DOUBLY COMMUTING SUBMODULES OF THE HARDY MODULE OVER POLYDISCS

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Abstract. In this note we establish a vector-valued version of Beurling's Theorem (the Lax-Halmos Theorem) for the polydisc. As an application of the main result, we provide necessary and sufficient conditions for the "weak" completion problem in $H^\infty(\mathbb{D}^n)$.

1. Introduction and Statement of Main Results

In [B], Beurling described all the invariant subspaces for the operator M_z of "multiplication by z" on the Hilbert space $H^2(\mathbb{D})$ of the disc. In [L], Peter Lax extended Beurling's result to the (finite-dimensional) vector-valued case (while also considering the Hardy space of the half plane). Lax's vectorial case proof was further extended to infinite-dimensional vector spaces by Halmos, see [NF]. The characterization of M_z-invariant subspaces obtained is the following famous result.

Theorem 1.1 (Beurling-Lax-Halmos)**.** *Let $\mathcal{S}$ be a closed nonzero subspace of $H^2_{E_*}(\mathbb{D})$. Then $\mathcal{S}$ is invariant under multiplication by z if and only if there exists a Hilbert space E and an inner function $\Theta \in H^\infty_{E\to E_*}(\mathbb{D})$ such that $\mathcal{S} = \Theta H^2_E(\mathbb{D})$.*

For $n \in \mathbb{N}$ and E_* a Hilbert space, $H^2_{E_*}(\mathbb{D}^n)$ is the set of all E_*-valued holomorphic functions in the polydisc $\mathbb{D}^n$, where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ (with boundary $\mathbb{T}$) such that

$$\|f\|_{H^2_{E_*}(\mathbb{D}^n)} := \sup_{0<r<1} \Big(\int_{\mathbb{T}^n} \|f(r\mathbf{z})\|^2_{E_*} d\mathbf{z} \Big)^{1/2} < +\infty.$$

On the other hand, if $\mathcal{L}(E, E_*)$ denotes the set of all continuous linear transformations from E to E_*, then $H^\infty_{E\to E_*}(\mathbb{D}^n)$ denotes the set of all $\mathcal{L}(E, E_*)$-valued holomorphic functions with $\|f\|_{H^\infty_{E\to E_*}(\mathbb{D}^n)} := \sup_{\mathbf{z}\in\mathbb{D}^n} \|f(\mathbf{z})\|_{\mathcal{L}(E,E_*)} < \infty$.

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An operator-valued $\Theta \in H^\infty_{E\to E_*}(\mathbb{D}^n)$ *inner* if the pointwise a.e. boundary values are isometries:

$$(\Theta(\zeta))^*\Theta(\zeta) = I_E \text{ for almost all } \zeta \in \mathbb{T}^n.$$

A natural question is then to ask what happens in the case of several variables, for example when one considers the Hardy space $H^2_{E_*}(\mathbb{D}^n)$ of the polydisc $\mathbb{D}^n$. It is known that in general, a Beurling-Lax-Halmos type characterization of subspaces of the Hardy Hilbert space is not possible [R]. It is however, easy to see that the Hardy space on the polydisc $H^2_{E_*}(\mathbb{D}^n)$, when $n > 1$, satisfies the *doubly commuting* property, that is, for all $1 \leq i < j \leq n$

$$M^*_{z_i} M_{z_j} = M_{z_j} M^*_{z_i}.$$

We impose this additional assumption to the submodules of $H^2_{E_*}(\mathbb{D}^n)$ and call that class of submodules as doubly commuting submodules. More precisely:

Definition 1.2. A commuting family of bounded linear operators $\{T_1, \ldots, T_n\}$ on some Hilbert space $\mathcal{H}$ is said to be *doubly commuting* if

$$T_i T_j^* = T_j^* T_i,$$

for all $1 \leq i, j \leq n$ and $i \neq j$.

A closed subspace $\mathcal{S}$ of $H^2_E(\mathbb{D}^n)$ which is invariant under $M_{z_1}, \cdots, M_{z_n}$ is said to be a *doubly commuting submodule* if $\mathcal{S}$ is a submodule, that is, $M_{z_i}\mathcal{S} \subseteq \mathcal{S}$ for all i and the family of module multiplication operators $\{R_{z_1}, \ldots, R_{z_n}\}$ where

$$R_{z_i} := M_{z_i}|_{\mathcal{S}},$$

for all $1 \leq i \leq n$, is doubly commuting, that is,

$$R_{z_i} R^*_{z_j} = R^*_{z_j} R_{z_i},$$

for all $i \neq j$ in $\{1, \ldots, n\}$.

In this note we completely characterize the doubly commuting submodules of the vector-valued Hardy module $H^2_{E_*}(\mathbb{D}^n)$ over the polydisc, and this is the content of our main theorem. This result is an analogue of the classical Beurling-Lax-Halmos Theorem on the Hardy space over the unit disc.

Theorem 1.3. *Let $\mathcal{S}$ be a closed nonzero subspace of $H^2_{E_*}(\mathbb{D}^n)$. Then $\mathcal{S}$ is a doubly commuting submodule if and only if there exists a Hilbert space E with $E \subseteq E_*$, where the inclusion is up to unitary equivalence, and an inner function $\Theta \in H^\infty_{E\to E_*}(\mathbb{D}^n)$ such that*

$$\mathcal{S} = M_\Theta H^2_E(\mathbb{D}^n).$$

In the special scalar case $E_* = \mathbb{C}$ and when $n = 2$ (the bidisc), this characterization was obtained by Mandrekar in [M], and the proof given there relies on the Wold decomposition for two variables [S]. Our proof is based on the more natural language of Hilbert modules and a generalization of Wold decomposition for doubly commuting isometries [Sa].

As an application of this theorem, we can establish a version of the "Weak" Completion Property for the algebra $H^\infty(\mathbb{D}^n)$. Suppose that $E \subset E_c$. Recall that the *Completion Problem* for $H^\infty(\mathbb{D}^n)$ is the problem of characterizing the functions $f \in H^\infty_{E \to E_c}(\mathbb{D}^n)$ such that there exists an invertible function $F \in H^\infty_{E_c \to E_c}(\mathbb{D}^n)$ with $F|_E = f$.

In the case of $H^\infty(\mathbb{D})$, the Completion Problem was settled by Tolokonnikov in [To]. In that paper, it was pointed out that there is a close connection between the Completion Problem and the characterization of invariant subspaces of $H^2(\mathbb{D})$. Using Theorem 1.3 we then have the following analogue of the results in [To].

Theorem 1.4 (Tolokonnikov's Lemma for the Polydisc)**.** *Let $f \in H^\infty_{E \to E_c}(\mathbb{D}^n)$ with $E \subset E_c$ and $\dim E, \dim E_c < \infty$. Then the following statements are equivalent:*

(i) *There exists a function $g \in H^\infty_{E_c \to E}(\mathbb{D}^n)$ such that $gf \equiv I$ in $\mathbb{D}^n$ and the operators $M_{z_1}, \ldots, M_{z_n}$ doubly commute on the subspace $\ker M_g$.*

(ii) *There exists a function $F \in H^\infty_{E_c \to E_c}(\mathbb{D}^n)$ such that $F|_E = f$, $F|_{E_c \ominus E}$ is inner, and $F^{-1} \in H^\infty_{E_c \to E_c}(\mathbb{D}^n)$.*

Remark 1.5. Theorem 1.4 for the polydisc is different from Tolokonnikov's lemma in the disc in which one does not demand that the completion F has the property that $F|_{E_c \ominus E}$ is inner. But, from the proof of Tolokonnikov's lemma in the case of the disc (see [N]), one can see that the following statements are equivalent for $f \in H^\infty_{E \to E_c}(\mathbb{D})$ with $E \subset E_c$ and $\dim E < \infty$:

(i) There exists a function $g \in H^\infty_{E_c \to E}(\mathbb{D})$ such that $gf \equiv I$ in $\mathbb{D}$.

(ii) There exists a function $F \in H^\infty_{E_c \to E_c}(\mathbb{D})$ such that $F|_E = f$, and $F^{-1} \in H^\infty_{E_c \to E_c}(\mathbb{D})$.

(ii′) There exists a function $F \in H^\infty_{E_c \to E_c}(\mathbb{D})$ such that $F|_E = f$, $F|_{E_c \ominus E}$ is inner, and $F^{-1} \in H^\infty_{E_c \to E_c}(\mathbb{D})$.

In the polydisc case it is unclear how the conditions

(II) There exists a function $F \in H^\infty_{E_c \to E_c}(\mathbb{D}^n)$ such that $F|_E = f$, and $F^{-1} \in H^\infty_{E_c \to E_c}(\mathbb{D}^n)$.

(II′) There exists a function $F \in H^\infty_{E_c \to E_c}(\mathbb{D}^n)$ such that $F|_E = f$, $F|_{E_c \ominus E}$ is inner, and $F^{-1} \in H^\infty_{E_c \to E_c}(\mathbb{D}^n)$.

are related. We refer to the Completion Problem in *(II)* as the *Strong Completion Problem*, while the one in *(II′)* as the *Weak Completion Problem*. Whether the two are equivalent is an open problem.

We also remark that in the disc case, Tolokonnikov's Lemma was proved by Sergei Treil [T] without any assumptions about the finite dimensionality of E, E_c. However, our proof of Theorem 1.4 relies on Lemma 3.1, whose validity we do not know without the assumption on the finite dimensionality of E and E_c.

Example 1.6. As a simple illustration of Theorem 1.4, take $n = 3$, $\dim E = 1$, $\dim E_c = 3$ and

$$f := \begin{bmatrix} e^{z_1} \\ e^{z_2} \\ e^{z_3} \end{bmatrix} \in (H^\infty(\mathbb{D}^3))^{3\times 1}.$$

With $g := \begin{bmatrix} e^{-z_1} & 0 & 0 \end{bmatrix} \in (H^\infty(\mathbb{D}^2))^{1\times 3}$, we see that $gf = 1$. We have

$$\begin{aligned} \ker M_g &= \left\{ \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} \in (H^2(\mathbb{D}^3))^{3\times 1} : e^{-z_1}\varphi_1 = 0 \right\} \\ &= \left\{ \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} \in (H^2(\mathbb{D}^3))^{3\times 1} : \varphi_1 = 0 \right\} = \Theta(H^2(\mathbb{D}^2))^{2\times 1}, \end{aligned}$$

where Θ is the inner function

$$\Theta := \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \in (H^\infty(\mathbb{D}^3))^{3\times 2}.$$

As Θ is inner, it follows from Theorem 1.3 that $M_{z_1}, M_{z_2}, M_{z_3}$ doubly commute on the submodule $\Theta(H^2(\mathbb{D}^3))^{2\times 1} = \ker M_g$. Hence f can be completed to an invertible matrix. In fact, with

$$F := \begin{bmatrix} f & \Theta \end{bmatrix} = \begin{bmatrix} e^{z_1} & 0 & 0 \\ e^{z_2} & 1 & 0 \\ e^{z_3} & 0 & 1 \end{bmatrix},$$

one can easily see that F is invertible as an element of $(H^\infty(\mathbb{D}^3))^{3\times 3}$.

In Section 2 we give a proof of Theorem 1.3, and subsequently, in Section 3, we use this theorem to study the Weak Completion Problem for $H^\infty(\mathbb{D}^n)$, providing a proof of Theorem 1.4.

2. Beurling-Lax-Halmos Theorem for the Polydisc

In this section we present a complete characterization of "reducing submodules" and a proof of the Beurling-Lax-Halmos theorem for doubly commuting submodules of $H^2_E(\mathbb{D}^n)$.

Recall that a closed subspace $\mathcal{S} \subseteq H^2_E(\mathbb{D}^n)$ is said to be a *reducing submodule* of $H^2_E(\mathbb{D}^n)$ if $M_{z_i}\mathcal{S}$, $M_{z_i}^*\mathcal{S} \subseteq \mathcal{S}$ for all $i = 1, \ldots, n$.

We start by reviewing some definitions and some well-known facts about the vector-valued Hardy space over polydisc. For more details about reproducing kernel Hilbert spaces over domains in $\mathbb{C}^n$, we refer the reader to [DMS]. Let

$$\mathbb{S}(\boldsymbol{z}, \boldsymbol{w}) = \prod_{j=1}^{n} (1 - \overline{w}_j z_j)^{-1}. \qquad ((\boldsymbol{z}, \boldsymbol{w}) \in \mathbb{D}^n \times \mathbb{D}^n)$$

be the Cauchy kernel on the polydisc $\mathbb{D}^n$. Then for some Hilbert space E, the kernel function $\mathbb{S}_E$ of $H^2_E(\mathbb{D}^n)$ is given by

$$\mathbb{S}_E(\boldsymbol{z}, \boldsymbol{w}) = \mathbb{S}(\boldsymbol{z}, \boldsymbol{w}) I_E. \qquad ((\boldsymbol{z}, \boldsymbol{w}) \in \mathbb{D}^n \times \mathbb{D}^n)$$

In particular, $\{\mathbb{S}(\cdot, \boldsymbol{w})\eta : \boldsymbol{w} \in \mathbb{D}^n, \eta \in E\}$ is a *total subset* for $H^2_E(\mathbb{D}^n)$, that is,

$$\overline{\text{span}}\{\mathbb{S}(\cdot, \boldsymbol{w})\eta : \boldsymbol{w} \in \mathbb{D}^n, \eta \in E\} = H^2_E(\mathbb{D}^n),$$

where $\mathbb{S}(\cdot, \boldsymbol{w}) \in H^2(\mathbb{D}^n)$ and

$$(\mathbb{S}(\cdot, \boldsymbol{w}))(z) = \mathbb{S}(\boldsymbol{z}, \boldsymbol{w}),$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n$. Moreover, for all $f \in H^2_E(\mathbb{D}^n)$, $\boldsymbol{w} \in \mathbb{D}^n$ and $\eta \in E$ we have

$$\langle f, \mathbb{S}(\cdot, \boldsymbol{w})\eta \rangle_{H^2_E(\mathbb{D}^n)} = \langle f(\boldsymbol{w}), \eta \rangle_E.$$

Note also that for the multiplication operator M_{z_i} on $H^2_E(\mathbb{D}^n)$

$$M_{z_i}^*(\mathbb{S}(\cdot, \boldsymbol{w})\eta) = \bar{w}_i(\mathbb{S}(\cdot, \boldsymbol{w})\eta),$$

where $\boldsymbol{w} \in \mathbb{D}^n$, $\eta \in E$ and $1 \leq i \leq n$.

We also have

$$\mathbb{S}^{-1}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{0 \leq i_1 < \ldots < i_l \leq n} (-1)^l z_{i_1} \cdots z_{i_l} \bar{w}_{i_1} \cdots \bar{w}_{i_l},$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n$.
For $H^2_E(\mathbb{D}^n)$ we set

$$\mathbb{S}_E^{-1}(\boldsymbol{M_z}, \boldsymbol{M_z}) := \sum_{0 \leq i_1 < \ldots < i_l \leq n} (-1)^l M_{z_{i_1}} \cdots M_{z_{i_l}} M_{z_{i_1}}^* \cdots M_{z_{i_l}}^*.$$

The following Lemma is well-known in the study of reproducing kernel Hilbert spaces.

Lemma 2.1. *Let E be a Hilbert space. Then*

$$\mathbb{S}_E^{-1}(\boldsymbol{M_z}, \boldsymbol{M_z}) = P_E,$$

where P_E is the orthogonal projection of $H^2_E(\mathbb{D}^n)$ onto the space of all constant functions.

Proof. for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n$ and $\eta, \zeta \in E$ we have

$$\begin{aligned}
&\left\langle \mathbb{S}_E^{-1}(\boldsymbol{M_z}, \boldsymbol{M_z})\ (\mathbb{S}(\cdot, \boldsymbol{z})\eta), (\mathbb{S}(\cdot, \boldsymbol{w})\zeta)\right\rangle_{H^2_E(\mathbb{D}^n)} \\
&= \left\langle \sum_{0 \le i_1 < \ldots < i_l \le n} (-1)^l M_{z_{i_1}} \cdots M_{z_{i_l}} M^*_{z_{i_1}} \cdots M^*_{z_{i_l}} (\mathbb{S}(\cdot, \boldsymbol{z})\eta), (\mathbb{S}(\cdot, \boldsymbol{w})\zeta) \right\rangle_{H^2_E(\mathbb{D}^n)} \\
&= \sum_{0 \le i_1 < \ldots < i_l \le n} (-1)^l \left\langle M^*_{z_{i_1}} \cdots M^*_{z_{i_l}} (\mathbb{S}(\cdot, \boldsymbol{z})\eta), M^*_{z_{i_1}} \cdots M^*_{z_{i_l}} (\mathbb{S}(\cdot, \boldsymbol{w})\zeta) \right\rangle_{H^2_E(\mathbb{D}^n)} \\
&= \sum_{0 \le i_1 < \ldots < i_l \le n} (-1)^l \bar{z}_{i_1} \cdots \bar{z}_{i_l} w_{i_1} \cdots w_{i_l} \langle \mathbb{S}(\cdot, \boldsymbol{z}), \mathbb{S}(\cdot, \boldsymbol{w})\rangle_{H^2(\mathbb{D}^n)} \langle \eta, \zeta\rangle_E \\
&= \mathbb{S}^{-1}(\boldsymbol{w}, \boldsymbol{z}) \mathbb{S}(\boldsymbol{w}, \boldsymbol{z}) \langle \eta, \zeta\rangle_E \\
&= \langle \eta, \zeta\rangle_E \\
&= \langle P_E \mathbb{S}(\cdot, \boldsymbol{z})\eta, \mathbb{S}(\cdot, \boldsymbol{w})\zeta\rangle_{H^2_E(\mathbb{D}^n)}
\end{aligned}$$

Since $\{\mathbb{S}(\cdot, \boldsymbol{z})\eta : \boldsymbol{z} \in \mathbb{D}^n, \eta \in E\}$ is a total subset of $H^2_E(\mathbb{D}^n)$, we have that

$$\mathbb{S}_E^{-1}(\boldsymbol{M_z}, \boldsymbol{M_z}) = P_E.$$

This completes the proof. ■

In the following proposition we characterize the reducing submodules of $H^2_E(\mathbb{D}^n)$.

Proposition 2.2. *Let $\mathcal{S}$ be a closed subspace of $H^2_E(\mathbb{D}^n)$. Then $\mathcal{S}$ is a reducing submodule of $H^2_E(\mathbb{D}^n)$ if and only if*

$$\mathcal{S} = H^2_{E_*}(\mathbb{D}^n),$$

for some closed subspace E_* of E.

Proof. Let $\mathcal{S}$ be a reducing submodule of $H^2_E(\mathbb{D}^n)$, that is, for all $1 \le i \le n$ we have

$$M_{z_i} P_{\mathcal{S}} = P_{\mathcal{S}} M_{z_i}.$$

By Lemma 2.1

$$P_E P_{\mathcal{S}} = \mathbb{S}_E^{-1}(\boldsymbol{M_z}, \boldsymbol{M_z}) P_{\mathcal{S}} = P_{\mathcal{S}} \mathbb{S}_E^{-1}(\boldsymbol{M_z}, \boldsymbol{M_z}) = P_{\mathcal{S}} P_E.$$

In particular, that $P_{\mathcal{S}} P_E$ is an orthogonal projection and

$$P_{\mathcal{S}} P_E = P_E P_{\mathcal{S}} = P_{E_*},$$

where $E_* := E \cap \mathcal{S}$. Hence, for any

$$f = \sum_{\boldsymbol{k} \in \mathbb{N}^n} a_{\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}} \in \mathcal{S},$$

where $a_{\boldsymbol{k}} \in E$ for all $\boldsymbol{k} \in \mathbb{N}^n$, we have

$$f = P_{\mathcal{S}} f = P_{\mathcal{S}} \left(\sum_{\boldsymbol{k} \in \mathbb{N}^n} M_z^{\boldsymbol{k}} a_{\boldsymbol{k}} \right) = \sum_{\boldsymbol{k} \in \mathbb{N}^n} M_z^{\boldsymbol{k}} P_{\mathcal{S}} a_{\boldsymbol{k}}.$$

But $P_{\mathcal{S}} a_{\boldsymbol{k}} = P_{\mathcal{S}} P_E a_{\boldsymbol{k}} \in E_*$. Consequently, $M_z^{\boldsymbol{k}} P_{\mathcal{S}} a_{\boldsymbol{k}} \in H^2_{E_*}(\mathbb{D}^n)$ for all $\boldsymbol{k} \in \mathbb{N}^n$ and hence $f \in H^2_{E_*}(\mathbb{D}^n)$. That is, $\mathcal{S} \subseteq H^2_{E_*}(\mathbb{D}^n)$. For the reverse inclusion, it is enough to observe that $E_* \subseteq \mathcal{S}$ and that $\mathcal{S}$ is a reducing submodule. The converse part is immediate. Hence the lemma follows. ∎

Let $\mathcal{S}$ be a doubly commuting submodule of $H^2_E(\mathbb{D}^n)$. Then

$$R_{z_i} R_{z_i}^* = M_{z_i} P_{\mathcal{S}} M_{z_i}^* P_{\mathcal{S}} = M_{z_i} P_{\mathcal{S}} M_{z_i}^*,$$

implies that $R_{z_i} R_{z_i}^*$ is an orthogonal projection of $\mathcal{S}$ onto $z_i \mathcal{S}$ and hence $I_{\mathcal{S}} - R_{z_i} R_{z_i}^*$ is an orthogonal projection of $\mathcal{S}$ onto $\mathcal{S} \ominus z_i \mathcal{S}$, that is,

$$I_{\mathcal{S}} - R_{z_i} R_{z_i}^* = P_{\mathcal{S} \ominus z_i \mathcal{S}},$$

for all $i = 1, \ldots, n$. Define

$$\mathcal{W}_i = \operatorname{ran}(I_{\mathcal{S}} - R_{z_i} R_{z_i}^*) = \mathcal{S} \ominus z_i \mathcal{S},$$

for all $i = 1, \ldots, n$, and

$$\mathcal{W} = \bigcap_{i=1}^n \mathcal{W}_i.$$

Now let $\mathcal{S}$ be a doubly commuting submodule of $H^2_E(\mathbb{D}^n)$. By doubly commutativity of $\mathcal{S}$ it follows that (also see [Sa])

$$(I_{\mathcal{S}} - R_{z_i} R_{z_i}^*)(I_{\mathcal{S}} - R_{z_j} R_{z_j}^*) = (I_{\mathcal{S}} - R_{z_j} R_{z_j}^*)(I_{\mathcal{S}} - R_{z_i} R_{z_i}^*),$$

for all $i \neq j$. Therefore $\{(I_{\mathcal{S}} - R_{z_i} R_{z_i}^*)\}_{i=1}^n$ is a family of commuting orthogonal projections and hence

(2.1)

$$\mathcal{W} = \bigcap_{i=1}^n \mathcal{W}_i = \bigcap_{i=1}^n (\mathcal{S} \ominus z_i \mathcal{S}) = \bigcap_{i=1}^n \operatorname{ran}(I_{\mathcal{S}} - R_{z_i} R_{z_i}^*) = \operatorname{ran}(\prod_{i=1}^n (I_{\mathcal{S}} - R_{z_i} R_{z_i}^*)).$$

Now we present a wandering subspace theorem concerning doubly commuting submodules of $H^2_E(\mathbb{D}^n)$. The result is a consequence of a several variables analogue of the classical Wold decomposition theorem as obtained by Gaspar and Suciu [GS]. We provide a direct proof (also see Corollary 3.2 in [Sa]).

Theorem 2.3. *Let $\mathcal{S}$ be a doubly commuting submodule of $H^2_E(\mathbb{D}^n)$. Then*

$$\mathcal{S} = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \oplus z^{\boldsymbol{k}} \mathcal{W}.$$

Proof. First, note that if $\mathcal{M}$ is a submodule of $H^2_E(\mathbb{D}^n)$ then

$$\bigcap_{k \in \mathbb{N}} R_{z_i}^{*k} \mathcal{M} \subseteq \bigcap_{k \in \mathbb{N}} M_{z_i}^{*k} H^2_E(\mathbb{D}^n) = \{0\},$$

for each $i = 1, \ldots, n$. Therefore, R_{z_i} is a shift, that is, the unitary part $\bigcap_{k \in \mathbb{N}} R_{z_i}^{*k} \mathcal{M}$ in the Wold decomposition (cf. [NF], [Sa]) of R_{z_i} on $\mathcal{M}$ is trivial for all $i = 1, \ldots, n$. Moreover, if $\mathcal{S}$ is doubly commuting then

$$R_{z_i}(I_{\mathcal{S}} - R_{z_j} R_{z_j}^*) = (I_{\mathcal{S}} - R_{z_j} R_{z_j}^*) R_{z_i},$$

for all $i \neq j$. Therefore $\mathcal{W}_j$ is a R_{z_i}-reducing subspace for all $i \neq j$. Note also that for all $1 \leq m < n$,

$$\begin{aligned}
\bigcap_{i=1}^{m+1} \mathcal{W}_i &= \operatorname{ran}(\prod_{i=1}^{m+1} (I_{\mathcal{S}} - R_{z_i} R_{z_i}^*)) \\
&= \operatorname{ran}(\prod_{i=1}^{m} (I_{\mathcal{S}} - R_{z_i} R_{z_i}^*) - R_{z_{m+1}} R_{z_{m+1}}^* \prod_{i=1}^{m} (I_{\mathcal{S}} - R_{z_i} R_{z_i}^*)) \\
&= \operatorname{ran}(\prod_{i=1}^{m} (I_{\mathcal{S}} - R_{z_i} R_{z_i}^*) - R_{z_{m+1}} \prod_{i=1}^{m} (I_{\mathcal{S}} - R_{z_i} R_{z_i}^*) R_{z_{m+1}}^*) \\
&= (\mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m) \ominus z_{m+1} (\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m),
\end{aligned}$$

and hence

$$(\mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m) \ominus z_{m+1} (\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m) = \bigcap_{i=1}^{m+1} \mathcal{W}_i.$$

We use mathematical induction to prove that for all $2 \leq m \leq n$, we have

$$\mathcal{S} = \sum_{\boldsymbol{k} \in \mathbb{N}^m} \oplus z^{\boldsymbol{k}} (\mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m).$$

First, by Wold decomposition theorem for the shift R_{z_1} on $\mathcal{S}$ we have

$$\mathcal{S} = \sum_{k_1 \in \mathbb{N}} \oplus R_{z_1}^{k_1} \mathcal{W}_1 = \sum_{k_1 \in \mathbb{N}} \oplus z_1^{k_1} \mathcal{W}_1.$$

Again by applying Wold decomposition for $R_{z_2}|_{\mathcal{W}_1} \in \mathcal{L}(\mathcal{W}_1)$ we have

$$\mathcal{W}_1 = \sum_{k_2 \in \mathbb{N}} \oplus R_{z_2}^{k_2} (\mathcal{W}_1 \ominus z_2 \mathcal{W}_1) = \sum_{k_2 \in \mathbb{N}} \oplus z_2^{k_2} (\mathcal{W}_1 \cap \mathcal{W}_2),$$

and hence

$$\mathcal{S} = \sum_{k_1 \in \mathbb{N}} \oplus z_1^{k_1} \Big(\sum_{k_2 \in \mathbb{N}} \oplus z_2^{k_2} (\mathcal{W}_1 \cap \mathcal{W}_2) \Big) = \sum_{k_1, k_2 \in \mathbb{N}} \oplus z_1^{k_1} z_2^{k_2} (\mathcal{W}_1 \cap \mathcal{W}_2).$$

Finally, let

$$\mathcal{S} = \sum_{\boldsymbol{k} \in \mathbb{N}^m} \oplus z^{\boldsymbol{k}} (\mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m),$$

for some $m < n$. Then we again apply the Wold decomposition on the isometry

$$R_{z_{m+1}}|_{\mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m} \in \mathcal{L}(\mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m)$$

to obtain

$$\begin{aligned}\mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m &= \sum_{k_{m+1} \in \mathbb{N}} \oplus z_{m+1}^{k_{m+1}} \Big((\mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m) \ominus z_{m+1} \mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m \Big) \\ &= \sum_{k_{m+1} \in \mathbb{N}} \oplus z_{m+1}^{k_{m+1}} (\mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m \cap \mathcal{W}_{m+1}),\end{aligned}$$

which yields

$$\mathcal{S} = \sum_{\boldsymbol{k} \in \mathbb{N}^{m+1}} \oplus z^{\boldsymbol{k}} (\mathcal{W}_1 \cap \ldots \cap \mathcal{W}_{m+1}).$$

This completes the proof. ∎

We now turn to the proof of Theorem 1.3.

Proof of Theorem 1.3. By Theorem 2.3 we have

$$\mathcal{S} = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \oplus z^{\boldsymbol{k}} (\bigcap_{i=1}^{n} \mathcal{W}_i). \tag{2.2}$$

Now define the Hilbert space E by

$$E = \bigcap_{i=1}^{n} \mathcal{W}_i,$$

and the linear operator $V : H^2_E(\mathbb{D}^n) \to H^2_{E_*}(\mathbb{D}^n)$ by

$$V \left(\sum_{\boldsymbol{k} \in \mathbb{N}^n} a_{\boldsymbol{k}} z^{\boldsymbol{k}} \right) = \sum_{\boldsymbol{k} \in \mathbb{N}^n} M_z^{\boldsymbol{k}} a_{\boldsymbol{k}},$$

where

$$\sum_{\boldsymbol{k} \in \mathbb{N}^n} a_{\boldsymbol{k}} z^{\boldsymbol{k}} \in H^2_E(\mathbb{D}^n)$$

and $a_{\boldsymbol{k}} \in E$ for all $\boldsymbol{k} \in \mathbb{N}^n$. Observe that

$$\| \sum_{\boldsymbol{k} \in \mathbb{N}^n} M_z^{\boldsymbol{k}} a_{\boldsymbol{k}} \|^2_{H^2_{E_*}(\mathbb{D}^n)} = \| \sum_{\boldsymbol{k} \in \mathbb{N}^n} z^{\boldsymbol{k}} a_{\boldsymbol{k}} \|^2_{H^2_{E_*}(\mathbb{D}^n)} = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \| z^{\boldsymbol{k}} a_{\boldsymbol{k}} \|^2_{H^2_{E_*}(\mathbb{D}^n)},$$

where the last equality follows from the orthogonal decomposition of $\mathcal{S}$ in (2.2). Therefore,

$$\begin{aligned}\| \sum_{\boldsymbol{k} \in \mathbb{N}^n} M_z^{\boldsymbol{k}} a_{\boldsymbol{k}} \|^2_{H^2_{E_*}(\mathbb{D}^n)} &= \sum_{\boldsymbol{k} \in \mathbb{N}^n} \| z^{\boldsymbol{k}} a_{\boldsymbol{k}} \|^2_{H^2_{E_*}(\mathbb{D}^n)} = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \| a_{\boldsymbol{k}} \|^2_{H^2_{E_*}(\mathbb{D}^n)} = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \| a_{\boldsymbol{k}} \|^2_E \\ &= \| \sum_{\boldsymbol{k} \in \mathbb{N}^n} z^{\boldsymbol{k}} a_{\boldsymbol{k}} \|^2_{H^2_E(\mathbb{D}^n)},\end{aligned}$$

and hence V is an isometry. Moreover, for all $\boldsymbol{k} \in \mathbb{N}^n$ and $\eta \in E$ we have

$$V M_{z_i} (z^{\boldsymbol{k}} \eta) = V(z^{\boldsymbol{k}+\boldsymbol{e}_i} \eta) = M_z^{\boldsymbol{k}+\boldsymbol{e}_i} \eta = M_{z_i} (M_z^{\boldsymbol{k}} \eta) = M_{z_i} V (z^{\boldsymbol{k}} \eta),$$

that is, $V M_{z_i} = M_{z_i} V$ for all $i = 1, \ldots, n$. Hence V is a module map. Therefore,

$$V = M_\Theta,$$

for some bounded holomorphic function $\Theta \in H^\infty_{E\to E_*}(\mathbb{D}^n)$ (cf. page 655 in [BLTT]). Moreover, since V is an isometry, we have

$$M_\Theta^* M_\Theta = I_{H^2_E(\mathbb{D}^n)},$$

that is, that Θ is an inner function. Also since $M_{z_i} E \subseteq \mathcal{S}$ for all $i = 1, \dots, n$ we have that

$$\mathrm{ran} V \subseteq \mathcal{S}.$$

Also by (2.2) that $\mathcal{S} \subseteq \mathrm{ran} V$. Hence it follows that

$$\mathrm{ran} V = \mathrm{ran} M_\Theta = \mathcal{S},$$

that is,

$$\mathcal{S} = \Theta H^2_E(\mathbb{D}^n).$$

Finally, for all $i = 1, \dots, n$, we have

$$\mathcal{S} \ominus z_i \mathcal{S} = \Theta H^2_E(\mathbb{D}^n) \ominus z_i \Theta H^2_E(\mathbb{D}^n) = \{\Theta f : f \in H^2_E(\mathbb{D}^n), M_{z_i}^* \Theta f = 0\},$$

and hence by (2.1)

$$\begin{aligned} E = \bigcap_{i=1}^n \mathcal{W}_i = \bigcap_{i=1}^n (\mathcal{S} \ominus z_i \mathcal{S}) &= \{\Theta f : M_{z_i}^* \Theta f = 0, f \in H^2_E(\mathbb{D}^n), \forall i = 1, \dots, n\} \\ &\subseteq \{g \in H^2_{E_*}(\mathbb{D}^n) : M_{z_i}^* g = 0, \forall i = 1, \dots, n\} = E_*, \end{aligned}$$

that is,

$$E \subseteq E_*.$$

To prove the converse part, let $\mathcal{S} = M_\Theta H^2_E(\mathbb{D}^n)$ be a submodule of $H^2_{E_*}(\mathbb{D}^n)$ for some inner function $\Theta \in H^\infty_{E\to E_*}(\mathbb{D}^n)$. Then

$$P_{\mathcal{S}} = M_\Theta M_\Theta^*,$$

and hence for all $i \neq j$,

$$\begin{aligned} M_{z_i} P_{\mathcal{S}} M_{z_j}^* &= M_{z_i} M_\Theta M_\Theta^* M_{z_j}^* = M_\Theta M_{z_i} M_{z_j}^* M_\Theta^* = M_\Theta M_{z_j}^* M_{z_i} M_\Theta^* \\ &= M_\Theta M_{z_j}^* M_\Theta^* M_\Theta M_{z_i} M_\Theta^* = M_\Theta M_\Theta^* M_{z_j}^* M_{z_i} M_\Theta M_\Theta^* \\ &= P_{\mathcal{S}} M_{z_j}^* M_{z_i} P_{\mathcal{S}}. \end{aligned}$$

This implies

$$R_{z_j}^* R_{z_i} = P_{\mathcal{S}} M_{z_j}^* P_{\mathcal{S}} M_{z_i}|_{\mathcal{S}} = P_{\mathcal{S}} M_{z_j}^* M_{z_i}|_{\mathcal{S}} = M_{z_i} P_{\mathcal{S}} M_{z_j}^* = R_{z_i} R_{z_j}^*,$$

that is, $\mathcal{S}$ is a doubly commuting submodule. This completes the proof. ■

3. Tolokonnikov's Lemma for the Polydisc

We will need the following lemma, which is a polydisc version of a similar result proved in the case of the disc in Nikolski's book [N]*p.44-45. Here we use the notation M_g for the multiplication operator on H^2_E induced by $g \in H^\infty_{E\to E_*}$.

Lemma 3.1 (Lemma on Local Rank). *Let E, E_c be Hilbert spaces, with $\dim E, \dim E_c < \infty$. Let $g \in H^\infty_{E_c\to E}(\mathbb{D}^n)$ be such that*

$$\ker M_g = \{h \in H^2_{E_c}(\mathbb{D}^n) \ : \ g(z)h(z) \equiv 0\} = \Theta H^2_{E_a}(\mathbb{D}^n),$$

where E_a is a Hilbert space and Θ is a $\mathcal{L}(E_a, E_c)$-valued inner function. Then

$$\dim E_c = \dim E_a + \operatorname{rank} g,$$

where $\operatorname{rank} g := \max_{\zeta\in\mathbb{D}^n} \operatorname{rank} g(\zeta)$.

Proof. We have $\ker M_g = \{h \in H^2_{E_c}(\mathbb{D}^n) : gh \equiv 0\}$. If $\zeta \in \mathbb{D}^n$, then let

$$[\ker M_g](\zeta) := \{h(\zeta) \ : \ h \in \ker M_g\}.$$

We claim that $[\ker M_g](\zeta) = \Theta(\zeta)E_a$. Indeed, let $v \in [\ker M_g](\zeta)$. Then $v = h(\zeta)$ for some element $h \in \ker M_g = \Theta H^2_{E_a}(\mathbb{D}^n)$. So $h = \Theta\varphi$, for some $\varphi \in H^2_{E_a}(\mathbb{D}^n)$. In particular, $v = h(\zeta) = \Theta(\zeta)\varphi(\zeta)$, where $\varphi(\zeta) \in E_a$. So

$$[\ker M_g](\zeta) \subset \Theta(\zeta)E_a. \tag{3.1}$$

On the other hand, if $w \in \Theta(\zeta)E_a$, then $w = \Theta(\zeta)x$, where $x \in E_a$. Consider the constant function $\mathbf{x}$ mapping $\mathbb{D} \ni \mathbf{z} \overset{\mathbf{x}}{\mapsto} x \in E_a$. Clearly $\mathbf{x} \in H^2_{E_a}(\mathbb{D}^n)$. So $h := \Theta\mathbf{x} \in \Theta H^2_{E_a}(\mathbb{D}^n) = \ker M_g$. Hence $w = \Theta(\zeta)x = (\Theta\mathbf{x})(\zeta) = h(\zeta)$, and so $w \in [\ker M_g](\zeta)$. So we also have that

$$\Theta(\zeta)E_a \subset [\ker M_g](\zeta). \tag{3.2}$$

Our claim that $[\ker M_g](\zeta) = \Theta(\zeta)E_a$ follows from (3.1) and (3.2).

Suppose that for a $\zeta \in \mathbb{D}^n$, $v \in [\ker M_g](\zeta)$. Then $v = h(\zeta)$ for some $h \in \ker M_g$. Thus $gh \equiv 0$ in $\mathbb{D}^n$, and in particular, $g(\zeta)v = g(\zeta)h(\zeta) = 0$. Thus $v \in \ker g(\zeta)$. So we have that $[\ker M_g](\zeta) \subset \ker g(\zeta)$. Hence $\dim[\ker M_g](\zeta) \le \dim\ker g(\zeta)$. Consequently

$$\dim \Theta(\zeta)E_a = \dim[\ker M_g](\zeta) \le \dim\ker g(\zeta) = \dim E_c - \operatorname{rank} g(\zeta),$$

where the last equality follows from the Rank-Nullity Theorem. Since Θ is inner, we have that the boundary values of Θ satisfy $\Theta(\zeta)^*\Theta(\zeta) = I_{E_c}$ for almost all $\zeta \in \mathbb{T}^n$. So there is an open set $U \subset \mathbb{D}^n$ such that for all $\zeta \in U$

$$\dim E_a = \dim \Theta(\zeta)E_a.$$

But from the definition of the rank of g, we know that there is a $\zeta_* \in \mathbb{D}^n$ such that we have $k := \text{rank } g = \text{rank } g(\zeta_*)$. So there is a $k \times k$ submatrix of $g(\zeta_*)$ which is invertible. Now look at the determinant of this $k \times k$ submatrix of g. This is a holomorphic function, and so it cannot be identically zero in the open set U. So there must exist a point $\zeta_1 \in U \subset \mathbb{D}^n$ such that rank $g =$ rank $g(\zeta_1)$ and $\dim E_a = \dim \Theta(\zeta_1) E_a$. Hence $\dim E_a \leq \dim E_c - \text{rank } g$.

For the proof of the opposite inequality, let us consider a principal minor $g_1(\zeta_1)$ of the matrix of the operator $g(\zeta_1)$ (with respect to two arbitrary fixed bases in E_c and E respectively). Then $\det g_1 \in H^\infty$, $\det g_1 \not\equiv 0$. Let $E_c = E_{c,1} \oplus E_{c,2}$, $E = E_1 \oplus E_2$ ($\dim E_{c,1} = \dim E_1 = \text{rank } g(\zeta_1)$) be the decompositions of the spaces E_c and E corresponding to this minor, and let

$$g(\zeta) = \begin{bmatrix} g_1(\zeta) & g_2(\zeta) \\ \gamma_1(\zeta) & \gamma_2(\zeta) \end{bmatrix}, \quad \zeta \in \mathbb{D}^n,$$

be the matrix representation of $g(\zeta)$ with respect to this decomposition. Owing to our assumption on the rank, it follows that there is a matrix function $\zeta \mapsto W(\zeta)$ such that

$$\begin{bmatrix} \gamma_1(\zeta) & \gamma_2(\zeta) \end{bmatrix} = W(\zeta) \begin{bmatrix} g_1(\zeta) & g_2(\zeta) \end{bmatrix}.$$

So $\gamma_2(\zeta) = W(\zeta) g_2(\zeta) = (\gamma_1(\zeta)(g_1(\zeta))^{-1}) g_2(\zeta)$. Thus with $g_1^{\text{co}} := (\det g_1) g_1^{-1}$, we have

$$\gamma_2 \det g_1 = \gamma_1 g_1^{\text{co}} g_2,$$

and using this we get the inclusion $M_\Omega H^2_{E_{c,2}}(\mathbb{D}^n) \subset \ker M_g$, where $\Omega \in H^\infty_{E_{c,2} \to E_c}(\mathbb{D}^n)$ is given by

$$\Omega = \begin{bmatrix} g_1^{\text{co}} g_2 \\ -\det g_1 \end{bmatrix}.$$

We have rank $\Omega = \dim E_{c,2} = \dim E_c - \text{rank } g = \dim \ker(g(\zeta_1))$. Consequently, we obtain $\dim[\ker M_g](\zeta_1) \geq \dim \ker(g(\zeta_1))$. ∎

We now turn to the extension of Tolokonnikov's Lemma to the polydisc.

Proof of Theorem 1.4. (ii) $\Rightarrow$ (i): If $g := P_E F^{-1}$, then $gf = I$. It only remains to show that the operators $M_{z_1}, \ldots, M_{z_n}$ are doubly commuting on the space $\ker M_g$. Let Θ, Γ be such that:

$$F = \begin{bmatrix} f & \Theta \end{bmatrix} \quad \text{and} \quad F^{-1} = \begin{bmatrix} g \\ \Gamma \end{bmatrix}.$$

Since $FF^{-1} = I_{E_c}$, it follows that $fg + \Theta\Gamma = I_{E_c}$. Thus if $h \in H^2_{E_c}(\mathbb{D}^n)$ is such that $gh = 0$, then $\Theta(\Gamma h) = h$, and so $h \in \Theta H^2_{E_c \ominus E)}(\mathbb{D}^n)$. Hence $\ker M_g \subset \text{ran } M_\Theta$. Also, since $F^{-1}F = I$, it follows that $g\Theta = 0$, and so $\text{ran } M_\Theta \subset \ker M_g$. So $\ker M_g = \text{ran } M_\Theta = \Theta H^2_{E_c \ominus E}(\mathbb{D}^2)$. By Theorem 1.3, the operators $M_{z_1}, \ldots, M_{z_n}$ must doubly commute on the subspace $\ker M_g$.

(i) $\Rightarrow$ (ii): Let

$$\mathcal{S} := \{h \in H^2_{E_c}(\mathbb{D}^n) \ : \ g(z)h(z) \equiv 0\} = \ker g.$$

$\mathcal{S}$ is a closed non-zero invariant subspace of $H^2_{E_c}(\mathbb{D}^n)$. Also, by assumption, $M_{z_1}, \ldots, M_{z_n}$ are doubly commuting operators on $\mathcal{S}$. Then by the above Theorem 1.3, there exists an auxiliary Hilbert space E_a and an inner function $\widetilde{\Theta}$ with values in $\mathcal{L}(E_a, E_c)$ with $\dim E_a \leq \dim E_c$ such that

$$\mathcal{S} = \widetilde{\Theta} H^2_{E_a}(\mathbb{D}^n).$$

By the Lemma on Local Rank, $\dim E_a = \dim E_c - \text{rank } g = \dim E_c - \dim E = \dim(E_c \ominus E)$. Let U be a (constant) unitary operator from $E_c \ominus E$ to E_a and define $\Theta := \widetilde{\Theta} U$. Then Θ is inner, and we have that $\ker g = \Theta H^2_{E_c \ominus E}(\mathbb{D}^n)$. To get $F \in H^\infty_{E_c \to E_c}(\mathbb{D}^n)$ define the function F for $z \in \mathbb{D}^n$ by

$$F(z)e := \begin{cases} f(z)e & \text{if } e \in E \\ \Theta(z)e & \text{if } e \in E_c \ominus E. \end{cases}$$

We note that $F \in H^\infty(\mathbb{D}^n)$ and $F|_E = f$. We now show that F is invertible. With this in mind, we first observe that

$$(I - fg)H^2_{E_c}(\mathbb{D}^n) \subset \Theta H^2_{E_c \ominus E}(\mathbb{D}^n) = \ker M_g.$$

This follows since $g(I - fg)h = gh - gh = 0$ for all $h \in H^2_{E_c}(\mathbb{D}^n)$. Thus we have that $\Theta^*(I - fg) \in H^\infty_{E_c \to E_c \ominus E}(\mathbb{D}^n)$. Now, define $\Omega = g \oplus \Theta^*(I - fg)$. Clearly $\Omega \in H^\infty_{E_c \to E_c}(\mathbb{D}^n)$. Next, note that

$$F\Omega = fg + \Theta\Theta^*(I - fg) = I.$$

Similarly,

$$\begin{aligned} \Omega F &= gf\mathbb{P}_E + \Theta^*(I - fg)(f\mathbb{P}_E + \Theta\mathbb{P}_{E_c \ominus E}) \\ &= \mathbb{P}_E + \Theta^*(f\mathbb{P}_E - fgf\mathbb{P}_E + \Theta\mathbb{P}_{E_c \ominus E}) \\ &= \mathbb{P}_E + \Theta^*\Theta\mathbb{P}_{E_c \ominus E} = I. \end{aligned}$$

Thus we have that $F^{-1} \in H^\infty(\mathbb{D}^n; E_c \to E_c)$. ∎

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