AN INTRODUCTION TO BASIC FUNCTIONAL ANALYSIS

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1. Introduction

This is a write-up of the lectures given at the Advanced Instructional School (AIS) on ‘ Functional Analysis and Harmonic Analysis ’ during the first week of July 2006. All of the material is standard and can be found in many basic text books on Functional Analysis.

As the prerequisites, I take the liberty of assuming 1) Basic Metric space theory, including completeness and the Baire category theorem. 2) Basic Lebesgue measure theory and some abstract (σ-finite) measure theory including the Radon-Nikodym theorem and the completeness of L^p-spaces. I thank Professor A. Mangasuli for carefully proof reading these notes.

2. Banach spaces and Examples.

Let X be a vector space over the real or complex scalar field. Let $\|.\|$: $X \to R^+$ be a function such that

(1) $\|x\| = 0 \Leftrightarrow x = 0$
(2) $\|\alpha x\| = |\alpha|\|x\|$
(3) $\|x + y\| \leq \|x\| + \|y\|$

for all $x, y \in X$ and scalars α.

Such a function is called a norm on X and $(X, \|.\|)$ is called a normed linear space. Most often we will be working with only one specific $\|.\|$ on any given vector space X thus we omit writing $\|.\|$ and simply say that X is a normed linear space.

It is an easy exercise to show that $d(x, y) = \|x - y\|$ defines a metric on X and thus there is an associated notion of topology and convergence.

If X is a normed linear space and $Y \subset X$ is a subspace then by restricting the norm to Y, we can consider Y as a normed linear space.

Clearly $|\|x\| - \|y\|| \leq \|x - y\|$ and hence for any sequence $\{x_n\}_{n \geq 1} \subset X$, $x_n \to x \Rightarrow \|x_n\| \to \|x\|$.

When the metric d is complete, i.e., every Cauchy sequence in X converges in X, we say that X is a Banach space.

It is easy to show that the Euclidian spaces R^n and C^n are Banach spaces.

A subspace $Y \subset X$ is a Banach space if and only if it is a closed subset of X.

Example 1. *Let K be a compact Hausdorff space, let $C(K)$ denote the vector space of complex-valued continuous functions on K. For any $f \in C(K)$, define $\|f\| = sup\{|f(k)| : k \in K\}$. This is called the supremum norm and $C(K)$ is a Banach space.*

More concretely, let $\Delta = \{z \in C : |z| \leq 1\}$ and $\Gamma = \{z : |z| = 1\}$. Let $A = \{f \in C(\Delta) : f \text{ is analytic in } \Delta^o\}$. It is easy to see that A is a closed subspace and hence a Banach space. On the other hand $\mathcal{P} = \{p \in C(\Delta) : p \text{ is a polynomial in } z\}$ is not a closed set and hence is not a Banach space.

If one only considers a locally compact set K then important spaces associated with this are, $C_0(K) = \{f : K \to C : f \text{ is continuous and vanishes at infinity}\}$ (recall that f vanishes at infinity if for every $\epsilon > 0$ there is a compact set $C \subset K$ such that $|f| < \epsilon$ outside C) and $C_c(K) = \{f : K \to C : f \text{ is continuous and compactly supported}\}$ (recall that f is compactly supported if $\{k : f(k) \neq 0\}^-$ is a compact set). Both these spaces are equipped with the supremum norm.

It is easy to see that $C_0(K)$ is a Banach space and is the completion (in the metric space sense) of the space $C_c(K)$.

Example 2. *Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space (more concretely take $\Omega = R$, $\mathcal{A}$ to be the σ-field of Lebesgue measurable sets, μ to be the Lebesgue measure). In what follows let us take two measurable functions that are equal almost every where (a.e) with respect to the measure μ as 'equal'. For $1 \leq p < \infty$, let $L^p(\mu) = \{f : \Omega \to C, f \text{ is measurable and } \int |f|^p \, d\mu < \infty\}$. For $f \in L^p(\mu)$, let $\|f\| = (\int |f|^p \, d\mu)^{\frac{1}{p}}$. Then $L^p(\mu)$ is a Banach space.*

A measurable function f is said to be essentially bounded if there exist a $\alpha > 0$ such that $\mu((|f|)^{-1}((\alpha, \infty]) = 0$. α is called an essential bound.

Let $L^\infty(\mu) = \{f : \Omega \to C, f \text{ is measurable and essentially bounded}\}$. For $f \in L^\infty(\mu)$ define $\|f\| = inf\{\alpha : \alpha \text{ is an essential bound}\}$. Then $L^\infty(\mu)$ is a Banach space.

A specific situation of the above example and an easy generalization leads to a procedure for generating new classes of Banach spaces. We will only consider countable index sets.

Example 3. *Let* $\{X_n\}_{n\geq 1}$ *be a sequence of Banach spaces. Let* $1 \leq p < \infty$. *Let* $X = \{x = \{x_n\} \in \prod X_n : \sum_1^\infty \|x_n\|^p < \infty\}$. *For* $x \in X$, *define* $\|x\| = (\sum_1^\infty \|x_n\|^p)^{\frac{1}{p}}$. *Then* X *is a Banach space. Next let* $X = \{x = \{x_n\} : sup|x_n| < \infty\}$. *Define* $\|x\| = sup|x_n|$. *This is again called the supremum norm. This is again a Banach space.*

When all the X_n *are taken as the scalar field, the corresponding space* X *is denoted by* ℓ^p *and* ℓ^∞ *respectively.*

Other important sequence spaces that are Banach spaces are, c, the space of convergent sequences of scalars and its subspace c_0 *of sequences converging to* 0. *Both the spaces are equipped with the supremum norm. They can also be treated as special cases of Example 1 for the discrete space* N *and its one point compactification.*

Next example again gives a way of generating new Banach spaces from a given collection.

Example 4. *Let* X *be a Banach space, let* $Y \subset X$ *be a closed subspace. Consider the quotient vector space* $X|Y$. *Elements of this space are denoted by* $[x] = \{x + y : y \in Y\}$. $\pi : X \to X|Y$ *defined by* $\pi(x) = [x]$ *is called the canonical quotient map.*

Define $\|[x]\| = d(x, Y) = inf\{\|x + y\| : y \in Y\}$. *Now* $X|Y$ *is a Banach space.*

3. Operators on normed linear spaces, Hahn-Banach theorem

Let X be a normed linear space. $X_1 = \{x \in X : \|x\| \leq 1\}$,the ball with center at 0 and radius 1 is called the closed unit ball. By dilating by an $r > 0$ and translating by an $x_0 \in X$ we get the ball $B(x_0, r) = x_0 + rX_1$.

A set $A \subset X$ is said to be bounded if there exists a $r > 0$ such that $A \subset rX_1$. Equivalently $\|x\| \leq r$ for all $x \in A$.

Clearly any compact set and in particular any convergent sequence, is a bounded set.

Let X, Y be normed linear spaces, let $T : X \to Y$ be a linear map. T is said to be bounded if it maps bounded sets to bounded sets. Such a T is called a bounded operator.

Using linearity of T, we have that T is bounded if and only if $T(X_1)$ is a bounded set if and only if there exists an $r > 0$ such that $\|T(x)\| \leq r\|x\|$ for all $x \in X$.

It is easily seen that a linear map is bounded if and only if it is continuous at 0 and then using the continuity of the vector space operations w. r. t the norm, one has that any bounded linear map is continuous.

Let X be a normed linear space and let $Y \subset X$ be a closed subspace. Let $\pi : X \to X|Y$ be the quotient map. Then $\|\pi\| = 1$.

The following remarks which are like starred exercises, are designed to illustrate the role of finite dimensional spaces.

For the Euclidean spaces, any linear map $T : C^n \to C^m$ is continuous.

A linear, one-to-one and onto map $T : X \to Y$ is said to be an isomorphism if both T and T^{-1} are bounded.

T is an isomorphism if and only if there exists constants $m, M > 0$ such that $m\|x\| \leq \|T(x)\| \leq M\|x\|$.

When $m = M = 1$ we say that T is an isometry.

Any finite dimensional normed linear space is isomorphic to some Euclidean space. So it is easy to see that any finite dimensional normed linear space is a Banach space.

Let X be a normed linear space and $Y \subset X$ a finite dimensional space. Then Y is closed.

We denote by $\mathcal{L}(X, Y)$ the vector space of bounded linear operators.

For $T \in \mathcal{L}(X, Y)$ define $\|T\| = sup\{\|T(x)\| : x \in X_1\}$. This is a norm. Also for any $x \in X$, $\|T(x)\| \leq \|T\|\|x\|$.

When Y is a Banach space, $\mathcal{L}(X,Y)$ is a Banach space. For $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y,Z)$, $S \circ T \in \mathcal{L}(X,Z)$ and $\|S \circ T\| \leq \|S\|\|T\|$.

When Y is the scalar field, a linear map $f : X \to C$ is called a linear functional. The space of continuous linear functionals is denoted by X^*. For $f \in X^*$, $\|f\| = sup\{|f(x)| : x \in X_1\}$. X^* is a Banach space.

Let X be an infinite dimensional normed linear space. Using the presence of infinitely many independent unit vectors one can see that there exists a linear map $F : X \to C$ that is not bounded.

Thus it is not clear in an abstract situation how one exhibits non-trivial bounded operators or bounded linear functionals?

Theorem 5. *(Hahn-Banach theorem): Let X be a normed linear space and let $M \subset X$ be a subspace. Let $f \in M^*$. There exists a $g \in X^*$ such that $f = g$ on M and $\|f\| = \|g\|$. In particular for any $x \in X$ there exists an $f \in X^*$ such that $\|x\| = f(x)$.*

Proof. We will assume, by normalizing if necessary, that $\|f\| = 1$.

We shall prove the theorem first when the scalar field is real.

Let $x_0 \notin M$ and let $N = span\{x_0, M\} = \{\alpha x_0 + m : m \in M,\ \alpha \in R\}$.

We will get a $g \in N^*$ satisfying the assertions of the theorem.

Define $g : N \to R$ by $g(\alpha x_0 + m) = \alpha\alpha_0 + f(m)$ where we will choose (below) α_0 to make g bounded by 1. Clearly g is linear and $g = f$ on M.

In order to achieve $\|g\| \leq 1$, we need to show, $|\alpha\alpha_0 + f(m)| \leq \|\alpha x_0 + m\|$ for all α and m.

Dividing by $|\alpha|$, this is same as, $|\alpha_0 + f(\frac{m}{\alpha})| \leq \|x_o + \frac{m}{\alpha}\|$.

Since M is a subspace we only need to achieve $|\alpha_0 + f(m)| \leq \|x_0 + m\|$.

This is equivalent to,

$-\|x_0 + m\| \leq \alpha_0 + f(m) \leq \|x_0 + m\|$

or $-f(m) - \|x_0 + m\| \leq \alpha_0 \leq -f(m) + \|x_0 + m\|$.

But for any $x, y \in M$,

$|f(x) - f(y)| = |f(x - y)| \leq \|x - y\| \leq \|x + x_0\| + \|x_0 + y\|$. So that $-f(y) - \|x_0 + y\| \leq -f(x) + \|x_0 + x\|$. Thus choose α_0 such that

$-f(m) - \|x_0 + m\| \leq \alpha_0 \leq -f(m) + \|x_0 + m\|$. This gives $g \in N^*$ with $\|g\| \leq 1$. As $g = f$ *on* M, $\|g\| = 1$.

To extend f to all of X we next invoke a version of the Axiom of choice called Zorn's Lemma.

Let $\mathcal{G} = \{(N, g) : M \subset N \subset X \textit{ is a subspace, } g \in N_1^*, \; g = f \textit{ on } M\}$. This is a non-empty set. Define a partial ordering here by 'inclusion' in the first coordinate and functionals agreeing on the smaller space. It is easy to see that every chain here has an upper bound. Thus by the Zorn's Lemma $\mathcal{G}$ has a maximal element, say $((N, g)$. To complete the proof we next note that $N = X$. If not, let $x_0 \notin N$. By the procedure outlined above we get an $(N', g') \in \mathcal{G}$ which is bigger than the maximal element (N, g), leading to a contradiction.

In the case of complex scalar field, we recall that for any linear functional f, its real part ref is a real linear functional . Also, given a real linear functional u, $f(x) = u(x) - iu(ix)$ is a complex linear functional whose real part is u.

To see the norm condition, note that $|f(x)| = \alpha f(x)$ for some $\alpha \in \Gamma$. So that $|f(x)| = f(\alpha x) = ref(\alpha x) \leq |ref(\alpha x)| \leq \|\alpha x\| = \|x\|$.

To see the last assertion of the theorem, for $x \in X$, let $M = span\{x\}$. Define $h : M \to C$ by $h(\alpha x) = \alpha\|x\|$. Then $|h(\alpha x)| = |\alpha|\|x\| = \|\alpha x\|$. Thus $\|h\| = 1$ and $h(x) = \|x\|$. By what we have done above, we get $f \in X_1^*$ with $f(x) = \|x\|$.

□

Corollary 6. *Let X be a normed linear space and $Y \subset X$ a closed subspace. Let $x_0 \notin Y$ with $\|x_0\| = 1$. There exists an $f \in X^*$ with $f(Y) = 0$ and $f(x_0) = d(x_0, Y)$.*

Proof. Let $Z = span\{Y, x_0\}$. Define $g : Z \to C$ by $g(y + \alpha x_0) = \alpha d(x_0, Y)$. Clearly g is a linear map with $g(Y) = 0$ and $g(x_0) = d(x_0, Y)$. $|g(y+\alpha x_0)| = |\alpha| d(x_0, Y) = d(\alpha x_0, Y) \leq \|\alpha x_0 + y\|$. Thus there is an $f \in X^*$ with the required properties. □

This allows us to associate with a closed subspace

$Y \subset X$, $Y^\perp = \{f \in X^* : f(Y) = 0\}$. This is called the annihilator of Y and is a closed subspace of X^*.

Let X, Y be normed linear spaces. Now fix a $x_0^* \in X^*$ and $y_0 \in Y$. Define $x_0^* \otimes y_0 : X \to Y$ as $(x_0^* \otimes y_0)(x) = x_0^*(x)y_0$. $x_0^* \otimes y_0$ is an operator with norm $\|x_0^*\|\|y_0\|$ whose range is $span\{y_0\}$. This is called a rank one operator.

The vector space spanned by the set of operators of rank one is the space of finite rank operators and is denoted by $\mathcal{F}$.

The following trick of quotienting an operator is quite useful.

Let $T \in \mathcal{L}(X,Y)$. $ker(T)$ is a closed subspace of X. Define $T' : X|ker(T) \to Y$ by $T'([x]) = T(x)$. Then T' is a one-one and continuous operator. If T is onto so is T'.

4. Concrete dual spaces and Banach-Alaoglu theorem

We now identify X^* for some concrete Banach spaces considered in Section 1. Most details can be seen in references [1] and [2].

Let K be a compact Hausdorff space, let $\mathcal{B}$ be the Borel σ-field. For a complex measure μ on $\mathcal{B}$, we recall that the total variation measure $|\mu|$ is defined for $E \in \mathcal{B}$ by $|\mu|(E) = sup\{\sum_1^n |\mu(E_i)| : \{E_i\}\ a\ Borel\ partition\ of\ E\}$. μ is said to be regular if $|\mu|(E) = inf\{|\mu|(V) : V open, E \subset V\}$.

The space of complex regular Borel measures on K with $\|\mu\| = |\mu|(K)$, is a Banach space.

For such a measure μ, $\Lambda : C(X) \to C$ defined by $\Lambda(f) = \int f d\mu$ for $f \in C(K)$, is a bounded linear functional with $\|\Lambda\| \leq \|\mu\|$.

Theorem 7. *(Riesz Representation theorem) : Let $\Lambda \in C(X)^*$. There exists a unique regular Borel measure μ such that $\Lambda(f) = \int f d\mu$ for $f \in C(K)$ and $\|\Lambda\| = \|\mu\|$.*

One can see that μ is a probability measure if and only if $\|\mu\| = 1 = \mu(1)$.

There are also versions of the above theorem for locally compact spaces.

Of particular interest are the properties of the Haar Measure μ in the case of a locally compact abelian group.

Now let $1 \leq p \leq \infty$, let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. q is called the conjugate exponent of p. When $p = 1$ we take $q = \infty$.

Let $1 < p \leq \infty$. Let $g \in L^q(\mu)$. Define $\Lambda : L^p(\mu) \to C$ by $\Lambda(f) = \int fg d\mu$ for $f \in L^p(\mu)$. Then $\Lambda \in (L^p(\mu))^*$ and $\|\Lambda\| \leq \|g\|_q$.

Theorem 8. *Let $1 \leq p < \infty$. Let $\Lambda \in (L^p(\mu))^*$. There exists a unique $g \in L^q(\mu)$ such that $\Lambda(f) = \int fg d\mu$ and $\|\Lambda\| = \|g\|_q$.*

We have formulated the above theorem for complex measures, but it is also true for positive σ-finite measures.

For the Haar measure μ, $L^1(\mu)^* = L^\infty(\mu)$ always holds.

Using the discrete version of the above theorem, one can easily identify the dual of the space in Example 3.

Turning to the discussion of the space in Example 4, let $Y \subset X$ be a closed subspace. Define $\Phi : Y^\perp \to (X|Y)^*$ by $\Phi(\Lambda)([x]) = \Lambda(x)$. It is easy to see that Φ is an isometry. For $\tau \in (X|Y)^*$ define $\Lambda : X \to C$ by $\Lambda(x) = \tau([x])$. Then $\Lambda \in Y^\perp$ and $\Phi(\Lambda) = \tau$.

Next define $\Psi : Y^* \to X^*|Y^\perp$ by $\Psi(\Lambda) = [z^*]$ where $z^* \in X^*$ is a norm preserving extension of Λ given by the Hahn-Banach theorem. Then Ψ is an isometry. For $[x^*] \in X^*|Y^\perp$ define $\Lambda : Y \to C$ by $\Lambda(y) = x^*(y)$. Then $\Lambda \in Y^*$ and $\Psi(\Lambda) = [x^*]$.

These are called the canonical duals of quotient space and subspace.

Consider now X^* as a Banach space. Its dual is denoted by X^{**}. Define $\Psi : X \to X^{**}$ by $\Psi(x)(x^*) = x^*(x)$. Using the Hahn-Banach theorem we have that Ψ is an isometry. This is called the canonical embedding in the bidual. Some times we ignore this embedding and consider $X \subset X^{**}$. A Banach space is said to be reflexive if Ψ is onto.

Clearly any finite dimensional space is reflexive. One has that for $1 < p < \infty$, $L^p(\mu)$ is a reflexive space.

In the infinite dimensional case, there is an example of a $\Lambda \in L^\infty(\mu)^*$ that is not given by a $g \in L^1(\mu)$ so that $\Lambda(f) = \int fgd\mu$.

Thus an infinite dimensional $L^1(\mu)$ and $L^\infty(\mu)$ are not reflexive spaces. The same is true for $C(K)$.

We now discuss briefly a topology on X^* which is weaker than the norm topology and in the case of an infinite dimensional space this topology is not induced by any norm.

Say a net $\{x^*_\alpha\}_{\alpha \in I} \subset X^*$, $x^*_\alpha \to x^* \in X^*$ in the weak*- topology , if for every $x \in X$, $x^*_\alpha(x) \to x^*(x)$.

This defines a topology on X^* and is the smallest topology on X^* w.r.t which $\Psi(X)$ is continuous.

Here weak*-neighborhoods of 0 are of the form $\{x^* : |x^*(x_i)| < \epsilon_i, 1 \le i \le n\}$ where $x_i \in X$ and $\epsilon_i > 0$. The vector space operations are continuous in this topology.

Also the identity map $I : (X^*, \|.\|) \to (X^*, weak^*)$ is continuous. We recall that for any normed linear space X, X_1 denotes the closed unit ball.

Theorem 9. *(Banach- Alaoglu) : X^*_1 is compact in the weak*-topology. Any weak*- closed and norm bounded set is weak*-compact.*

Proof. Consider the product space $\prod_X \|x\|\Delta$ with the product topology. By the Tychonoff's theorem, this is a compact set. Define $\Phi : (X^*_1, weak^*) \to \prod_X \|x\|\Delta$ by $\Phi(x^*)(x) = x^*(x)$ for $x^* \in X^*_1$ and $x \in X$. Φ is a one-one continuous map.

We now show that $\Phi(X_1^*)$ is a closed set and Φ^{-1} is continuous. Thus Φ is homeomorphism onto its range and as the range is a closed subset of a compact set we conclude that X_1^* is weak*-compact.

Let $\Phi(x_\alpha^*) \to f \in \prod_X \|x\|\Delta$. Since for any $x \in X$, $f(x) = lim x_\alpha^*(x)$ we have that f is linear and $|f(x)| \leq \|x\|$. Thus $f \in X_1^*$ and $\Phi(f) = f$ so that $\Phi(X_1^*)$ is a closed set.

Also, if $\Phi(x_\alpha^*) \to \Phi(x^*)$ in the product topology, clearly $x_\alpha^* \to x^*$ in the weak*-topology. Therefore Φ is a homeomorphism onto its range.

$\square$

Let X be a Banach space, consider X_1^* equipped with the weak*-topology. Define $\Phi : X \to C(X_1^*)$ defined by $\Phi(x)(x^*) = x^*(x)$. Then Φ is an isometry. Some times we ignore this embedding and consider $X \subset C(X_1^*)$.

Similar to the weak*-topology, one can also consider the smallest topology on X which makes X^* continuous. This is called the weak topology. This is also a vector space topology and the neighborhoods of 0 are of the form $\{x : |x_i^*(x)| < \epsilon_i \, 1 \leq i \leq n\}$ where $x_i^* \in X^*$ and $\epsilon_i > 0$. This topology on X is clearly weaker than the norm topology. Also $x_\alpha \to x$ in weak topology if and only if $x^*(x_\alpha) \to x^*(x)$ for every $x^* \in X^*$. In the case of reflexive spaces the weak and weak*-topology coincide and thus X_1 is a compact set in the weak topology. If X is such that in X^* the weak and weak*-topologies coincide then X is reflexive.

5. Open mapping, closed graph and closed range theorems

In this section, all the spaces are assumed to be Banach spaces. Let X_1^o denote the open unit ball.

Theorem 10. *(Open mapping theorem) : Let X, Y be Banach spaces and let $T : X \to Y$ be an onto, bounded operator. There exists a $\delta > 0$ such that $\delta Y_1^o \subset T(X_1^o)$. In particular, T is an open mapping. If T is also one-one then T is an isomorphism.*

Proof. Since $Y = \bigcup_{k\geq 1} T(kX_1^0)$ by the Baire category theorem, there is a $k \geq 1$ and an open set $W \subset Y$ such that $W \subset T(kX_1^o)^-$.

Let $y_0 \in W$ and let $\eta > 0$ such that $y_0 + \eta Y_1^o \subset W$. Thus for any $y \in \eta Y_1^o$ we can choose a sequence $\{x_i\} \subset 2kX_1^o$ with $T(x_i) \to y$.

Now let $\delta = \frac{\eta}{2k}$. Then for any $\epsilon > 0$ and $y \in \delta Y_1^o$ there is a $x \in X_1^o$ with $\|T(x) - y\| < \epsilon$.

Now we use the completeness of X to get a x with $T(x) = y$.

Fix $y \in \delta Y_1^o$. Let $x_1 \in X_1^o$ with $\|y - T(x_1)\| < \frac{1}{2}\delta\epsilon$.

By induction we choose x_{n+1} with $\|x_{n+1}\| < \frac{\epsilon}{2^n}$ and $\|y - \Sigma T(x_i)\| < \frac{\delta\epsilon}{2^{n+1}}$.

Now if $s_n = x_1 + ... + x_n$ then it is a Cauchy sequence in X and therefore $s_n \to x \in X$. Since T is continuous, $T(x) = y$ and $\|x\| < 1 + \epsilon$. Since this holds for any $\epsilon > 0$, we have $\delta Y_1^o \subset T(X_1^o)$.

Again by the linearity of T we have that T maps open sets to open sets. □

For a $T : X \to Y$, let $G(T) = \{(x, T(x)) : x \in X\}$. This is called the graph of T. When T is a bounded operator, this is a closed subspace of the product space $X \times Y$.

Theorem 11. *(Closed graph theorem): Let X, Y be Banach spaces and let $T : X \to Y$ be a linear map with $G = G(T)$ closed. Then T is continuous.*

Proof. Make the product space $X \times Y$ a Banach space under the norm $\|(x, y)\| = \|x\| + \|y\|$, so that G is a Banach space. Define $\Phi : G \to X$ by $\Phi(x, T(x)) = x$. This is a linear one-one and onto map. Since projection is continuous w. r. t product topology, we have that Φ is continuous. Therefore, by the open mapping theorem, we have that Φ is an isomorphism.

Thus there exists an $m > 0$ such that $\|x\| + \|T(x)\| = \|(x, T(x))\| \leq m\|x\|$. Thus $\|T(x)\| \leq (m - 1)\|x\|$ and hence T is continuous.

□

A linear map $P : X \to X$ is said to be a projection if $P(P(x)) = P(x)$. Note that in this case $X = R(P) \oplus ker(P)$ (direct sum).

Let $Y \subset X$ be a finite dimensional subspace. Then there exists a continuous projection $P : X \to Y$ with $R(P) = Y$. Also if $Y \subset X$ is such that $X|Y$ is finite dimensional the same conclusion holds.

A projection P is continuous if and only if $R(P)$ and $ker(P)$ are closed.

For any $T \in \mathcal{L}(X, Y)$. Define $T^* : Y^* \to X^*$ by $T^*(y^*)(x) = y^*(T(x))$. T^* is a linear map and $\|T^*(y^*)\| \leq \|T\|\|y^*\|$.

Similarly $T^{**} : X^{**} \to Y^{**}$ is defined and agrees with T on the canonical embedding of X in X^{**}. Thus $\|T\| \leq \|T^{**}\| \leq \|T^*\| \leq \|T\|$. So that $\|T\| = \|T^*\|$.

Note that $T^* : Y^* \to X^*$ is also a continuous map when the domain and the range are equipped with the weak*-topology.

We assume the following consequences of the general version of the Hahn-Banach theorem that we have not proved here.

A subspace $Y \subset X^*$ is weak *-dense if and only if $x \in X$, $y^*(x) = 0$ for all $y^* \in Y$ implies $x = 0$.

Let $A \subset X$ be a closed, convex and balanced set. If $x_0 \notin A$ then there exists an $x^* \in X^*$ such that $|x^*(x)| \leq 1$ for all $x \in A$ but $x^*(x_0) > 1$.

Theorem 12. *(Closed Range Theorem): Let X, Y be Banach spaces. Let $T \in \mathcal{L}(X, Y)$. The range $R(T)$ is closed if and only if $R(T^*)$ is closed if and only if $R(T^*)$ is weak*-closed.*

Proof. Suppose $R(T^*)$ is closed. By replacing Y by $R(T)^-$ if necessary, we may assume that $R(T)$ is dense in Y. Note that this implies that T^* is one-one. Therefore $T^* : Y^* \to R(T^*)$ is an isomorphism by the open mapping theorem. Thus there exists a $c > 0$ such that $c\|y^*\| \leq \|T^*(y^*)\|$ for $y^* \in Y^*$. We next show that $cY_1^0 \subset \overline{T(X_1^0)}$. By an argument similar to the one given during the proof of the Open mapping theorem, this implies that T is onto.

Let $y_0 \in Y$. If $y_0 \notin \overline{T(X_1^0)}$, as the later set is closed, convex and balanced, by the separation theorem quoted above, there exists a $y^* \in Y^*$ such that $|y^*(y)| \leq 1 < |y^*(y_0)|$ for all $y \in \overline{T(X_1^0)}$. For $x \in X_1^0$, $|T^*(y^*)(x)| = |y^*(T(x))| \leq 1$. Thus $c\|y^*\| \leq \|T^*(y^*)\| \leq 1$.

On the other hand $1 < |y^*(y_0)| \leq \|y_0\|\|y^*\| < \frac{1}{c}\|y_0\|$. So $\|y_0\| > c$. Thus $cY_1^0 \subset \overline{T(X_1^0)}$.

Conversely suppose that $R(T)$ is closed. By replacing Y by $R(T)$, we may assume that T is onto. By passing through a quotient if necessary we may assume that T is one-one. Thus T is an isomorphism and hence so is T^*. Therefore $R(T^*)$ is norm as well as weak*-closed.

□

Theorem 13. *(Uniform boundedness principle) : Let* $\{T_n\}_{n\geq 1} \subset \mathcal{L}(X, Y)$. *Suppose for every* $x \in X$, $\{T_n(x)\}_{n\geq 1}$ *is bounded. Then* $\{T_n\}_{n\geq 1}$ *is bounded.*

Proof. By hypothesis $X = \bigcup_{k\geq 1}\{x : \|T_n(x)\| \leq k \ for \ all \ n\}$. Since X is complete, by the Baire category theorem, there exist k_0, x_0 and $r > 0$ such that $\|x - x_0\| \leq r \Rightarrow \|T_n(x)\| \leq k_o$ for all n. Let M be the bound for $\{T_n(x_0)\}_{n\geq 1}$.

Now for $\|x\| \leq 1$, $\|T_n(x_0 - rx)\| \leq k_o$.

Thus $r\|T_n(x)\| \leq \|T_n(rx - x_0)\| + \|T_n(x_0)\| \leq k_0 + M$.

Hence $\|T_n\| \leq \frac{k_0+M}{r}$.

□

6. Theory of Operators on Banach spaces

A linear map $T : X \to Y$ is said to be compact if $T(X_1)^-$ is a compact set. We also say that $T(X_1)$ is precompact.

Clearly T is a bounded map. Note that T is compact if and only if every bounded sequence $\{x_n\}_{n\geq 1}$ has a subsequence $\{x_{n_k}\}$ such that $T(x_{n_k}) \to y$ for some $y \in Y$.

It is easy to see that any finite rank operator is compact.

If T is a compact operator with closed range, then T is a finite rank operator.

If $T : X \to Y$ is compact and $Z \subset X$ is a closed subspace then the restriction operator, $T|Z \ : Z \to Y$ is compact.

If $T : X \to X$ is compact and $\lambda \neq 0$ then $ker(T - \lambda I)$ is finite dimensional.

Let $\mathcal{K}(X, Y)$ denote the vector space of compact operators.

If $T : X \to Y$ is compact and $S : Y \to W$ is bounded operator then $S \circ T$ is compact.

Theorem 14. *(Schauder): T is compact if and only if T^* is compact .*

Proof. Suppose T is compact. Note that since T^* is weak*-continuous, $T^*(Y_1^*)$ is weak*-compact and hence norm-closed. To show that $T^*(Y_1^*)$ is compact, let $\{y_n^*\}_{n\geq 1} \subset Y_1^*$. Since $y_n^* \in C(T(X_1)^-)$ are uniformly bounded and linear maps, it is easy to see that $\{y_n^*\}_{n\geq 1}$ is equicontinuous . Now using the Ascoli's theorem, we can conclude that there is a subsequence $y_{n_k}^*$ converging uniformly on $T(X_1)$.

Further,

$\|T^*(y_{n_k}^*) - T^*(y_{n_l}^*)\| = sup\{|(y_{n_k}^* - y_{n_l}^*)(y)| : y \in T(X_1)\} \to 0$. Since X^* is complete, we get that $T^*(y_{n_k}^*)$ converges. Therefore T^* is compact.

Now suppose T^* is compact. By what we have proved above, T^{**} is compact. Since $T = T^{**}|X$ we have that T is compact. $\square$

When T is compact let us note that for any net, $\{y_\alpha^*\} \subset Y_1^*$, $y_\alpha^* \to y^*$ in the weak*-topology implies, $T^*(y_\alpha^*) \to T^*(y^*)$ in the norm topology (i.e., T^* is completely continuous).

Theorem 15. *$\mathcal{K}(X, Y)$ is a closed subspace of $\mathcal{L}(X, Y)$ and hence is a Banach space.*

Proof. Let $\{T_n\}_{n\geq 1} \subset \mathcal{K}(X, Y)$ and $T_n \to T$. We shall show that $T(X_1)$ is totally bounded. Then, as Y is a Banach space, we have that $T(X_1)^-$ is a compact set. Let $\epsilon > 0$ and let n_0 be such that $\|T_{n_0} - T\| < \epsilon$. Since $T_{n_0}(X_1)$ is a totally bounded set, it is now easy to see that $T(X_1)$ can be covered by finitely many balls with radius 3ϵ. Thus $T \in \mathcal{K}(X,Y)$.

□

Thus, if $\mathcal{F}(X,Y) \subset \mathcal{K}(X,Y)$ denote the finite rank operators then $\mathcal{F}(X,Y)^- \subset \mathcal{K}(X,Y)$. From now on we take $X = Y$ so that, $\mathcal{L}(X)$ is an algebra with I as identity and, $\mathcal{K}(X)$ is a two-sided ideal. When X is infinite dimensional, $I \notin \mathcal{K}(X)$. For any $T \in \mathcal{L}(X)$, its spectrum $\sigma(T) = \{\lambda : T - \lambda I \textit{ is not invertible}\}$. λ is said to be an eigenvalue if $0 \neq x \in ker(T - \lambda I)$. When X is infinite dimensional and T compact, $0 \in \sigma(T)$.

Proposition 16. *For a compact operator T and a non-zero eigenvalue λ , $R(T - \lambda I) \neq X$. $E_r = \{\lambda : |\lambda| > r, \lambda \textit{ an eigenvalue}\}$ is a finite set and thus E, the set of eigenvalues ,is a countable set.*

Proof. Suppose the conclusions fail. We claim that there exists a sequence $\{M_n\}_{n \geq 1}$ of strictly increasing closed subspaces of X with $T(M_n) \subset M_n$ and $(T - \lambda_n I)(M_n) \subset M_{n-1}$, where $\{\lambda_n\}_{n \geq 1} \subset E$ is bounded below by an $r > 0$.

Assuming the claim, as M_{n-1} is a proper subspace of M_n, let $y_n \in M_n$ with $\|y_n\| \leq 2$ and $d(y_n, M_{n-1}) = 1$.

For $2 \leq m < n$, let $z = T(y_m) - (T - \lambda_n I)(y_n)$. Then,

$\|T(y_n - y_m)\| = \|\lambda_n y_n - z\| = |\lambda_n| \|y_n - \frac{z}{\lambda_n}\| \geq |\lambda_n| > r$. This contradicts the compactness of T.

Thus, suppose $range(S = (T - \lambda I)) = X$. Let $M_n = Ker(S^n)$. Since λ is an eigenvalue, there exists a $0 \neq x_1 \in M_1$ and as $range(S) = X$, there exists a sequence $\{x_n\}_{n \geq 1} \subset X$ such that $S(x_{n+1}) = x_n$. Now $S^n(x_{n+1}) = x_1 \neq 0$ but $S^{n+1}(x_{n+1}) = S(x_1) = 0$. Thus M_n is a proper closed subspace of M_{n+1}. The other assertions are valid with $\lambda_n = \lambda$.

On the other hand, suppose for an $r > 0$ there is a distinct sequence $\{\lambda_n\}_{n \geq 1} \subset E$ with $|\lambda_n| > r$. Let $\{e_n\}_{n \geq 1}$ be a corresponding sequence of independent eigenvectors. Let $M_n = span\{e_i\}_{1 \leq i \leq n}$. Then $T(M_n) \subset M_n$ and $(T - \lambda_n I)(M_n) \subset M_{n-1}$.

□

We recall that for an operator T, $R(T)$ stands for its range.

Theorem 17. *Let $T \in \mathcal{K}(X)$, $\lambda \neq 0$. The numbers, $dim(ker(T-\lambda I)) = dim(X|R(T-\lambda I)) = dim(ker(T^*-\lambda I))$. If $0 \neq \lambda \in \sigma(T)$ then λ is an eigenvalue of $T$and T^*. $\sigma(T)$ is countable, compact with at most one limit point,* 0.

Proof. Let $S = T - \lambda I$ then $S^* = T^* - \lambda I$. We will first show that $R(S)$ is closed. Since $ker(S)$ is finite dimensional, let $P : X \to ker(S)$ be a bounded projection. It is enough to show that S is bounded below on $ker(P)$ where it is one-one. If not, there exists a sequence $\{x_n\}_{n\geq 1} \subset ker(P)$ of unit vectors such that $S(x_n) \to 0$. Since T is compact we may assume that $x_n \to x_0 \in ker(P)$, so that $\lambda x_n \to \lambda x_o$. Since S is one-one on $ker(P)$, $S(x_0) = 0 \Rightarrow x_0 = 0$. This contradicts that x_n's are unit vectors and $\lambda \neq 0$. Now by duality, $kerS^* = R(S)^{\perp} = (X|R(S))^*$, so that
$dim(X|R(T-\lambda I)) = dim(ker(T^*-\lambda I))$.

Now suppose $dim(ker(S)) > dim(X|R(S))$. Suppose $X = ker(S) \oplus E = F \oplus R(S)$. Let P be the projection with range $ker(S)$.

Since $dim(ker(S)) > dim(F)$, let $\phi : ker(S) \to F$ be linear onto and $\phi(x_0) = 0$ for some $x_0 \neq 0$. Let $\Phi = T + \phi \circ P$. Clearly Φ is compact and $\Phi - \lambda I = S + \phi \circ P$. As $\Phi(x_0) = \lambda x_0$, $R(\Phi - \lambda I) \neq X$. Since $P = 0$ on E, $(\Phi - \lambda I)(E) = R(S)$. Also as $P = I$ on $ker(S)$, $(\Phi - \lambda I)(ker(S) = F$. Therefore, $R(\Phi - \lambda I) = X$, a contradiction.

Thus $dim(ker(T-\lambda I)) \leq dim(X|R(T-\lambda I))$ and by duality we have equality.

We already know that $\sigma(T) \cup \{0\}$ is countable and compact. If X is finite dimensional, $\sigma(T)$ is a finite set and if X is infinite dimensional $0 \in \sigma(T)$.

$\square$

7. Hilbert spaces

Let H be a complex linear space. $<,>$: $H \times H \to C$ such that, $<,>$ is linear in the first variable with $< x, x > \geq 0$ and $< x, x >= 0$ if and only if $x = 0$, $< x, y >=< y, x >^{-}$, is called an inner product on H.

For a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$ and $p = 2$, for $f, g \in L^2(\mu)$,

$< f, g >= \int fg^{-} d\mu$ is an inner product.

Similarly, in the discrete case ℓ^2 the space of square summable complex sequences, $< x, y >= \sum x(n)y(n)^{-}$ is an inner product.

If $< x, y >= 0$ we say, x is orthogonal to y, and write $x \perp y$.

Let $\|x\| = (< x, x >)^{\frac{1}{2}}$. Put $\alpha =< x, y >$.

For any $\lambda \in C$, $0 \leq \|\lambda x + y\|^2 = |\lambda|^2 \|x\|^2 + 2re(\alpha\lambda) + \|y\|^2$.

Thus, if $< x, y >= 0$ then $\|y\| \leq \|\lambda x + y\|$ for all $\lambda \in C$.

Also when $0 \neq x$, by taking $\lambda = -\frac{\alpha^{-}}{\|x\|^2}$ we see that,

$0 \leq \|\lambda x + y\| = \|y\|^2 - \frac{|\alpha|^2}{\|x\|^2}$.

Thus, $| < x, y > | \leq \|x\|\|y\|$. This is called the Schwarz inequality.

Also if $< x, y > \neq 0$ for the above choice of λ, $\|y\| > \|\lambda x + y\|$. Hence $x \perp y$ if and only if $\|y\| \leq \|\lambda x + y\|$ for all $\lambda \in C$.

Thus orthogonality can be described only using $\|.\|$. In a Banach space, according to G. Birkhoff, $x \perp y$ if $\|\lambda x + y\| \geq \|y\|$ for all $\lambda \in C$.

It is easy to see now from the Schwarz inequality that $\|x+y\| \leq \|x\|+\|y\|$. Thus $\|x\| = (< x, x >)^{\frac{1}{2}}$ defines a norm on H.

H is a Hilbert space when it is a Banach space with this norm.

$L^2(\mu)$ and ℓ^2 are Hilbert spaces with norms, $\|f\|_2 = (\int |f|^2 d\mu)^{\frac{1}{2}}$ and $\|x\|^2 = (\sum |x(n)|^2)^{\frac{1}{2}}$.

The equation $\|x + y\|^2 + \|x - y\|^2 = 2(\|x||^2 + \|y\|^2)$ is called the the parallelogram law.

For $M \subset H$ define $M^{\perp} = \{x \in H :< x, y >= 0 \ for \ y \in M\}$. This is a closed subspace. Also $M \cap M^{\perp} = 0$.

Fix a $y \in H$. Define $f : H \to C$ by $f(x) =< x, y >$. Then $f \in H^*$ and $\|f\| = \|y\|$.

Theorem 18. *Every non-empty closed convex set $A \subset H$ has a unique vector of smallest norm. For any closed subspace $M \subset H$ there is a unique projection $P \in \mathcal{L}(H)$ such that $ker(P) = M^{\perp}$, $R(P) = M$ and $\|x\|^2 = \|P(x)\|^2 + \|(I - P)(x)\|^2$. Let $f \in H^*$. There exists a unique $y \in H$ such that $f(x) =< x, y >$.*

Proof. We may assume that $0 \notin A$. Let $d = d(0, A)$. Let $\{a_n\}_{n\geq 1} \subset A$ and $d = lim\|a_n\|$. As $\frac{1}{2}(a_n + a_m) \in A$, $\|a_n + a_m\|^2 \geq 4d^2$. Thus by the Parallelogram law, $\|a_n - a_m\|^2 \to 0$. therefore $a_n \to a \in A$ and hence $d = \|a\|$. If $b \in A$ and $\|b\| = d$ then the sequence $a_n = a$ *or* b alternately must converge by the above argument. So $a = b$.

Let $x \in H$ and let $d(x, M) = \|x - y\|$ for a unique $y \in M$. Define $P(x) = y$. Now $\|x - P(x)\| \leq \|x - P(x) + m\|$ for all $m \in M$. Therefore $x - P(x) \in M^{\perp}$. Thus P is a projection with the above properties.

Let $0 \neq f \in H^*$. Applying the above conclusion to $M = ker(f)$, let $0 \neq z \in M^{\perp}$. Since $f(x)z - f(z)x \in ker(f)$, $f(x) < z, z >= f(z) < x, z >$. Now by taking $y = \frac{f(z)^-}{<z,z>} z$ we have $f(x) =< x, y >$. Uniqueness is easy to see.

□

The P in the above theorem is called an orthogonal projection.

Note that $f \to y$ is a conjugate linear isometry of H^* with H. Thus a Hilbert space is reflexive and the unit ball is weakly compact.

We can relate $M^{\perp}$ to the annihilator of M discussed in Section 3.

We saw, in the case of Banach spaces, $\sum_{n=1}^{\infty} \|x_n\| < \infty$ implies $\sum_{n=1}^{\infty} x_n = lim \sum_{i=1}^{n} x_i$ exists. This property, in the case of pair-wise orthogonal vectors, can be completely described in terms of the norm in a Hilbert space.

Theorem 19. *Let $\{x_n\}_{n\geq 1} \subset H$ be a sequence of pair-wise orthogonal vectors. $\sum x_n$ converges if and only if $\sum \|x_n\|^2 < \infty$. This is also equivalent to $\sum < x_n, y > < \infty$ for any $y \in H$.*

Proof. As the vectors are pair-wise orthogonal, by the parallelogram law we have, $\sum_{i=k}^{l} \|x_i\|^2 = \|x_k + ... + x_l\|^2$. Thus $\sum \|x_n\|^2 < \infty$ implies the convergence of the series of vectors. Also by the Schwarz' inequality, $\sum x_n$ converges, implies $\sum < x_n, y > < \infty$ for all $y \in H$.

Conversely, suppose $\sum < x_n, y > < \infty$ for all $y \in H$. Define $\Lambda_n : H \to C$ by $\Lambda_n(x) = \sum_{i=1}^{n} < x_i, y >$. Then $\Lambda_n \in H^*$ and $\|\Lambda_n\| = \sum_{i=1}^{n} \|x_i\|^2$. Also for any $y \in H$, $\{\Lambda_n(y)\}$ is a bounded sequence. Thus, by the uniform boundedness principle, we get that $\sum \|x_n\|^2 < \infty$.

□

In the following we assume that H is a separable space.

A sequence $\{x_n\}_{n\geq 1}$ of unit vectors that are pair-wise orthogonal is called an orthonormal sequence. These are clearly independent vectors. There can be only countably many such vectors in a separable space.

Note that in the case of ℓ^2, $x_n = e_n$ the coordinate vectors form an orthonormal sequence and

$\|x\|^2 = \sum | < x, e_i > |^2$ and $< x, y >= \sum < x, e_n >< e_n, y >$.

An orthonormal sequence satisfies the Bessel's inequality,

$\sum | < x, x_n > |^2 \leq \|x\|^2$.

To see this, let $M_n = span\{\{x_i\}_{1\leq i\leq n}\}$. Then

$d(x, M) = \|x - \sum < x, x_i > x_i\|$.

Also, we know

$\|x\|^2 = \|x - \sum < x, x_i > x_i\|^2 + \| \sum < x, x_i > x_i\|^2$.

But $\| \sum < x, x_i > x_i\|^2 = \sum | < x, x_i > |^2$.

Thus $\Phi : H \to \ell^2$ defined by $\Phi(x) = \{< x, x_n >\}$ is a linear onto map with $\|\Phi\| = 1$.

We will next see conditions under which Φ is an isometry and preserves the inner product.

An orthonormal sequence is said to be maximal if $< x, x_n >= 0$ for all n implies $x = 0$.

One can show that $\{x_n\}_{n\geq 1}$ is a maximal orthonormal sequence if and only if $M = span\{x_n\}_{n\geq 1}$ is dense in H.

Theorem 20. $\{x_n\}_{n\geq 1}$ *is a maximal orthonormal sequence if and only if* $\|x\|^2 = \sum | < x, x_n > |^2$. *In this case,* $< x, y >= \sum < x, x_n >< x_n, y >$.

Proof. Suppose $\{x_n\}_{n\geq 1}$ is a maximal orthonormal sequence. Let $\epsilon > 0$. There is a $z \in M_n$ for some n such that $\|x - z\| < \epsilon$. Therefore

$\|x - \sum_1^n < x, x_i > x_i\| = d(x, M_n) \leq \|x - z\| < \epsilon$.

Thus,

$(\|x\| - \epsilon)^2 < \| \sum_1^n < x, x_i > x_i\|^2 = \sum_1^n | < x, x_i > |^2 \leq \sum_1^\infty | < x, x_i > |^2$.

Hence $\|x\|^2 = \sum | < x, x_n > |^2$. The converse is trivial. The last assertion follows since $\Phi : H \to \ell^2$ has this property. More precisely, by our hypothesis we have $< x, x >= \sum < x, x_n >< x_n, x >$. For x, y, $\lambda \in C$,

$< x + \lambda y, x + \lambda y >=< \Phi(x) + \lambda\phi(y), \Phi(x) + \lambda\Phi(y) >$. Thus,

$\lambda^- < x, y > + \lambda < y, x >= \lambda^- < \Phi(x), \Phi(y) > + \lambda < \Phi(y), \Phi(x) >$. Taking $\lambda = 1, i$ in this relation shows that $< x, y >=< \Phi(x), \Phi(y) >$ as they have the same real and imaginary parts.

$\square$

Finally to generate a maximal orthonormal set (also called a complete orthonormal basis) we invoke Zorn's Lemma. Consider the class of all orthonormal sets and order it by inclusion. Every chain here has a maximal element. Thus the maximal element given by the Zorn's Lemma is a maximal orthonormal set.

8. Operators on Hilbert spaces

Let $T \in \mathcal{L}(H)$. If $< T(x), x >= 0$ for all $x \in H$ then $T = 0$.

To see this, apply the idea from the last section to get $< T(x), y >= 0$ for all $x, y \in H$. Taking $y = T(x)$ thus gives the conclusion.

For each $y \in H$, by Theorem 18, there exists a unique $S(y) \in H$ with $\|S(y)\| \leq \|T(x)\|\|y\|$, such that $< T(x), y >=< x, S(y) >$. It is easy to see that S is linear and thus $S \in \mathcal{L}(H)$ and $\|S\| \leq \|T\|$. We denote this S by T^* and is unique with the property $< T(x), y >=< x, T^*(y) >$. It is called the adjoint of T.

Applying the same idea to $T^* \in \mathcal{L}(H)$,

$< T(x), y >=< T^*(y), x >^- =< y, T^{**}(x) >^- =< T^{**}(x), y >$

so that by uniqueness $T^{**} = T$ and $\|T^*\| = \|T\|$.

Also $\|T(x)\|^2 =< T(x), T(x) >=< T^*(T(x)), x >\leq \|T^*T\|\|x\|^2$. So $\|T\|^2 \leq \|T^*T\| \leq \|T\|\|T^*\| = \|T\|^2$. Therefore $\|T\|^2 = \|T^*T\|$.

$T \to T^*$ is a conjugate linear map and $(TS)^* = S^*T^*$.

In the case of Hilbert space, we have that T is compact if and only if $T(H_1)$ is a compact set. T is compact if and only if T^* is compact.

For $\alpha \in \ell^\infty$, define $M_\alpha : \ell^2 \to \ell^2$ by $M_\alpha(x) = (\alpha(n)x(n))$. It is easy to see that $\|M_\alpha\| = \|\alpha\|$. One can easily compute M_α^*, and also decide when is M_α compact ?

Let H be separable with a complete orthonormal basis $\{e_n\}_{n\geq 1}$. Let P_n be the orthogonal projection onto $span\{e_1, ..., e_n\}$. Then $P_n(x) \to x$. If $T \in \mathcal{K}(H)$ then $P_nT \to T$. Thus $\mathcal{F}(H)^- = \mathcal{K}(H)$.

The duality between the range of an operator and the kernel of the adjoint, in Section 5, in the case of Hilbert spaces is given by, $Ker(T^*) = R(T)^\perp$ and $Ker(T) = R(T^*)^\perp$.

T is said to be normal if $TT^* = T^*T$, self-adjoint if $T^* = T$ and unitary if $T^*T = I = TT^*$.

For a normal operator we have,

$\|T(x)\|^2 =< T(x), T(x) >=< T^*T(x), x >=< TT^*(x), x >$. Also

$< TT^*(x), x >=< T^*(x), T^*(x) >= \|T^*(x)\|^2$.

Thus $\|T(x)\| = \|T^*(x)\|$.

M_α defined above is a normal operator. Define $U : \ell^2 \to \ell^2$ by

$U(x_1, ...x_n, ..) = (0, x_1, x_2, ...)$. It is easy to compute U^*. This is called the shift operator.

We see that an operator U is unitary if and only if U is onto and preserves the norm or inner product.

Let $P \in \mathcal{H}$ be a projection, i.e., $P^2 = P$. Then if P is normal then P is an orthogonal projection (i.e., $ker(P)^\perp = R(P)$) and P is self adjoint. To see this we note that for a normal operator T, $ker(T) = ker(T^*) = R(T)^\perp$. Now since $R(P) = ker(I - P)$, we have that P is an orthogonal projection.

Now $\|P(x)\|^2 =< P(x), P(x) >=< P(x), (I - P)(x) + P(x) >$.

Similarly, $\|P(x)\|^2 =< P(x) + (I - P)(x), P(x) >$. Thus,

$< P(x), x >=< x, P(x) >$. Hence an orthogonal projection is self adjoint.

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