ANDO DILATIONS, VON NEUMANN INEQUALITY, AND DISTINGUISHED VARIETIES

B. KRISHNA DAS AND JAYDEB SARKAR

Abstract. Let $\mathbb{D}$ denote the unit disc in the complex plane $\mathbb{C}$ and let $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$ be the unit bidisc in $\mathbb{C}^2$. Let (T_1, T_2) be a pair of commuting contractions on a Hilbert space $\mathcal{H}$. Let $\dim \operatorname{ran}(I_{\mathcal{H}} - T_j T_j^*) < \infty$, $j = 1, 2$, and let T_1 be a pure contraction. Then there exists a variety $V \subseteq \overline{\mathbb{D}}^2$ such that for any polynomial $p \in \mathbb{C}[z_1, z_2]$, the inequality

$$\|p(T_1, T_2)\|_{\mathcal{B}(\mathcal{H})} \leq \|p\|_V$$

holds. If, in addition, T_2 is pure, then

$$V = \{(z_1, z_2) \in \mathbb{D}^2 : \det(\Psi(z_1) - z_2 I_{\mathbb{C}^n}) = 0\}$$

is a distinguished variety, where Ψ is a matrix-valued analytic function on $\mathbb{D}$ that is unitary on $\partial\mathbb{D}$. Our results comprise a new proof, as well as a generalization, of Agler and McCarthy's sharper von Neumann inequality for pairs of commuting and strictly contractive matrices.

Notation

$\mathbb{D}$	Open unit disc in the complex plane $\mathbb{C}$.
$\mathbb{D}^2$	Open unit bidisc in $\mathbb{C}^2$.
$\mathcal{H}$, $\mathcal{E}$	Hilbert spaces.
$\mathcal{B}(\mathcal{H})$	The space of all bounded linear operators on $\mathcal{H}$.
$H^2_{\mathcal{E}}(\mathbb{D})$	$\mathcal{E}$-valued Hardy space on $\mathbb{D}$.
M_z	Multiplication operator by the coordinate function z.
$H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$	Set of $\mathcal{B}(\mathcal{E})$-valued bounded analytic functions on $\mathbb{D}$.

All Hilbert spaces are assumed to be over the complex numbers. For a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, we denote by $P_{\mathcal{S}}$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{S}$.

1. Introduction

The famous von Neumann inequality [8] states that: if T is a linear operator on a Hilbert space $\mathcal{H}$ of norm one or less (that is, T is a contraction), then for any polynomial $p \in \mathbb{C}[z]$, the inequality

$$\|p(T)\|_{\mathcal{B}(\mathcal{H})} \leq \|p\|_{\mathbb{D}}$$

holds. Here $\|p\|_{\mathbb{D}}$ denotes the supremum of $|p(z)|$ over the unit disc $\mathbb{D}$.

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In 1953, Sz.-Nagy [6] proved that a linear operator on a Hilbert space is a contraction if and only if the operator has a unitary dilation. This immediately gives a simple and elegant proof of the von Neumann inequality.

In 1963, Ando [3] proved the following generalization of Sz.-Nagy's dilation theorem: Any pair of commuting contractions has a commuting unitary dilation. As an immediate consequence, we obtain the following two variables von Neumann inequality:
Theorem (Ando): Let (T_1, T_2) be a pair of commuting contractions on a Hilbert space $\mathcal{H}$. Then for any polynomial $p \in \mathbb{C}[z_1, z_2]$, the inequality

$$\|p(T_1, T_2)\|_{\mathcal{B}(\mathcal{H})} \leq \|p\|_{\mathbb{D}^2}$$

holds.

However, for three or more commuting contractions the above von Neumann type inequality is not true in general (see [4], [12]). An excellent source of further information on von Neumann inequality is the monograph by Pisier [11].

In a recent seminal paper, Agler and McCarthy [2] proved a sharper version of von-Neumann inequality for pairs of commuting and strictly contractive matrices (see Theorem 3.1 in [2]): Two variables von Neumann inequality can be improved in the case of a pair of commuting and strictly contractive operators (T_1, T_2) on a finite dimensional Hilbert space $\mathcal{H}$ to

$$\|p(T_1, T_2)\|_{\mathcal{B}(\mathcal{H})} \leq \|p\|_V \qquad (p \in \mathbb{C}[z_1, z_2]),$$

where V is a distinguished variety depending on the pair (T_1, T_2). Again, $\|p\|_V$ is the supremum of $|p(z_1, z_2)|$ over V. The proof of this result involves many different techniques including isometric dilation of (T_1, T_2) to a vector-valued Hardy space and approximation of commuting matrices by digonalizable commuting matrices.

We recall that a non-empty set V in $\mathbb{C}^2$ is a *distinguished variety* if there is a polynomial $p \in \mathbb{C}[z_1, z_2]$ such that

$$V = \{(z_1, z_2) \in \mathbb{D}^2 : p(z_1, z_2) = 0\},$$

and V exits the bidisc through the distinguished boundary, that is,

$$\overline{V} \cap \partial\mathbb{D}^2 = \overline{V} \cap (\partial\mathbb{D} \times \partial\mathbb{D}).$$

Here $\partial\mathbb{D}^2$ and $\partial\mathbb{D} \times \partial\mathbb{D}$ denote the boundary and the distinguished boundary of the bidisc repectively, and $\overline{V}$ is the closure of V in $\overline{\mathbb{D}^2}$. We denote by ∂V the set $\overline{V} \cap \partial\mathbb{D}^2$, the boundary of V within the zero set of the polynomial p and $\overline{\mathbb{D}^2}$.

In the same paper [2], Agler and McCarthy proved that a distinguished variety can be represented by a rational matrix inner function in the following sense (see Theorem 1.12 in [2]): Let $V \subseteq \mathbb{C}^2$. Then V is a distinguished variety if and only if there exists a rational matrix inner function $\Psi \in H^\infty_{\mathcal{B}(\mathbb{C}^n)}(\mathbb{D})$, for some $n \geq 1$, such that

$$V = \{(z_1, z_2) \in \mathbb{D}^2 : \det(\Psi(z_1) - z_2 I_{\mathbb{C}^n}) = 0\}.$$

The proof uses dilation and model theoretic techniques (see page 140 in [2]) in the sense of Sz.-Nagy and Foias [7]. See Knese [5] for another proof. For similar results in the symmetrized bidisc setting see also [9].

In this paper, in Theorem 3.1, we obtain an explicit way to construct isometric dilations of a large class of commuting pairs of contractions: Let (T_1, T_2) be a pair of commuting operators on a Hilbert space $\mathcal{H}$ and $\|T_j\| \leq 1$, $j = 1, 2$. Let $\dim \operatorname{ran}(I_{\mathcal{H}} - T_j T_j^*) < \infty$, $j = 1, 2$, and T_1 be a pure contraction (that is, $\lim_{m\to\infty} \|T_1^{*m} h\| = 0$ for all $h \in \mathcal{H}$). Set $\mathcal{E} = \operatorname{ran}(I_{\mathcal{H}} - T_1 T_1^*)$. Then there is an $\mathcal{B}(\mathcal{E})$-valued inner function Ψ such that the commuting isometric pair (M_z, M_Ψ) on the $\mathcal{E}$-valued Hardy space $H^2_{\mathcal{E}}(\mathbb{D})$ is an isometric dilation of (T_1, T_2).
We actually prove a more general dilation result in Theorem 3.2.

Then in Theorem 4.3, we prove: There exists a variety $V \subseteq \overline{\mathbb{D}}^2$ such that

$$\|p(T_1, T_2)\|_{\mathcal{B}(\mathcal{H})} \leq \|p\|_V \qquad (p \in \mathbb{C}[z_1, z_2]).$$

If, in addition, T_2 is pure, then V can be taken to be a distinguished variety.

It is important to note that every distinguished variety, by definition, is a subset of the bidisc $\mathbb{D}^2$.

Our results comprise both a new proof, as well as a generalization, of Agler and McCarthy's sharper von Neumann inequality for pairs of commuting and strictly contractive matrices (see the final paragraph in Section 4).

The remainder of this paper is built as follows. In Section 2, we first recall some basic definitions and results. We then proceed to prove an important lemma which will be used in the sequel. Dilations of pairs of commuting contractions are studied in Section 3. In Section 4, we use results from the previous section to show a sharper von Neumann inequality for pairs of pure commuting contractive tuples with finite dimensional defect spaces. In the concluding section, Section 5, among other things, we prove that the distinguished variety in our von Neumann inequality is independent of the choice of (T_1, T_2) and (T_2, T_1).

2. Preliminaries and a Correlation lemma

First we recall some definitions of objects we are going to use and fix few notations.

Let T be a contraction on a Hilbert space $\mathcal{H}$ (that is, $\|Tf\| \leq \|f\|$ for all $f \in \mathcal{H}$ or, equivalently, if $I_{\mathcal{H}} - TT^* \geq 0$). Recall again that T is *pure* if $\lim_{m\to 0} \|T^{*m} f\| = 0$ for all $f \in \mathcal{H}$.

Let T be a contraction and $\mathcal{E}$ be a Hilbert space. An isometry $\Gamma : \mathcal{H} \to H^2_{\mathcal{E}}(\mathbb{D})$ is called an isometric dilation of T (cf. [10]) if

$$\Gamma T^* = M_z^* \Gamma.$$

If, in addition,

$$H^2_{\mathcal{E}}(\mathbb{D}) = \overline{\operatorname{span}}\{z^m \Gamma f : m \in \mathbb{N}, f \in \mathcal{H}\},$$

then we say that $\Gamma : \mathcal{H} \to H^2_{\mathcal{E}}(\mathbb{D})$ is a minimal isometric dilation of T.

Now let T be a pure contraction on a Hilbert space $\mathcal{H}$. Set

$$\mathcal{D}_T = \overline{\operatorname{ran}}(I_{\mathcal{H}} - TT^*), \qquad D_T = (I_{\mathcal{H}} - TT^*)^{\frac{1}{2}}.$$

Then $\Pi : \mathcal{H} \to H^2_{\mathcal{D}_T}(\mathbb{D})$ is a minimal isometric dilation of T (cf. [10]), where

$$(\Pi h)(z) = D_T (I_{\mathcal{H}} - zT^*)^{-1} h \qquad (z \in \mathbb{D}, h \in \mathcal{H}). \tag{2.1}$$

In particular, $\mathcal{Q} := \mathrm{ran}\Pi$ is a M_z^*-invariant subspace of $H^2_{\mathcal{D}_T}(\mathbb{D})$ and

$$T \cong P_{\mathcal{Q}} M_z|_{\mathcal{Q}}.$$

Finally, recall that a contraction T on $\mathcal{H}$ is said to be *completely non-unitary* if there is no non-zero T-reducing subspace $\mathcal{S}$ of $\mathcal{H}$ such that $T|_{\mathcal{S}}$ is a unitary operator. It is well known that for every contraction T on a Hilbert space $\mathcal{H}$ there exists a unique *canonical decomposition* $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ of $\mathcal{H}$ reducing T, such that $T|_{\mathcal{H}_0}$ is unitary and $T|_{\mathcal{H}_1}$ is completely non-unitary. We therefore have the following decomposition of T:

$$T = \begin{bmatrix} T|_{\mathcal{H}_0} & 0 \\ 0 & T|_{\mathcal{H}_1} \end{bmatrix}.$$

We now turn to the study of contractions with finite dimensional defect spaces. Let (T_1, T_2) be a pair of commuting contractions and $\dim \mathcal{D}_{T_j} < \infty$, $j = 1, 2$. Since

$$(I_{\mathcal{H}} - T_1T_1^*) + T_1(I_{\mathcal{H}} - T_2T_2^*)T_1^* = T_2(I_{\mathcal{H}} - T_1T_1^*)T_2^* + (I_{\mathcal{H}} - T_2T_2^*),$$

it follows that

$$\|D_{T_1}h\|^2 + \|D_{T_2}T_1^*h\|^2 = \|D_{T_1}T_2^*h\|^2 + \|D_{T_2}h\|^2 \quad (h \in \mathcal{H}).$$

Thus

$$U : \{D_{T_1}h \oplus D_{T_2}T_1^*h : h \in \mathcal{H}\} \to \{D_{T_1}T_2^*h \oplus D_{T_2}h : h \in \mathcal{H}\}$$

defined by

$$U\left(D_{T_1}h, D_{T_2}T_1^*h\right) = \left(D_{T_1}T_2^*h, D_{T_2}h\right) \qquad (h \in \mathcal{H}), \tag{2.2}$$

is an isometry. Moreover, since $\dim \mathcal{D}_{T_j} < \infty$, $j = 1, 2$, it follows that U extends to a unitary, denoted again by U, on $\mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2}$. In particular, there exists a unitary operator

$$U := \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2} \to \mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2}, \tag{2.3}$$

such that (2.2) holds.

The following lemma plays a key role in our considerations.

Lemma 2.1. *Let (T_1, T_2) be a pair of commuting contractions on a Hilbert space $\mathcal{H}$. Let T_1 be pure and dim $\mathcal{D}_{T_j} < \infty$, $j = 1, 2$. Then with $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ as above we have*

$$D_{T_1}T_2^* = AD_{T_1} + \sum_{n=0}^{\infty} BD^nCD_{T_1}T_1^{*n+1},$$

where the series converges in the strong operator topology.

Proof. For each $h \in \mathcal{H}$ we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} D_{T_1}h \\ D_{T_2}T_1^*h \end{bmatrix} = \begin{bmatrix} D_{T_1}T_2^*h \\ D_{T_2}h \end{bmatrix},$$

that is,

$$D_{T_1}T_2^*h = AD_{T_1}h + BD_{T_2}T_1^*h, \tag{2.4}$$

and

$$D_{T_2}h = CD_{T_1}h + DD_{T_2}T_1^*h. \tag{2.5}$$

By replacing h by T_1^*h in (2.5), we have

$$D_{T_2}T_1^*h = CD_{T_1}T_1^*h + DD_{T_2}T_1^{*2}h.$$

Then by (2.4), we have

$$\begin{aligned} D_{T_1}T_2^*h &= AD_{T_1}h + BD_{T_2}T_1^*h \\ &= AD_{T_1}h + B(CD_{T_1}T_1^*h + DD_{T_2}T_1^{*2}h), \end{aligned}$$

that is,

$$D_{T_1}T_2^*h = AD_{T_1}h + BCD_{T_1}T_1^*h + BDD_{T_2}T_1^{*2}h. \tag{2.6}$$

In (2.6), again replacing $D_{T_2}T_1^{*2}h$ by $CD_{T_1}T_1^{*2}h + DD_{T_2}T_1^{*3}h$, we have

$$D_{T_1}T_2^*h = AD_{T_1}h + BCD_{T_1}T_1^*h + BDCD_{T_1}T_1^{*2}h + BD^2D_{T_2}T_1^{*3}h.$$

Going on in this way, we obtain

$$D_{T_1}T_2^*h = AD_{T_1}h + \sum_{n=0}^{m} BD^nCD_{T_1}T_1^{*n+1}h + (BD^{m+1}D_{T_2}T_1^{*m+2}h) \quad (m \in \mathbb{N}).$$

By $\|D\| \leq 1$ and

$$\lim_{m\to\infty} T_1^{*m}h = 0,$$

it follows that

$$\lim_{m\to\infty}(BD^mD_{T_2}T_1^{*m+1}h) = 0.$$

Finally, the convergence of the desired series follows from the fact that

$$\begin{aligned} \|D_{T_1}T_2^*h - AD_{T_1}h - \sum_{n=0}^{m} BD^nCD_{T_1}T_1^{*n+1}h\| &= \|BD^{m+1}CD_{T_2}T_1^{*m+2}h\| \\ &\leq \|T_1^{*m+2}h\|, \end{aligned}$$

for all $m \in \mathbb{N}$ and again that $\lim_{m\to\infty} T_1^{*m}h = 0$. This completes the proof of the lemma. □

Remark 2.1. With the above assumptions, the conclusion of Lemma 2.1 remains valid if we add the possibility

$$\dim \mathcal{D}_{T_1} = \infty, \quad \textit{or} \quad \dim \mathcal{D}_{T_2} = \infty.$$

To this end, let $\dim \mathcal{D}_{T_1} = \infty$, or $\dim \mathcal{D}_{T_2} = \infty$. Then there exists an infinite dimensional Hilbert space $\mathcal{D}$ such that the isometry

$$U : \{D_{T_1}h \oplus D_{T_2}T_1^*h : h \in \mathcal{H}\} \oplus \{0_{\mathcal{D}}\} \to \{D_{T_1}T_2^*h \oplus D_{T_2}h : h \in \mathcal{H}\} \oplus \{0_{\mathcal{D}}\}$$

defined by

$$U\left(D_{T_1}h, D_{T_2}T_1^*h, 0_{\mathcal{D}}\right) = \left(D_{T_1}T_2^*h, D_{T_2}h, 0_{\mathcal{D}}\right) \qquad (h \in \mathcal{H}),$$

extends to a unitary, denoted again by U, on $\mathcal{D}_{T_1} \oplus (\mathcal{D}_{T_2} \oplus \mathcal{D})$. Now we proceed similarly with the unitary matrix

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathcal{D}_{T_1} \oplus (\mathcal{D}_{T_2} \oplus \mathcal{D})),$$

to obtain the same conclusion as in Lemma 2.1.

3. Ando Dilations

One of the aims of this section is to obtain an explicit isometric dilation of a pair of commuting tuple of pure contractions (see Theorem 3.2). In particular, the construction of isometric dilations of pairs of commuting contractions with finite defect indices (see Theorem 3.1) will be important to us in the sequel.

We begin by briefly recalling some standard facts about transfer functions (cf. [1]). Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, and

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2),$$

be a unitary operator. Then the $\mathcal{B}(\mathcal{H}_1)$-valued analytic function τ_U on $\mathbb{D}$ defined by

$$\tau_U(z) := A + zB(I - zD)^{-1}C \qquad (z \in \mathbb{D}),$$

is called the *transfer function* of U. Using $U^*U = I$, a standard computation yields (cf. [1])

$$I - \tau_U(z)^*\tau_U(z) = (1 - |z|^2)C^*(I - \bar{z}D^*)^{-1}(I - zD)^{-1}C \qquad (z \in \mathbb{D}). \tag{3.1}$$

Now let $\mathcal{H}_1 = \mathbb{C}^m$ and $\mathcal{H}_2 = \mathbb{C}^n$, and let U be as above. Then τ_U is a contractive rational matrix-valued function on $\mathbb{D}$. Moreover, τ_U is unitary on $\partial\mathbb{D}$ (see page 138 in [2]). Thus τ_U is a rational matrix-valued inner function.

We now turn our attention to the study of the transfer function of the unitary matrix U^* in (2.3). Set

$$\Psi(z) := \tau_{U^*}(z) = A^* + zC^*(I - zD^*)^{-1}B^* \qquad (z \in \mathbb{D}). \tag{3.2}$$

Then Ψ is a $\mathcal{B}(\mathcal{D}_{T_1})$-valued inner function on $\mathbb{D}$. Thus the multiplication operator M_Ψ on $H^2_{\mathcal{D}_{T_1}}(\mathbb{D})$ defined by

$$(M_\Psi f)(w) = \Psi(w)f(w) \qquad (w \in \mathbb{D}, f \in H^2_{\mathcal{D}_{T_1}}(\mathbb{D})),$$

is an isometry (cf. [7]).

Now, we are ready to prove our first main result.

Theorem 3.1. *Let (T_1, T_2) be a pair of commuting contractions on a Hilbert space $\mathcal{H}$. Let T_1 be pure and dim $\mathcal{D}_{T_j} < \infty$, $j = 1, 2$. Then there exist an isometry $\Pi : \mathcal{H} \to H^2_{\mathcal{D}_{T_1}}(\mathbb{D})$ and an inner function $\Psi \in H^\infty_{\mathcal{B}(\mathcal{D}_{T_1})}(\mathbb{D})$ such that*

$$\Pi T_1^* = M_z^* \Pi,$$

and

$$\Pi T_2^* = M_\Psi^* \Pi.$$

Moreover

$$T_1 \cong P_{\mathcal{Q}} M_z|_{\mathcal{Q}}, \quad \textit{and} \qquad T_2 \cong P_{\mathcal{Q}} M_\Psi|_{\mathcal{Q}},$$

where $\mathcal{Q} := \mathit{ran}\,\Pi$ *is a joint* (M_z^*, M_Ψ^*)*-invariant subspace of* $H^2_{\mathcal{D}_{T_1}}(\mathbb{D})$.

Proof. Let $\Pi : \mathcal{H} \to H^2_{\mathcal{D}_{T_1}}(\mathbb{D})$ be the minimal isometric dilation of T_1 as defined in (2.1) and let Ψ be as in (3.2). Now it is enough to show that Π intertwine T_2^* and M_Ψ^*. Let $h \in \mathcal{H}$, $n \geq 0$ and $\eta \in \mathcal{D}_{T_1}$. Then

$$\begin{aligned}
\langle M_\Psi^* \Pi h, z^n \eta\rangle &= \langle \Pi h, \Psi(z^n\eta)\rangle \\
&= \langle D_{T_1}(I_{\mathcal{H}} - zT_1^*)^{-1}h, (A^* + C^* \sum_{q=0}^{\infty} D^{*q} B^* z^{q+1})(z^n\eta)\rangle \\
&= \langle D_{T_1} \sum_{p=0}^{\infty} z^p T_1^{*p} h, (A^* + C^* \sum_{q=0}^{\infty} D^{*q} B^* z^{q+1})(z^n\eta)\rangle \\
&= \langle D_{T_1} T_1^{*n} h, A^*\eta\rangle + \sum_{q=0}^{\infty} \langle D_{T_1} T_1^{*q+n+1} h, C^* D^{*q} B^* \eta\rangle \\
&= \langle A D_{T_1} T_1^{*n} h, \eta\rangle + \sum_{q=0}^{\infty} \langle B D^q C D_{T_1} T_1^{*q+n+1} h, \eta\rangle \\
&= \langle (A D_{T_1} + \sum_{q=0}^{\infty} B D^q C D_{T_1} T_1^{*q+1}) T_1^{*n} h, \eta\rangle.
\end{aligned}$$

Then by Lemma 2.1, we have

$$\langle M_\Psi^* \Pi h, z^n\eta\rangle = \langle D_{T_1} T_2^* (T_1^{*n} h), \eta\rangle.$$

Hence

$$\begin{aligned}
\langle \Pi T_2^* h, z^n\eta\rangle &= \langle D_{T_1}(I_{\mathcal{H}} - zT_1^*)^{-1} T_2^* h, z^n\eta\rangle \\
&= \langle D_{T_1} \sum_{p=0}^{\infty} z^p T_1^{*p} T_2^* h, z^n\eta\rangle \\
&= \langle D_{T_1} T_1^{*n} T_2^* h, \eta\rangle \\
&= \langle D_{T_1} T_2^* (T_1^{*n} h), \eta\rangle \\
&= \langle M_\Psi^* \Pi h, z^n\eta\rangle.
\end{aligned}$$

This implies

$$M_\Psi^* \Pi = \Pi T_2^*.$$

The second claim is an immediate consequence of the first part. This completes the proof of the theorem. □

Theorem 3.1 remains valid if we drop the assumption that $\dim \mathcal{D}_{T_j} < \infty$, $j = 1, 2$. Indeed, the only change needed in the proof of Theorem 3.1 is to replace the transfer function Ψ in

(3.2) by the transfer function of U^* in Remark 2.1. In this case, however, the new transfer function will be a contractive multiplier. Thus we have proved the following dilation result.

Theorem 3.2. *Let (T_1, T_2) be a pair of commuting contractions on a Hilbert space $\mathcal{H}$ and let T_1 be a pure contraction. Then there exist an isometry $\Pi : \mathcal{H} \to H^2_{\mathcal{D}_{T_1}}(\mathbb{D})$ and a contractive multiplier $\Psi \in H^\infty_{\mathcal{B}(\mathcal{D}_{T_1})}(\mathbb{D})$ such that*

$$\Pi T_1^* = M_z^* \Pi,$$

and

$$\Pi T_2^* = M_\Psi^* \Pi.$$

In particular,

$$T_1 \cong P_{\mathcal{Q}} M_z|_{\mathcal{Q}}, \quad \text{and} \qquad T_2 \cong P_{\mathcal{Q}} M_\Psi|_{\mathcal{Q}},$$

where $\mathcal{Q} := ran\ \Pi$ is a joint (M_z^*, M_Ψ^*)-invariant subspace of $H^2_{\mathcal{D}_{T_1}}(\mathbb{D})$.

4. von Neumann inequality

This section is devoted mostly to the study of von Neumann inequality for the class of pure and commuting contractive pair of operators with finite defect spaces. However, we treat this matter in a slightly general setting. It is convenient to be aware of the results and constructions of Section 3.

We begin by noting the following proposition.

Proposition 4.1. *Let $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a unitary matrix on $\mathcal{H} \oplus \mathcal{K}$ and let A be a completely non-unitary contraction. Then for all $z \in \mathbb{D}$, $\tau_U(z)$ does not have any unimodular eigenvalues.*

Proof. Let $z \in \mathbb{D}$ and A be a completely non-unitary contraction. Suppose, by contradiction,

$$(\tau_U(z))v = \lambda v,$$

for some non-zero vector $v \in \mathcal{H}$ and for some $\lambda \in \partial\mathbb{D}$. Since $\tau_U(z)$ is a contraction, we have

$$(\tau_U(z))^* v = \bar{\lambda} v,$$

and hence, by (3.1)

$$Cv = 0.$$

This and the definition of τ_U implies

$$Av = (\tau_U(z))v = \lambda v.$$

Then A has a non-trivial unitary part. This contradiction establishes the proposition. $\square$

Now we examine the role of the unitary part of the contraction A to the transfer function τ_U.

Proposition 4.2. *Let $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a unitary matrix on $\mathcal{H} \oplus \mathcal{K}$ and let $A = \begin{bmatrix} W & 0 \\ 0 & A' \end{bmatrix} \in \mathcal{B}(\mathcal{H}_0 \oplus \mathcal{H}_1)$ be the canonical decomposition of A into the unitary part W on $\mathcal{H}_0$ and the* *completely non-unitary part A' on $\mathcal{H}_1$. Then $U' = \begin{bmatrix} A' & B \\ C|_{\mathcal{H}_1} & D \end{bmatrix}$ is a unitary operator on $\mathcal{H}_1 \oplus \mathcal{K}$ and*

$$\tau_U(z) = \begin{bmatrix} W & 0 \\ 0 & \tau_{U'}(z) \end{bmatrix} \in \mathcal{B}(\mathcal{H}_0 \oplus \mathcal{H}_1) \qquad (z \in \mathbb{D}).$$

Proof. Let us observe first that if U is a unitary operator, then

$$A^*A + C^*C = I_{\mathcal{H}},$$

and

$$AA^* + BB^* = I_{\mathcal{H}}.$$

On account of $A^*A|_{\mathcal{H}_0} = AA^*|_{\mathcal{H}_0} = I_{\mathcal{H}_0}$, the first equality implies

$$C^*C|_{\mathcal{H}_0} = 0,$$

and hence

$$\mathcal{H}_0 \subseteq \ker C,$$

while the second equality yields

$$\begin{aligned} \overline{ran} B &= \overline{ran}(BB^*) \\ &= \overline{ran}(I_{\mathcal{H}} - AA^*) \\ &= \overline{ran}(I_{\mathcal{H}_1} - AA^*|_{\mathcal{H}_1}) \\ &\subseteq \mathcal{H}_1. \end{aligned}$$

Consequently, it follows that $U' = \begin{bmatrix} A' & B \\ C|_{\mathcal{H}_1} & D \end{bmatrix}$ is a unitary operator on $\mathcal{H}_1 \oplus \mathcal{K}$, and hence

$$\tau_U(z) = W \oplus \tau_{U'}(z) \qquad (z \in \mathbb{D}),$$

follows from the definition of transfer functions. $\square$

We now return to the study of rational inner functions. Let $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a unitary on $\mathbb{C}^m \oplus \mathbb{C}^n$. Let $A = \begin{bmatrix} W & 0 \\ 0 & E \end{bmatrix}$ on $\mathbb{C}^m = H_0 \oplus H_1$ be the canonical decomposition of A into the unitary part W on H_0 and the completely non-unitary part E on H_1. Then by the previous proposition, we have

$$\tau_U(z) = \begin{bmatrix} \Psi_0(z) & 0 \\ 0 & \Psi_1(z) \end{bmatrix} \in \mathcal{B}(H_0 \oplus H_1) \qquad (z \in \mathbb{D}),$$

where

$$\Psi_0(z) = W \qquad (z \in \mathbb{D})$$

is a $\mathcal{B}(H_0)$-valued unitary constant, for some unitary $W \in \mathcal{B}(H_0)$, and

$$\Psi_1(z) = E + zB(I - zD)^{-1}C \qquad (z \in \mathbb{D})$$

is a $\mathcal{B}(H_1)$-valued rational inner function. It should be noted, however, that the distinguished varieties corresponding to the rational inner functions τ_U and Ψ_1 are the same, that is,

$$\{(z_1,z_2)\in\mathbb{D}^2:\det(\tau_U(z_1)-z_2I_{\mathbb{C}^m})=0\}=\{(z_1,z_2)\in\mathbb{D}^2:\det(\Psi_1(z_1)-z_2I_{H_1})=0\}.$$

This follows from the observation that the unitary summand Ψ_0 would add sheets to the variety corresponding to Ψ_1 of the type $\mathbb{C}\times\{\lambda\}$, for some $\lambda\in\partial\mathbb{D}$, which is disjoint from $\mathbb{D}^2$. However, we want to stress here that, by Proposition 4.1 and the fact that Ψ_1 is unitary on $\partial\mathbb{D}$,

$$\overline{V}_{\Psi_1}=\{(z_1,z_2)\in\overline{\mathbb{D}^2}:\det(\Psi_1(z_1)-z_2I_{H_1})=0\},$$

where V_{Ψ_1} is the distinguished variety corresponding to the inner multiplier Ψ_1 and $\overline{V}_{\Psi_1}$ is the closure of V_{Ψ_1} in $\overline{\mathbb{D}^2}$.

We now have all the ingredients in place to prove a von Neumann type inequality for pairs of commuting contractions.

Theorem 4.3. *Let (T_1,T_2) be a pair of commuting contractions on a Hilbert space $\mathcal{H}$. Let T_1 be pure and dim $\mathcal{D}_{T_j}<\infty$, $j=1,2$. Then there exists a variety $V\subseteq\overline{\mathbb{D}^2}$ such that*

$$\|p(T_1,T_2)\|\leq\|p\|_V\qquad(p\in\mathbb{C}[z_1,z_2]).$$

If, in addition, T_2 is pure, then V can be taken to be a distinguished variety.

Proof. By Theorem 3.1, there is a rational inner function $\Psi\in H^\infty_{\mathcal{B}(\mathcal{D}_{T_1})}(\mathbb{D})$ and a joint (M_z^*,M_Ψ^*)-invariant subspace $\mathcal{Q}$ of $H^2_{\mathcal{D}_{T_1}}(\mathbb{D})$ such that

$$T_1\cong P_{\mathcal{Q}}M_z|_{\mathcal{Q}},\quad\text{and}\quad T_2\cong P_{\mathcal{Q}}M_\Psi|_{\mathcal{Q}}.$$

Here

$$\Psi(z)=\tau_{U^*}(z)=A^*+zC^*(I-zD^*)^{-1}B^*,\qquad(z\in\mathbb{D})$$

is the transfer function of the unitary $U^*=\begin{bmatrix}A^* & C^*\\ B^* & D^*\end{bmatrix}$ in $\mathcal{B}(\mathcal{D}_{T_1}\oplus\mathcal{D}_{T_2})$ as defined in (2.3). Let $A^*=\begin{bmatrix}W & 0\\ 0 & E^*\end{bmatrix}\in\mathcal{B}(H_0\oplus H_1)$ on $\mathcal{D}_{T_1}=H_0\oplus H_1$ be the canonical decomposition of A^* in to the unitary part W on H_0 and the completely non-unitary part E^* on H_1. Now from Proposition 4.2 (or the discussion following Proposition 4.2) it follows that

$$\Psi(z)=\begin{bmatrix}\Psi_0(z) & 0\\ 0 & \Psi_1(z)\end{bmatrix}\in\mathcal{B}(H_0\oplus H_1)\qquad(z\in\mathbb{D}),$$

where

$$\Psi_0(z)=W\in\mathcal{B}(H_0)\quad(z\in\mathbb{D}),$$

and

$$\Psi_1(z)=E^*+zC^*(I-zD^*)^{-1}B^*\in\mathcal{B}(H_1)\qquad(z\in\mathbb{D}).$$

Let us set

$$V=V_0\cup V_1,$$

where

$$V_0=\{(z_1,z_2)\in\mathbb{D}\times\overline{\mathbb{D}}:\det(\Psi_0(z_1)-z_2I_{H_0})=0\},$$

and

$$V_1 = \{(z_1, z_2) \in \mathbb{D}^2 : \det(\Psi_1(z_1) - z_2 I_{H_1}) = 0\}.$$

Clearly

$$V_0 = \{(z_1, z_2) \in \mathbb{D} \times \overline{\mathbb{D}} : \det(W - z_2 I_{H_0}) = 0\} = \bigcup_{j=1}^{l} \mathbb{D} \times \{\lambda_j\} \subseteq \mathbb{D} \times \partial\mathbb{D},$$

where $\{\lambda_j\}_{j=1}^{l} = \sigma(W) \subseteq \partial\mathbb{D}$. Now for each $p \in \mathbb{C}[z_1, z_2]$, we have

$$\begin{aligned}
\|p(T_1, T_2)\|_{\mathcal{H}} &= \|P_{\mathcal{Q}} p(M_z, M_\Psi)|_{\mathcal{Q}}\|_{\mathcal{Q}} \\
&\leq \|p(M_z, M_\Psi)\|_{H^2_{\mathcal{D}_{T_1}}(\mathbb{D})} \\
&\leq \|p(M_{e^{i\theta}}, M_{\Psi(e^{i\theta})})\|_{L^2_{\mathcal{D}_{T_1}}(\mathbb{T})} \\
&= \|M_{p(e^{i\theta} I_{\mathcal{D}_{T_1}}, \Psi(e^{i\theta}))}\|_{L^2_{\mathcal{D}_{T_1}}(\mathbb{T})} \\
&= \sup_\theta \|p(e^{i\theta} I_{\mathcal{D}_{T_1}}, \Psi(e^{i\theta}))\|_{\mathcal{B}(\mathcal{D}_{T_1})} \\
&= \sup_\theta \|p(e^{i\theta} I_{H_0}, \Psi_0(e^{i\theta})) \oplus p(e^{i\theta} I_{H_1}, \Psi_1(e^{i\theta}))\|_{\mathcal{B}(H_0) \oplus \mathcal{B}(H_1)} \\
&\leq \max\{\sup_\theta \|p(e^{i\theta} I_{H_0}, \Psi_0(e^{i\theta}))\|_{\mathcal{B}(H_0)}, \|p(e^{i\theta} I_{H_1}, \Psi_1(e^{i\theta}))\|_{\mathcal{B}(H_1)}\}.
\end{aligned}$$

But now, since $\Psi(e^{i\theta})$ is unitary on $\partial\mathbb{D}$, for each fixed $e^{i\theta} \in \partial\mathbb{D}$ and $j = 1, 2$, we have

$$\begin{aligned}
\|p(e^{i\theta} I_{H_j}, \Psi_j(e^{i\theta}))\|_{\mathcal{B}(H_j)} &= \sup\{|p(e^{i\theta}, \lambda)| : \lambda \in \sigma(\Psi_j(e^{i\theta}))\} \\
&= \sup\{|p(e^{i\theta}, \lambda)| : \det(\Psi_j(e^{i\theta}) - \lambda I_{H_j}) = 0\} \\
&\leq \|p\|_{\partial V_j},
\end{aligned}$$

and hence, by continuity, we obtain

$$\|p(T_1, T_2)\|_{\mathcal{H}} \leq \|p\|_V.$$

This completes the proof of the first part.

For the second part, it is enough to show that $\Psi_0(z) = W = 0$. For this, we show that A^* is a completely non-unitary operator. To this end, first notice that $\mathcal{Q} \subseteq H^2_{H_1}(\mathbb{D})$. Indeed, for each $f \in H^2_{H_0}(\mathbb{D})$ we have

$$f_n := M_\Psi^{*n} f = W^{*n} f \in H^2_{H_0}(\mathbb{D}) \qquad (n \in \mathbb{N}),$$

and hence for $g \in \mathcal{Q}$ we have

$$|\langle f, g\rangle| = |\langle M_\Psi^n f_n, g\rangle| = |\langle f_n, M_\Psi^{*n} g\rangle| = |\langle f_n, T_2^{*n} g\rangle| \leq \|f_n\| \|T_2^{*n} g\| = \|f\| \|T_2^{*n} g\| \quad (n \in \mathbb{N}).$$

Since T_2 is a pure contraction, $\langle f, g\rangle = 0$. This implies that $\mathcal{Q} \subseteq H^2_{H_1}(\mathbb{D})$.
On the other hand, note that M_z is the minimal isometric dilation of T_1, that is,

$$\bigvee_{n \geq 0} M_z^n \mathcal{Q} = H^2_{\mathcal{D}_{T_1}}(\mathbb{D}),$$

and $H^2_{H_1}(\mathbb{D})$ is a M_z-reducing subspace of $H^2_{\mathcal{D}_{T_1}}(\mathbb{D})$. Therefore

$$H^2_{H_0}(\mathbb{D}) = \{0\}.$$

This shows $H_0 = \{0\}$ and completes the proof of this theorem. □

In the special case where $\dim \mathcal{H} < \infty$ and where (T_1, T_2) is a commuting pair of pure contractions, we have

$$\sigma(T_j) \cap \partial\mathbb{D} = \emptyset \qquad (j = 1, 2).$$

Hence, in this particular case, we recover Agler and McCarthy's sharper von Neumann inequality for commuting pairs of strictly contractive matrices (see Theorem 3.1 in [2]). Moreover, the present proof is more direct and explicit than the one by Agler and McCarthy (see, for instance, case (ii) in page 145 [2]).

5. Concluding remarks

Uniqueness of varieties: Let (T_1, T_2) be a pair of pure commuting contractions on a Hilbert space $\mathcal{H}$ and $\dim \mathcal{D}_{T_j} < \infty$, $j = 1, 2$. Theorem 3.1 implies that there exists a rational inner function $\Psi \in H^\infty_{\mathcal{B}(\mathcal{D}_{T_1})}(\mathbb{D})$ such that

$$\Pi T_1^* = M_z^* \Pi, \quad \text{and} \quad \Pi T_2^* = M_\Psi^* \Pi,$$

where $\Pi : \mathcal{H} \to H^2_{\mathcal{D}_{T_1}}(\mathbb{D})$ is the minimal isometric dilation of T_1 (see (2.1) in Section 2). Furthermore, by the second assertion of Theorem 4.3, we have

$$\|p(T_1, T_2)\|_{\mathcal{H}} \le \|p\|_V \qquad (p \in \mathbb{C}[z_1, z_2]),$$

where

$$V = \{(z_1, z_2) \in \mathbb{D}^2 : \det(\Psi(z_1) - z_2 I) = 0\}$$

is a distinguished variety.

Now let $\tilde{\Pi} : \mathcal{H} \to H^2_{\mathcal{D}_{T_2}}(\mathbb{D})$ be the minimal isometric dilation of T_2. Then Theorem 3.1 applied to (T_2, T_1) yields a rational inner multiplier $\tilde{\Psi} \in H^\infty_{\mathcal{B}(\mathcal{D}_{T_2})}(\mathbb{D})$ such that

$$\tilde{\Pi} T_2^* = M_z^* \tilde{\Pi}, \quad \text{and} \quad \tilde{\Pi} T_1^* = M_{\tilde{\Psi}}^* \tilde{\Pi}.$$

Therefore $(M_{\tilde{\Psi}}, M_z)$ on $H^2_{\mathcal{D}_{T_2}}(\mathbb{D})$ is also an isometric dilation of (T_1, T_2). Furthermore,

$$\Psi = \tau_{U^*}, \quad \text{and} \quad \tilde{\Psi} = \tau_{\tilde{U}},$$

where

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \text{and} \quad \tilde{U} = \begin{bmatrix} D^* & B^* \\ C^* & A^* \end{bmatrix}.$$

Again applying the second assertion of Theorem 4.3 to (T_2, T_1), we see that

$$\|p(T_1, T_2)\|_{\mathcal{H}} \le \|p\|_{\tilde{V}} \qquad (p \in \mathbb{C}[z_1, z_2]),$$

where

$$\tilde{V} = \{(z_1, z_2) \in \mathbb{D}^2 : \det(\tilde{\Psi}(z_2) - z_1 I) = 0\}$$

is a distinguished variety. Now it follows from Lemma 1.7 in [2] that

$$V = \tilde{V}.$$

Joint eigenspaces: Let (T_1, T_2) and the variety V be as in Theorem 4.3. Then the joint eigenspace of (T_1^*, T_2^*) is contained in the distinguished variety V. Indeed, let

$$T_j^* v = \bar{\lambda}_j v \qquad (j = 1, 2),$$

for some $(\lambda_1, \lambda_2) \in \mathbb{D}^2$ and for some non-zero vector $v \in \mathcal{H}$. Then $(\lambda_1, \lambda_2) \in V$. Equation (2.2) gives

$$U(D_{T_1} v, \bar{\lambda}_1 D_{T_2} v) = (\bar{\lambda}_2 D_{T_1} v, D_{T_2} v).$$

Hence, by (2.3)

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} D_{T_1} v \\ \bar{\lambda}_1 D_{T_2} v \end{bmatrix} = \begin{bmatrix} \bar{\lambda}_2 D_{T_1} v \\ D_{T_2} v \end{bmatrix}.$$

Then by Lemma 2.1,

$$(A + \bar{\lambda}_1 B(I - \bar{\lambda}_1 D)^{-1} C)(D_{T_1} v) = \bar{\lambda}_2 (D_{T_1} v),$$

and hence

$$\left(A^* + \lambda_1 C^* (I - \lambda_1 D^*)^{-1} B^*\right)^* D_{T_1} v = \bar{\lambda}_2 D_{T_1} v,$$

that is, $\det(\Psi(\lambda_1) - \lambda_2 I) = 0$. Thus the claim follows.

An example: We conclude this paper by pointing out a simple but illustrative example of our sharper von Neumann inequality. Let M_z be the shift operator on $H^2_{\mathbb{C}^m}(\mathbb{D})$ and

$$(T_1, T_2) = (M_z, M_z).$$

Since $D_{M_z} = P_{\mathbb{C}^m}$, by a simple calculation it follows that the unitary U in (2.3) has the form

$$U = \begin{bmatrix} 0 & W \\ I_{\mathbb{C}^m} & 0 \end{bmatrix},$$

where $W \in \mathcal{B}(\mathbb{C}^m)$ is an arbitrary unitary operator. Let us choose a unitary W in $\mathcal{B}(\mathbb{C}^m)$. In this case,

$$\tau_{U^*}(z) = \Psi(z) = zW^* \qquad (z \in \mathbb{D}).$$

Let $\{\lambda_1, \ldots, \lambda_k\}$, $1 \leq k \leq m$, be the set of distinct eigenvalues of W^* and

$$p(z_1, z_2) := \prod_{i=1}^{k} (z_2 - \lambda_i z_1).$$

Then the distinguished variety V in the second assertion of Theorem 4.3 is given by

$$\begin{aligned} V &= \{(z_1, z_2) \in \mathbb{D}^2 : \det(z_1 W^* - z_2 I_{\mathbb{C}^m}) = 0\} \\ &= \{(z_1, z_2) \in \mathbb{D}^2 : p(z_1, z_2) = 0\}, \end{aligned}$$

and hence for any $p \in \mathbb{C}[z_1, z_2]$, the inequality

$$\|p(M_z, M_z)\|_{\mathcal{B}(H^2_{\mathbb{C}^m}(\mathbb{D}))} \leq \|p\|_V$$

holds. In particular, if we choose $W = I_{\mathbb{C}^m}$, then the distinguished variety V is given by

$$V = \{(z, z) : z \in \mathbb{D}\}.$$

This observation also follows by a direct calculation.

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Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai, 400076, India

E-mail address: `dasb@math.iitb.ac.in, bata436@gmail.com`

Indian Statistical Institute, Statistics and Mathematics Unit, 8th Mile, Mysore Road, Bangalore, 560059, India

E-mail address: `jay@isibang.ac.in, jaydeb@gmail.com`