DOUBLY COMMUTING SUBMODULES OF THE HARDY MODULE OVER POLYDISCS

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ABSTRACT. In this note we establish a vector-valued version of Beurling’s Theorem (the Lax-Halmos Theorem) for the polydisc. As an application of the main result, we provide necessary and sufficient conditions for the “weak” completion problem in \( H^\infty(\mathbb{D}^n) \).

1. Introduction and Statement of Main Results

In [B], Beurling described all the invariant subspaces for the operator \( M_z \) of “multiplication by \( z \)” on the Hilbert space \( H^2(\mathbb{D}) \) of the disc. In [L], Peter Lax extended Beurling’s result to the (finite-dimensional) vector-valued case (while also considering the Hardy space of the half plane). Lax’s vectorial case proof was further extended to infinite-dimensional vector spaces by Halmos, see [NF]. The characterization of \( M_z \)-invariant subspaces obtained is the following famous result.

**Theorem 1.1 (Beurling-Lax-Halmos).** Let \( \mathcal{S} \) be a closed nonzero subspace of \( H^2_{E^*}(\mathbb{D}) \). Then \( \mathcal{S} \) is invariant under multiplication by \( z \) if and only if there exists a Hilbert space \( E \) and an inner function \( \Theta \in H^\infty_{E \to E}(\mathbb{D}) \) such that \( \mathcal{S} = \Theta H^2_{E}(\mathbb{D}) \).

For \( n \in \mathbb{N} \) and \( E^* \) a Hilbert space, \( H^2_{E^*}(\mathbb{D}^n) \) is the set of all \( E^* \)-valued holomorphic functions in the polydisc \( \mathbb{D}^n \), where \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) (with boundary \( \mathbb{T} \)) such that

\[
\|f\|_{H^2_{E^*}(\mathbb{D}^n)} := \sup_{0 < r < 1} \left( \int_{\mathbb{T}^n} \|f(rz)\|^2_{E^*} \, dz \right)^{1/2} < +\infty.
\]

On the other hand, if \( \mathcal{L}(E, E^*) \) denotes the set of all continuous linear transformations from \( E \) to \( E^* \), then \( H^\infty_{E \to E^*}(\mathbb{D}^n) \) denotes the set of all \( \mathcal{L}(E, E^*) \)-valued holomorphic functions with \( \|f\|_{H^\infty_{E \to E^*}(\mathbb{D}^n)} := \sup_{z \in \mathbb{D}^n} \|f(z)\|_{\mathcal{L}(E, E^*)} < \infty \).

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An operator-valued $\Theta \in H_{E \to E_*}^\infty(\mathbb{D}^n)$ is inner if the pointwise a.e. boundary values are isometries:

$$(\Theta(\zeta))^*\Theta(\zeta) = I_E \text{ for almost all } \zeta \in \mathbb{T}^n.$$ 

A natural question is then to ask what happens in the case of several variables, for example when one considers the Hardy space $H^2_{E_*}(\mathbb{D}^n)$ of the polydisc $\mathbb{D}^n$. It is known that in general, a Beurling-Lax-Halmos type characterization of subspaces of the Hardy Hilbert space is not possible [R]. It is however, easy to see that the Hardy space on the polydisc $H^2_{E_*}(\mathbb{D}^n)$, when $n > 1$, satisfies the doubly commuting property, that is, for all $1 \leq i < j \leq n$

$$M_{z_i}^*M_{z_j} = M_{z_j}M_{z_i}^*.$$ 

We impose this additional assumption to the submodules of $H^2_{E_*}(\mathbb{D}^n)$ and call that class of submodules as doubly commuting submodules. More precisely:

**Definition 1.2.** A commuting family of bounded linear operators $\{T_1, \ldots, T_n\}$ on some Hilbert space $\mathcal{H}$ is said to be doubly commuting if

$$T_i T_j^* = T_j^* T_i,$$

for all $1 \leq i, j \leq n$ and $i \neq j$.

A closed subspace $S$ of $H^2_E(\mathbb{D}^n)$ which is invariant under $M_{z_1}, \ldots, M_{z_n}$ is said to be a doubly commuting submodule if $S$ is a submodule, that is, $M_{z_i}S \subseteq S$ for all $i$ and the family of module multiplication operators $\{R_{z_1}, \ldots, R_{z_n}\}$ where

$$R_{z_i} := M_{z_i}|S,$$

for all $1 \leq i \leq n$, is doubly commuting, that is,

$$R_{z_i} R_{z_j}^* = R_{z_j}^* R_{z_i},$$

for all $i \neq j$ in $\{1, \ldots, n\}$.

In this note we completely characterize the doubly commuting submodules of the vector-valued Hardy module $H^2_{E_*}(\mathbb{D}^n)$ over the polydisc, and this is the content of our main theorem. This result is an analogue of the classical Beurling-Lax-Halmos Theorem on the Hardy space over the unit disc.

**Theorem 1.3.** Let $S$ be a closed nonzero subspace of $H^2_{E_*}(\mathbb{D}^n)$. Then $S$ is a doubly commuting submodule if and only if there exists a Hilbert space $E$ with $E \subseteq E_*$, where the inclusion is up to unitary equivalence, and an inner function $\Theta \in H_{E \to E_*}^\infty(\mathbb{D}^n)$ such that

$$S = M_\Theta H^2_{E}(\mathbb{D}^n).$$
In the special scalar case $E_\ast = \mathbb{C}$ and when $n = 2$ (the bidisc), this characterization was obtained by Mandrekar in [M], and the proof given there relies on the Wold decomposition for two variables [S]. Our proof is based on the more natural language of Hilbert modules and a generalization of Wold decomposition for doubly commuting isometries [Sa].

As an application of this theorem, we can establish a version of the “Weak” Completion Property for the algebra $H^\infty(\mathbb{D}^n)$. Suppose that $E \subset E_c$. Recall that the Completion Problem for $H^\infty(\mathbb{D}^n)$ is the problem of characterizing the functions $f \in H^\infty_{E\to E_c}(\mathbb{D}^n)$ such that there exists an invertible function $F \in H^\infty_{E\to E_c}(\mathbb{D}^n)$ with $F|_E = f$.

In the case of $H^\infty(\mathbb{D})$, the Completion Problem was settled by Tolokonnikov in [To]. In that paper, it was pointed out that there is a close connection between the Completion Problem and the characterization of invariant subspaces of $H^2(\mathbb{D})$. Using Theorem 1.3 we then have the following analogue of the results in [To].

**Theorem 1.4 (Tolokonnikov’s Lemma for the Polydisc).** Let $f \in H^\infty_{E\to E_c}(\mathbb{D}^n)$ with $E \subset E_c$ and $\dim E, \dim E_c < \infty$. Then the following statements are equivalent:

(i) There exists a function $g \in H^\infty_{E\to E_c}(\mathbb{D}^n)$ such that $gf \equiv I$ in $\mathbb{D}^n$ and the operators $M_{z_1}, \ldots, M_{z_n}$ doubly commute on the subspace $\ker M_g$.

(ii) There exists a function $F \in H^\infty_{E\to E_c}(\mathbb{D}^n)$ such that $F|_E = f$, $F|_{E_c \oplus E}$ is inner, and $F^{-1} \in H^\infty_{E\to E_c}(\mathbb{D}^n)$.

**Remark 1.5.** Theorem 1.4 for the polydisc is different from Tolokonnikov’s lemma in the disc in which one does not demand that the completion $F$ has the property that $F|_{E_c \oplus E}$ is inner. But, from the proof of Tolokonnikov’s lemma in the case of the disc (see [N]), one can see that the following statements are equivalent for $f \in H^\infty_{E\to E_c}(\mathbb{D})$ with $E \subset E_c$ and $\dim E < \infty$:

(i) There exists a function $g \in H^\infty_{E\to E_c}(\mathbb{D})$ such that $gf \equiv I$ in $\mathbb{D}$.

(ii) There exists a function $F \in H^\infty_{E\to E_c}(\mathbb{D})$ such that $F|_E = f$, and $F^{-1} \in H^\infty_{E\to E_c}(\mathbb{D})$.

(ii') There exists a function $F \in H^\infty_{E\to E_c}(\mathbb{D})$ such that $F|_E = f$, $F|_{E_c \oplus E}$ is inner, and $F^{-1} \in H^\infty_{E\to E_c}(\mathbb{D})$.

In the polydisc case it is unclear how the conditions

(II) There exists a function $F \in H^\infty_{E\to E_c}(\mathbb{D}^n)$ such that $F|_E = f$, and $F^{-1} \in H^\infty_{E\to E_c}(\mathbb{D}^n)$.

(II') There exists a function $F \in H^\infty_{E\to E_c}(\mathbb{D}^n)$ such that $F|_E = f$, $F|_{E_c \oplus E}$ is inner, and $F^{-1} \in H^\infty_{E\to E_c}(\mathbb{D}^n)$. 


are related. We refer to the Completion Problem in (II) as the \textit{Strong Completion Problem}, while the one in (II′) as the \textit{Weak Completion Problem}. Whether the two are equivalent is an open problem.

We also remark that in the disc case, Tolokonnikov’s Lemma was proved by Sergei Treil [T] without any assumptions about the finite dimensionality of $E, E_c$. However, our proof of Theorem 1.4 relies on Lemma 3.1, whose validity we do not know without the assumption on the finite dimensionality of $E$ and $E_c$.

**Example 1.6.** As a simple illustration of Theorem 1.4, take $n = 3$, $\dim E = 1$, $\dim E_c = 3$ and

$$f := \begin{bmatrix} e^{z_1} \\ e^{z_2} \\ e^{z_3} \end{bmatrix} \in (\mathcal{H}^\infty(\mathbb{D}^3))^{3 \times 1}.$$  

With $g := \begin{bmatrix} e^{-z_1} & 0 & 0 \end{bmatrix} \in (\mathcal{H}^\infty(\mathbb{D}^2))^{1 \times 3}$, we see that $gf = 1$. We have

$$\ker M_g = \left\{ \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} \in (\mathcal{H}^2(\mathbb{D}^3))^{3 \times 1} : e^{-z_1} \varphi_1 = 0 \right\}$$

$$= \left\{ \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} \in (\mathcal{H}^2(\mathbb{D}^3))^{3 \times 1} : \varphi_1 = 0 \right\} = \Theta(\mathcal{H}^2(\mathbb{D}^2))^{2 \times 1},$$

where $\Theta$ is the inner function

$$\Theta := \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \in (\mathcal{H}^\infty(\mathbb{D}^3))^{3 \times 2}.$$

As $\Theta$ is inner, it follows from Theorem 1.3 that $M_{z_1}, M_{z_2}, M_{z_3}$ doubly commute on the submodule $\Theta(\mathcal{H}^2(\mathbb{D}^3))^{2 \times 1} = \ker M_g$. Hence $f$ can be completed to an invertible matrix. In fact, with

$$F := \begin{bmatrix} f & \Theta \end{bmatrix} = \begin{bmatrix} e^{z_1} & 0 & 0 \\ e^{z_2} & 1 & 0 \\ e^{z_3} & 0 & 1 \end{bmatrix},$$

one can easily see that $F$ is invertible as an element of $(\mathcal{H}^\infty(\mathbb{D}^3))^{3 \times 3}$.

In Section 2 we give a proof of Theorem 1.3, and subsequently, in Section 3, we use this theorem to study the Weak Completion Problem for $\mathcal{H}^\infty(\mathbb{D}^n)$, providing a proof of Theorem 1.4.

2. \textsc{Beurling-Lax-Halmos Theorem for the Polydisc}

In this section we present a complete characterization of “reducing submodules” and a proof of the Beurling-Lax-Halmos theorem for doubly commuting submodules of $H^2_E(\mathbb{D}^n)$. 


Recall that a closed subspace $S \subseteq H^2_E(\mathbb{D}^n)$ is said to be a reducing submodule of $H^2_E(\mathbb{D}^n)$ if $M_z S, M^*_z S \subseteq S$ for all $i = 1, \ldots, n$.

We start by reviewing some definitions and some well-known facts about the vector-valued Hardy space over polydisc. For more details about reproducing kernel Hilbert spaces over domains in $\mathbb{C}^n$, we refer the reader to [DMS]. Let

$$S(z, w) = \prod_{j=1}^n \left(1 - \bar{w}_j z_j\right)^{-1}. \quad ((z, w) \in \mathbb{D}^n \times \mathbb{D}^n)$$

be the Cauchy kernel on the polydisc $\mathbb{D}^n$. Then for some Hilbert space $E$, the kernel function $S_E$ of $H^2_E(\mathbb{D}^n)$ is given by

$$S_E(z, w) = S(z, w) I_E. \quad ((z, w) \in \mathbb{D}^n \times \mathbb{D}^n)$$

In particular, $\{S(\cdot, w) \eta : w \in \mathbb{D}^n, \eta \in E\}$ is a total subset for $H^2_E(\mathbb{D}^n)$, that is,

$$\operatorname{span}\{S(\cdot, w) \eta : w \in \mathbb{D}^n, \eta \in E\} = H^2_E(\mathbb{D}^n),$$

where $S(\cdot, w) \in H^2(\mathbb{D}^n)$ and

$$(S(\cdot, w))(z) = S(z, w),$$

for all $z, w \in \mathbb{D}^n$. Moreover, for all $f \in H^2_E(\mathbb{D}^n)$, $w \in \mathbb{D}^n$ and $\eta \in E$ we have

$$\langle f, S(\cdot, w) \eta \rangle_{H^2_E(\mathbb{D}^n)} = \langle f(w), \eta \rangle_E.$$

Note also that for the multiplication operator $M_{z_i}$ on $H^2_E(\mathbb{D}^n)$

$$M_{z_i}^*(S(\cdot, w) \eta) = \bar{w}_i (S(\cdot, w) \eta),$$

where $w \in \mathbb{D}^n$, $\eta \in E$ and $1 \leq i \leq n$.

We also have

$$S^{-1}(z, w) = \sum_{0 \leq i_1 < \ldots < i_l \leq n} (-1)^l z_{i_1} \cdots z_{i_l} \bar{w}_{i_1} \cdots \bar{w}_{i_l},$$

for all $z, w \in \mathbb{D}^n$.

For $H^2_E(\mathbb{D}^n)$ we set

$$S^{-1}_E(M_z, M^*_z) := \sum_{0 \leq i_1 < \ldots < i_l \leq n} (-1)^l M_{z_{i_1}} \cdots M_{z_{i_l}} M^*_{z_{i_1}} \cdots M^*_{z_{i_l}}.$$

The following Lemma is well-known in the study of reproducing kernel Hilbert spaces.

**Lemma 2.1.** Let $E$ be a Hilbert space. Then

$$S^{-1}_E(M_z, M_z) = P_E,$$

where $P_E$ is the orthogonal projection of $H^2_E(\mathbb{D}^n)$ onto the space of all constant functions.
This completes the proof.

Proof. for all \( z, w \in \mathbb{D}^n \) and \( \eta, \zeta \in E \) we have
\[
\langle S^E_1(M_z, M_z) (S(\cdot, z)\eta), (S(\cdot, w)\zeta) \rangle_{H^2_E(\mathbb{D}^n)}
\]
\[
= \sum_{0 \leq i_1 < \ldots < i_n \leq n} (-1)^l M_{z_{i_1}} \cdots M_{z_{i_n}} M_{z_{i_1}}^* \cdots M_{z_{i_n}}^* (S(\cdot, z)\eta), (S(\cdot, w)\zeta) \rangle_{H^2_E(\mathbb{D}^n)}
\]
\[
= \sum_{0 \leq i_1 < \ldots < i_n \leq n} (-1)^l \langle M_{z_{i_1}}^* \cdots M_{z_{i_1}}^* (S(\cdot, z)\eta), M_{z_{i_1}}^* \cdots M_{z_{i_1}}^* (S(\cdot, w)\zeta) \rangle_{H^2_E(\mathbb{D}^n)}
\]
\[
= \sum_{0 \leq i_1 < \ldots < i_n \leq n} (-1)^l z_{i_1} \cdots z_{i_n} w_{i_1} \cdots w_{i_n} \langle S(\cdot, z), S(\cdot, w) \rangle_{H^2_E(\mathbb{D}^n)} \langle \eta, \zeta \rangle_{E}
\]
\[
= S^{-1}(w, z)S(w, z) \langle \eta, \zeta \rangle_{E}
\]
\[
= P_E S(\cdot, z)\eta, S(\cdot, w)\zeta \rangle_{H^2_E(\mathbb{D}^n)}
\]
Since \( \{S(\cdot, z)\eta : z \in \mathbb{D}^n, \eta \in E \} \) is a total subset of \( H^2_E(\mathbb{D}^n) \), we have that
\[
S^E_1(M_z, M_z) = P_E.
\]
This completes the proof. \( \blacksquare \)

In the following proposition we characterize the reducing submodules of \( H^2_E(\mathbb{D}^n) \).

**Proposition 2.2.** Let \( S \) be a closed subspace of \( H^2_E(\mathbb{D}^n) \). Then \( S \) is a reducing submodule of \( H^2_E(\mathbb{D}^n) \) if and only if
\[
S = H^2_E(\mathbb{D}^n),
\]
for some closed subspace \( E_* \) of \( E \).

**Proof.** Let \( S \) be a reducing submodule of \( H^2_E(\mathbb{D}^n) \), that is, for all \( 1 \leq i \leq n \) we have
\[
M_{z_i} P_S = P_S M_{z_i}.
\]
By Lemma 2.1
\[
P_E P_S = S^{-1}_E(M_z, M_z) P_S = P_S S^{-1}_E(M_z, M_z) = P_S P_E.
\]
In particular, that \( P_S P_E \) is an orthogonal projection and
\[
P_S P_E = P_E P_S = P_{E_*},
\]
where \( E_* := E \cap S \). Hence, for any
\[
f = \sum_{k \in \mathbb{N}^n} a_k z^k \in S,
\]
where \( a_k \in E \) for all \( k \in \mathbb{N}^n \), we have
\[
f = P_S f = P_S \left( \sum_{k \in \mathbb{N}^n} M_z^k a_k \right) = \sum_{k \in \mathbb{N}^n} M_z^k P_S a_k.
\]
But \( P \sigma_k = P \sigma E \sigma_k \in E_\star \). Consequently, \( M^k \sigma_k P \sigma E_k \in H^2_E(\mathbb{D}^n) \) for all \( k \in \mathbb{N}^n \) and hence \( f \in H^2_E(\mathbb{D}^n) \). That is, \( E \subseteq H^2_E(\mathbb{D}^n) \). For the reverse inclusion, it is enough to observe that \( E, \subseteq S \) and that \( S \) is a reducing submodule. The converse part is immediate. Hence the lemma follows.

Let \( S \) be a doubly commuting submodule of \( H^2_E(\mathbb{D}^n) \). Then
\[
R_{z_i} R_{z_i}^* = M_{z_i} P \sigma M_{z_i}^* P \sigma = M_{z_i} P \sigma M_{z_i}^*,
\]
implies that \( R_{z_i} R_{z_i}^* \) is an orthogonal projection of \( S \) onto \( z_i S \) and hence \( I_S - R_{z_i} R_{z_i}^* \) is an orthogonal projection of \( S \) onto \( S \ominus z_i S \), that is,
\[
I_S - R_{z_i} R_{z_i}^* = P \sigma z_i S,
\]
for all \( i = 1, \ldots, n \). Define
\[
W_i = \text{ran}(I_S - R_{z_i} R_{z_i}^*) = S \ominus z_i S,
\]
for all \( i = 1, \ldots, n \), and
\[
W = \bigcap_{i=1}^n W_i.
\]
Now let \( S \) be a doubly commuting submodule of \( H^2_E(\mathbb{D}^n) \). By doubly commutativity of \( S \) it follows that (also see [Sa])
\[
(I_S - R_{z_i} R_{z_i}^*) (I_S - R_{z_j} R_{z_j}^*) = (I_S - R_{z_j} R_{z_j}^*) (I_S - R_{z_i} R_{z_i}^*),
\]
for all \( i \neq j \). Therefore \( \{(I_S - R_{z_i} R_{z_i}^*)\}_{i=1}^n \) is a family of commuting orthogonal projections and hence
\[(2.1) \quad W = \bigcap_{i=1}^n W_i = \bigcap_{i=1}^n (S \ominus z_i S) = \bigcap_{i=1}^n \text{ran}(I_S - R_{z_i} R_{z_i}^*) = \text{ran}(\prod_{i=1}^n (I_S - R_{z_i} R_{z_i}^*)).\]

Now we present a wandering subspace theorem concerning doubly commuting submodules of \( H^2_E(\mathbb{D}^n) \). The result is a consequence of a several variables analogue of the classical Wold decomposition theorem as obtained by Gaspar and Suciu [GS]. We provide a direct proof (also see Corollary 3.2 in [Sa]).

**Theorem 2.3.** Let \( S \) be a doubly commuting submodule of \( H^2_E(\mathbb{D}^n) \). Then
\[
S = \sum_{k \in \mathbb{N}^n} \oplus z^k W.
\]

**Proof.** First, note that if \( M \) is a submodule of \( H^2_E(\mathbb{D}^n) \) then
\[
\bigcap_{k \in \mathbb{N}} R_{z_i}^k M \subseteq \bigcap_{k \in \mathbb{N}} M_{z_i}^k H^2_E(\mathbb{D}^n) = \{0\},
\]
for each \( i = 1, \ldots, n \). Therefore, \( R_{z_i} \) is a shift, that is, the unitary part 
\[
\bigcap_{k \in \mathbb{N}} R_{z_i}^{k} \mathcal{M}
\]
in the Wold decomposition (cf. [NF], [Sa]) of \( R_{z_i} \) on \( \mathcal{M} \) is trivial for all \( i = 1, \ldots, n \). Moreover, if \( S \) is doubly commuting then
\[
R_{z_i} (I_{S} - R_{z_j} R_{z_j}^{*}) = (I_{S} - R_{z_j} R_{z_j}^{*}) R_{z_i},
\]
for all \( i \neq j \). Therefore \( \mathcal{W}_j \) is a \( R_{z_i} \)-reducing subspace for all \( i \neq j \). Note also that for all \( 1 \leq m < n \),
\[
\bigcap_{i=1}^{m+1} \mathcal{W}_i = \text{ran} \left( \prod_{i=1}^{m+1} (I_{S} - R_{z_i} R_{z_i}^{*}) \right)
\]
\[
= \text{ran} \left( \prod_{i=1}^{m} (I_{S} - R_{z_i} R_{z_i}^{*}) - R_{z_{m+1}} R_{z_{m+1}}^{*} \prod_{i=1}^{m} (I_{S} - R_{z_i} R_{z_i}^{*}) \right)
\]
\[
= \text{ran} \left( \prod_{i=1}^{m} (I_{S} - R_{z_i} R_{z_i}^{*}) - R_{z_{m+1}} \prod_{i=1}^{m} (I_{S} - R_{z_i} R_{z_i}^{*}) R_{z_{m+1}}^{*} \right)
\]
\[
= (\mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m) \ominus z_{m+1}(\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m),
\]
and hence
\[
(\mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m) \ominus z_{m+1}(\mathcal{W}_1 \cap \cdots \cap \mathcal{W}_m) = \bigcap_{i=1}^{m+1} \mathcal{W}_i.
\]
We use mathematical induction to prove that for all \( 2 \leq m \leq n \), we have
\[
\mathcal{S} = \sum_{k \in \mathbb{N}^m} \oplus z^k (\mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m).
\]
First, by Wold decomposition theorem for the shift \( R_{z_1} \) on \( \mathcal{S} \) we have
\[
\mathcal{S} = \sum_{k_1 \in \mathbb{N}} \oplus R_{z_1}^{k_1} \mathcal{W}_1 = \sum_{k_1 \in \mathbb{N}} \oplus z_{1}^{k_1} \mathcal{W}_1.
\]
Again by applying Wold decomposition for \( R_{z_2} \mid \mathcal{W}_1 \in \mathcal{L}(\mathcal{W}_1) \) we have
\[
\mathcal{W}_1 = \sum_{k_2 \in \mathbb{N}} \oplus R_{z_2}^{k_2} (\mathcal{W}_1 \ominus z_2 \mathcal{W}_1) = \sum_{k_2 \in \mathbb{N}} \oplus z_{2}^{k_2} (\mathcal{W}_1 \cap \mathcal{W}_2),
\]
and hence
\[
\mathcal{S} = \sum_{k_1 \in \mathbb{N}} \oplus z_{1}^{k_1} \left( \sum_{k_2 \in \mathbb{N}} \oplus z_{2}^{k_2} (\mathcal{W}_1 \cap \mathcal{W}_2) \right) = \sum_{k_1, k_2 \in \mathbb{N}} \oplus z_{1}^{k_1} z_{2}^{k_2} (\mathcal{W}_1 \cap \mathcal{W}_2).
\]
Finally, let
\[
\mathcal{S} = \sum_{k \in \mathbb{N}^m} \oplus z^k (\mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m),
\]
for some \( m < n \). Then we again apply the Wold decomposition on the isometry
\[
R_{z_{m+1}} \mid \mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m \in \mathcal{L}(\mathcal{W}_1 \cap \ldots \cap \mathcal{W}_m)\]
to obtain
\[ W_1 \cap \ldots \cap W_m = \sum_{k_{m+1} \in \mathbb{N}} \bigoplus z_{m+1}^{k_{m+1}} \left( (W_1 \cap \ldots \cap W_m) \oplus z_{m+1} W_1 \cap \ldots \cap W_m \right) \]
\[ = \sum_{k_{m+1} \in \mathbb{N}} \bigoplus z_{m+1}^{k_{m+1}} (W_1 \cap \ldots \cap W_m \cap W_{m+1}) , \]
which yields
\[ S = \sum_{k \in \mathbb{N}^{m+1}} \bigoplus z^k (W_1 \cap \ldots \cap W_{m+1}). \]
This completes the proof.

We now turn to the proof of Theorem 1.3.

**Proof of Theorem 1.3.** By Theorem 2.3 we have
\[ S = \sum_{k \in \mathbb{N}^n} \bigoplus z^k (\bigcap_{i=1}^n W_i). \]

Now define the Hilbert space \( E \) by
\[ E = \bigcap_{i=1}^n W_i , \]
and the linear operator \( V : H^2_E(\mathbb{D}^n) \to H^2_{E_z}(\mathbb{D}^n) \) by
\[ V \left( \sum_{k \in \mathbb{N}^n} a_k z^k \right) = \sum_{k \in \mathbb{N}^n} M_z a_k , \]
where
\[ \sum_{k \in \mathbb{N}^n} a_k z^k \in H^2_E(\mathbb{D}^n) \]
and \( a_k \in E \) for all \( k \in \mathbb{N}^n \). Observe that
\[ \| \sum_{k \in \mathbb{N}^n} M_{z} a_k \|_{H^2_E(\mathbb{D}^n)}^2 = \| \sum_{k \in \mathbb{N}^n} z^k a_k \|_{H^2_{E_z}(\mathbb{D}^n)}^2 = \sum_{k \in \mathbb{N}^n} \| z^k a_k \|_{H^2_{E_z}(\mathbb{D}^n)}^2 , \]
where the last equality follows from the orthogonal decomposition of \( S \) in (2.2). Therefore,
\[ \| \sum_{k \in \mathbb{N}^n} M_{z} a_k \|_{H^2_E(\mathbb{D}^n)}^2 = \| \sum_{k \in \mathbb{N}^n} z^k a_k \|_{H^2_{E_z}(\mathbb{D}^n)}^2 = \sum_{k \in \mathbb{N}^n} \| z^k a_k \|_{H^2_{E_z}(\mathbb{D}^n)}^2 = \sum_{k \in \mathbb{N}^n} \| a_k \|_{E}^2 \]
and hence \( V \) is an isometry. Moreover, for all \( k \in \mathbb{N}^n \) and \( \eta \in E \) we have
\[ VM_{z_k}(z^k \eta) = V(z^{k+\epsilon_i} \eta) = M_z^{k+\epsilon_i} \eta = M_{z_k}(M_z^{k} \eta) = M_{z_k} V(z^k \eta) , \]
that is, \( VM_{z_i} = M_{z_i} V \) for all \( i = 1, \ldots , n \). Hence \( V \) is a module map. Therefore,
\[ V = M_{\Theta} , \]
for some bounded holomorphic function $\Theta \in H^\infty_{E \to E_n}(\mathbb{D}^n)$ (cf. page 655 in [BLTT]). Moreover, since $V$ is an isometry, we have

$$M_{\Theta}^*M_{\Theta} = I_{H^2_E(\mathbb{D}^n)},$$

that is, that $\Theta$ is an inner function. Also since $M_{z_i}E \subseteq S$ for all $i = 1, \ldots, n$ we have that

$$\text{ran} V \subseteq S.$$

Also by (2.2) that $S \subseteq \text{ran} V$. Hence it follows that

$$\text{ran} V = \text{ran} M_{\Theta} = S,$$

that is,

$$S = \Theta H^2_E(\mathbb{D}^n).$$

Finally, for all $i = 1, \ldots, n$, we have

$$S \ominus z_iS = \Theta H^2_E(\mathbb{D}^n) \ominus z_i\Theta H^2_E(\mathbb{D}^n) = \{\Theta f : f \in H^2_E(\mathbb{D}^n), M_{z_i}^*\Theta f = 0\},$$

and hence by (2.1)

$$E = \bigcap_{i=1}^n \mathcal{W}_i = \bigcap_{i=1}^n (S \ominus z_iS) = \{\Theta f : M_{z_i}^*\Theta f = 0, f \in H^2_E(\mathbb{D}^n), \forall i = 1, \ldots, n\}
\subseteq \{g \in H^2_E(\mathbb{D}^n) : M_{z_i}^*g = 0, \forall i = 1, \ldots, n\} = E_*,$$

that is,

$$E \subseteq E_*.$$

To prove the converse part, let $S = M_{\Theta}H^2_E(\mathbb{D}^n)$ be a submodule of $H^2_E(\mathbb{D}^n)$ for some inner function $\Theta \in H^\infty_{E \to E_n}(\mathbb{D}^n)$. Then

$$P_S = M_{\Theta}M_{\Theta}^*,$$

and hence for all $i \neq j$,

$$M_{z_i}P_SM_{z_j}^* = M_{z_i}M_{\Theta}M_{\Theta}^*M_{z_j} = M_{\Theta}M_{z_i}M_{z_j}^*M_{\Theta}^* = M_{\Theta}M_{z_j}^*M_{z_i}M_{\Theta}^* = M_{\Theta}M_{z_j}^*M_{\Theta}M_{z_i} = M_{\Theta}M_{z_j}^*M_{z_j}M_{\Theta}^* = P_SM_{z_j}^*M_{z_j}P_S.$$

This implies

$$R_{z_j}^*R_{z_i} = P_SM_{z_j}^*P_SM_{z_i}|_S = P_SM_{z_j}^*P_SM_{z_i}|_S = M_{z_i}P_SM_{z_j}^* = R_{z_i}R_{z_j}^*,$$

that is, $S$ is a doubly commuting submodule. This completes the proof. $\blacksquare$
3. TOLOKONNIKOV’S LEMMA FOR THE POLYDISC

We will need the following lemma, which is a polydisc version of a similar result proved in the case of the disc in Nikolski’s book [N]*p.44-45. Here we use the notation $M_g$ for the multiplication operator on $H^2_E$ induced by $g \in H^\infty_{E \to E}$.

**Lemma 3.1** (Lemma on Local Rank). Let $E, E_c$ be Hilbert spaces, with $\dim E, \dim E_c < \infty$. Let $g \in H^\infty_{E \to E}(\mathbb{D}^n)$ be such that

$$\ker M_g = \{ h \in H^2_{E_c}(\mathbb{D}^n) : gh(z) \equiv 0 \} = \Theta H^2_{E_a}(\mathbb{D}^n),$$

where $E_a$ is a Hilbert space and $\Theta$ is a $\mathcal{L}(E_a, E_c)$-valued inner function. Then

$$\dim E_c = \dim E_a + \text{rank } g,$$

where $\text{rank } g := \max_{\zeta \in \mathbb{D}^n} g(\zeta)$.

**Proof.** We have $\ker M_g = \{ h \in H^2_{E_c}(\mathbb{D}^n) : gh \equiv 0 \}$. If $\zeta \in \mathbb{D}^n$, then let

$$[\ker M_g](\zeta) := \{ h(\zeta) : h \in \ker M_g \}.$$

We claim that $[\ker M_g](\zeta) = \Theta(\zeta) E_a$. Indeed, let $v \in [\ker M_g](\zeta)$. Then $v = h(\zeta)$ for some element $h \in \ker M_g = \Theta H^2_{E_a}(\mathbb{D}^n)$. So $h = \Theta \varphi$, for some $\varphi \in H^2_{E_a}(\mathbb{D}^n)$. In particular, $v = h(\zeta) = \Theta(\zeta) \varphi(\zeta)$, where $\varphi(\zeta) \in E_a$. So

(3.1)  \[ [\ker M_g](\zeta) \subset \Theta(\zeta) E_a. \]

On the other hand, if $w \in \Theta(\zeta) E_a$, then $w = \Theta(\zeta) x$, where $x \in E_a$. Consider the constant function $x$ mapping $\mathbb{D} \ni z \mapsto x \in E_a$. Clearly $x \in H^2_{E_a}(\mathbb{D}^n)$. So $h := \Theta x \in \Theta H^2_{E_a}(\mathbb{D}^n) = \ker M_g$. Hence $w = \Theta(\zeta) x = (\Theta x)(\zeta) = h(\zeta)$, and so $w \in [\ker M_g](\zeta)$. So we also have that

(3.2)  \[ \Theta(\zeta) E_a \subset [\ker M_g](\zeta). \]

Our claim that $[\ker M_g](\zeta) = \Theta(\zeta) E_a$ follows from (3.1) and (3.2).

Suppose that for a $\zeta \in \mathbb{D}^n$, $v \in [\ker M_g](\zeta)$. Then $v = h(\zeta)$ for some $h \in \ker M_g$. Thus $gh \equiv 0$ in $\mathbb{D}^n$, and in particular, $g(\zeta)v = g(\zeta)h(\zeta) = 0$. Thus $v \in \ker g(\zeta)$. So we have that $[\ker M_g](\zeta) \subset \ker g(\zeta)$. Hence $\dim [\ker M_g](\zeta) \leq \dim \ker g(\zeta)$. Consequently

$$\dim \Theta(\zeta) E_a = \dim [\ker M_g](\zeta) \leq \dim \ker g(\zeta) = \dim E_c - \text{rank } g(\zeta),$$

where the last equality follows from the Rank-Nullity Theorem. Since $\Theta$ is inner, we have that the boundary values of $\Theta$ satisfy $\Theta(\zeta)^* \Theta(\zeta) = I_{E_a}$ for almost all $\zeta \in \mathbb{T}^n$. So there is an open set $U \subset \mathbb{D}^n$ such that for all $\zeta \in U$

$$\dim E_a = \dim \Theta(\zeta) E_a.$$
But from the definition of the rank of \( g \), we know that there is a \( \zeta_* \in \mathbb{D}^n \) such that we have \( k := \text{rank} \; g = \text{rank} \; g(\zeta_*) \). So there is a \( k \times k \) submatrix of \( g(\zeta_*) \) which is invertible. Now look at the determinant of this \( k \times k \) submatrix of \( g \). This is a holomorphic function, and so it cannot be identically zero in the open set \( U \). So there must exist a point \( \zeta_1 \in U \subset \mathbb{D}^n \) such that \( \text{rank} \; g = \text{rank} \; g(\zeta_1) \) and \( \dim E_a = \dim \Theta(\zeta_1)E_a \). Hence \( \dim E_a \leq \dim E_c - \text{rank} \; g \).

For the proof of the opposite inequality, let us consider a principal minor \( g_1(\zeta_1) \) of the matrix of the operator \( g(\zeta_1) \) (with respect to two arbitrary fixed bases in \( E_c \) and \( E \) respectively). Then \( \det g_1 \in \mathcal{H}^\infty \), \( \det g_1 \not\equiv 0 \). Let \( E_c = E_{c,1} \oplus E_{c,2} \), \( E = E_1 \oplus E_2 \) (\( \dim E_{c,1} = \dim E_1 = \text{rank} \; g(\zeta_1) \)) be the decompositions of the spaces \( E_c \) and \( E \) corresponding to this minor, and let

\[
g(\zeta) = \begin{bmatrix} g_1(\zeta) & g_2(\zeta) \\ \gamma_1(\zeta) & \gamma_2(\zeta) \end{bmatrix}, \quad \zeta \in \mathbb{D}^n,
\]

be the matrix representation of \( g(\zeta) \) with respect to this decomposition. Owing to our assumption on the rank, it follows that there is a matrix function \( \zeta \mapsto \mathcal{W}(\zeta) \) such that

\[
\begin{bmatrix} \gamma_1(\zeta) & \gamma_2(\zeta) \end{bmatrix} = \mathcal{W}(\zeta) \begin{bmatrix} g_1(\zeta) & g_2(\zeta) \end{bmatrix}.
\]

So \( \gamma_2(\zeta) = \mathcal{W}(\zeta)g_2(\zeta) = (\gamma_1(\zeta)(g_1(\zeta))^{-1})g_2(\zeta) \). Thus with \( g_1^{\text{co}} := (\det g_1)g_1^{-1} \), we have

\[
\gamma_2 \det g_1 = \gamma_1 g_1^{\text{co}} g_2,
\]

and using this we get the inclusion \( M_\Omega \mathcal{H}^2_{E_c \oplus E}(\mathbb{D}^n) \subset \ker M_g \), where \( \Omega \in \mathcal{H}_{E_c \to E}(\mathbb{D}^n) \) is given by

\[
\Omega = \begin{bmatrix} g_1^{\text{co}} g_2 \\ - \det g_1 \end{bmatrix}.
\]

We have \( \text{rank} \; \Omega = \dim E_{c,2} = \dim E_c - \text{rank} \; g = \dim \ker(\mathcal{W}(\zeta_1)) \). Consequently, we obtain \( \dim [\ker M_\Theta](\zeta_1) \geq \dim \ker(\mathcal{W}(\zeta_1)) \).

We now turn to the extension of Tolokonnikov’s Lemma to the polydisc.

**Proof of Theorem 1.4.** (ii) \( \Rightarrow \) (i): If \( g := P_E F^{-1} \), then \( gf = I \). It only remains to show that the operators \( M_{z_1}, \ldots, M_{z_n} \) are doubly commuting on the space \( \ker M_g \). Let \( \Theta, \Gamma \) be such that:

\[
F = \begin{bmatrix} f & \Theta \\ \Gamma \end{bmatrix} \quad \text{and} \quad F^{-1} = \begin{bmatrix} g \\ \Gamma \end{bmatrix}.
\]

Since \( FF^{-1} = I_{E_c} \), it follows that \( fg + \Theta \Gamma = I_{E_c} \). Thus if \( h \in \mathcal{H}^2_{E_c}(\mathbb{D}^n) \) is such that \( gh = 0 \), then \( \Theta(\Gamma h) = h \), and so \( h \in \Theta \mathcal{H}^2_{E_c \oplus E}(\mathbb{D}^n) \). Hence \( \ker M_g \subset \ker M_\Theta \). Also, since \( F^{-1} F = I \), it follows that \( g\Theta = 0 \), and so \( \ker M_\Theta \subset \ker M_g \). So \( \ker M_g = \ker M_\Theta = \Theta \mathcal{H}^2_{E_c \oplus E}(\mathbb{D}^n) \). By Theorem 1.3, the operators \( M_{z_1}, \ldots, M_{z_n} \) must doubly commute on the subspace \( \ker M_g \).
(i) $\Rightarrow$ (ii): Let \[ S := \{ h \in H^2_{E_c}(\mathbb{D}^n) : g(z)h(z) \equiv 0 \} = \ker g. \]

$S$ is a closed non-zero invariant subspace of $H^2_{E_c}(\mathbb{D}^n)$. Also, by assumption, $M_{z_1}, \ldots, M_{z_n}$ are doubly commuting operators on $S$. Then by the above Theorem 1.3, there exists an auxiliary Hilbert space $E_a$ and an inner function $\Theta$ with values in $L(E_a, E_c)$ with $\dim E_a \leq \dim E_c$ such that

$S = \Theta H^2_{E_a}(\mathbb{D}^n)$.

By the Lemma on Local Rank, $\dim E_a = \dim E_c - \text{rank } g = \dim E_c - \dim E = \dim (E_c \ominus E)$. Let $U$ be a (constant) unitary operator from $E_c \ominus E$ to $E_a$ and define $\Theta := \tilde{\Theta}U$. Then $\Theta$ is inner, and we have that $\ker g = \Theta H^2_{E_c \ominus E}(\mathbb{D}^n)$. To get $F \in H^\infty_{E_c \rightarrow E_c}(\mathbb{D}^n)$ define the function $F$ for $z \in \mathbb{D}^n$ by

\[
F(z)e := \begin{cases}
  f(z)e & \text{if } e \in E \\
  \Theta(z)e & \text{if } e \in E_c \ominus E.
\end{cases}
\]

We note that $F \in H^\infty(\mathbb{D}^n)$ and $F|_E = f$. We now show that $F$ is invertible. With this in mind, we first observe that

\[(I - fg)H^2_{E_c}(\mathbb{D}^n) \subset \Theta H^2_{E_c \ominus E}(\mathbb{D}^n) = \ker M_g.\]

This follows since $g(I - fg)h = gh - gh = 0$ for all $h \in H^2_{E_c}(\mathbb{D}^n)$. Thus we have that $\Theta^*(I - fg) \in H^\infty_{E_c \rightarrow E_c}(\mathbb{D}^n)$. Now, define $\Omega = g \Theta^*(I - fg)$. Clearly $\Omega \in H^\infty_{E_c \rightarrow E_c}(\mathbb{D}^n)$. Next, note that

\[F\Omega = fg + \Theta \Theta^*(I - fg) = I.\]

Similarly,

\[\Omega F = gfP_E + \Theta^*(I - fg)(fP_E + \Theta P_{E_c \ominus E}) = P_E + \Theta^*(fP_E - fgfP_E + \Theta P_{E_c \ominus E}) = P_E + \Theta^*\Theta P_{E_c \ominus E} = I.\]

Thus we have that $F^{-1} \in H^\infty(\mathbb{D}^n; E_c \rightarrow E_e)$. \[\blacksquare\]

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