and Šemrl classifications of orthogonality in the setting of infinite dimensional Hilbert spaces (see part (i) of Theorem 1.1). In addition, it is not completely clear if our results recovers the Bhatia and Šemrl classifications of orthogonality in the setting of finite matrices.

All in all, on one hand our results are valid for general Banach spaces and rather abstract, and on the other hand our approach is intimately related to the delicate structure of projective tensor product of Banach spaces (see Chapter 2 in [13]). We also believe that our approach to orthogonality via projective tensor product is of independent interest and may have other applications.

Finally, we remark that our main result Theorem 1.2 is also valid in full generality in the setting of normed linear spaces. The proof works verbatim. In this case, [17, page 443, Proposition 43.12. (b)] is the normed linear space counterpart to our key tool Theorem 1.3.

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