CHAPTER 1

Classical stuff - title to be changed later

1. Positive Definite Kernels

To start with something simple and elegant, we choose *positive definite kernels* which appear at every corner in functional analysis. This section does not require much background. A basic knowledge of Hilbert spaces and of operators on them will be assumed in the book.

A positive definite kernel k on a set X is a complex valued function on $X \times X$ such that for any positive integer n, any $x_1, x_2, \ldots, x_n$ from X and any n complex numbers $c_1, c_2, \ldots, c_n$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{c_i} c_j k(x_i, x_j) \geq 0.$$

Note that this means that the $n \times n$ matrix

$$((k(x_i, x_j)))$$

is a positive definite matrix. We shall use the term "positive definite matrix" even when the matrix has zero eigenvalues. Thus our positive definite matrices need not be invertible. In the case when the matrix is actually invertible, i.e., when the eigenvalues are all positive, we shall call it a strictly positive definite matrix.

Examples of positive definite kernels are as follows.

(1) If X is the finite set $\{1, 2, \ldots, n\}$, take an $n \times n$ positive definite matrix $A = ((a_{ij}))$ and define $k(i, j) = a_{ij}$ for all $i, j = 1, 2, \ldots, n$.
(2) Let $X = H$, a Hilbert space and $k(x, y) = \langle x, y \rangle$.
(3) If X is any set and $\lambda : X \to H$ is a function from X to a Hilbert space H the kernel $k : X \times X \to \mathbb{C}$ by

$$k(x, y) = \langle \lambda(x), \lambda(y) \rangle, \qquad x, y \in X \tag{1.1.1}$$

is positive definite.

In the last example, let $\mathcal{M}$ be the closed linear span of the set of vectors $\{\lambda(x) : x \in X\}$. Then $\mathcal{H}$ could be replaced by $\mathcal{M}$ and the function $\lambda : X \to \mathcal{M}$ then would have the property that the Hilbert space is generated as the closed linear span of the vectors $\{\lambda(x) : x \in X\}$. When this happens, the function λ is said to be *minimal* (for the positive definite kernel k). It turns out, as the following lemma illustrates, that this is a prototypical example of a positive definite kernel.

Before we go to the lemma, we would like to mention here that, on our Hilbert spaces, the inner product is linear in the second variable and conjugate linear in the first. This is a departure from the standard practice in functional analysis. However, as we shall see, this helps us regarding notations. Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, when we say $\mathcal{H}$ is contained in $\mathcal{K}$, usually denote by $\mathcal{K} \supset \mathcal{H}$ and $\mathcal{H} \subset \mathcal{K}$ we shall mean that $\mathcal{H}$ is physically contained in $\mathcal{K}$ as a closed subspace or that $\mathcal{H}$ is isometrically embedded into $\mathcal{K}$, i.e., there is a linear isometry V mapping $\mathcal{H}$ into $\mathcal{K}$. In the latter case, we shall identify $\mathcal{H}$ with the closed subspace Ran V of $\mathcal{K}$ and make no distinction between the elements h and Vh. Obviously, this then implies that if Ran V is the whole space $\mathcal{K}$, then an element k of $\mathcal{K}$ is identified with that unique element h of $\mathcal{H}$ which satisfies $Vh = k$. An isometry which is onto is called a unitary map. If V is a unitary map, then we say that $\mathcal{H}$ and $\mathcal{K}$ are unitarily equivalent. As we shall often see and as is often required, the unitary map between two Hilbert spaces also preserves some more structures.

LEMMA 1.1. *Given a positive definite kernel k on a set X, there is a unique (up to unitary isomorphism) Hilbert space $\mathcal{H}$ and a minimal map $\lambda : X \to \mathcal{H}$ such that equation (1.1.1) is satisfied.*

PROOF. Given a positive definite kernel $k : X \times X \to \mathbb{C}$, one can construct a Hilbert space $\mathcal{H}$ and a function $\lambda : X \to \mathcal{H}$ as follows. Consider the vector space $\mathbb{C}X$ of all complex valued functions $\xi : X \to \mathbb{C}$ with the property that $\xi(x) = 0$ for all but a finite number of $x \in X$. We can define a sesquilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{C}X$ by

$$\langle \xi, \eta \rangle = \sum_{x,y \in X} k(x, y)\bar{\xi}(x)\eta(y), \qquad \xi, \eta \in \mathbb{C}X.$$

The sesquilinear form $\langle \cdot, \cdot \rangle$ is positive semidefinite because of the hypothesis on k. An application of the Schwarz inequality shows that the set

$$N = \{\xi \in \mathbb{C}X : \langle \xi, \xi \rangle = 0\}$$

is a linear subspace of $\mathbb{C}X$, so this sesquilinear form can be promoted naturally to an inner product on the quotient $\mathbb{C}X/N$. The completion of the inner product space $\mathbb{C}X/N$ is a Hilbert space $\mathcal{H}(k)$. We shall usually omit k and denote it only by $\mathcal{H}$ when there is no chance of confusion. Define a function $\lambda : X \to \mathcal{H}$ as follows:

$$\lambda(x) = \delta_x + N, \qquad x \in X, \tag{1.1.2}$$

where δ_x is the indicator function of the singleton $\{x\}$. By construction, $k(x, y) = \langle \lambda(x), \lambda(y) \rangle$. Note too that this function λ is *minimal* for k just by construction.

The uniqueness statement means that given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with two minimal functions $\lambda_1 : X \to \mathcal{H}_1$ and $\lambda_2 : X \to \mathcal{H}_2$, both satisfying (1.1.1), there is a unitary operator

$U : \mathcal{H}_1 \to \mathcal{H}_2$ which satisfies

$$U(\lambda_1(x)) = \lambda_2(x), \qquad x \in X. \tag{1.1.3}$$

We could try to define U by (1.1.3) and extend it linearly to finite linear combinations of the $\lambda(x)$. Indeed, it then defines a linear operator on a dense subspace of $\mathcal{H}_1$ and we find that for points $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ from X

$$\begin{aligned}
\langle U(\sum_{i=1}^n c_i\lambda_1(x_i)), U(\sum_{j=1}^m d_j\lambda_1(y_j))\rangle_{\mathcal{H}_2} &= \sum_{i,j} \overline{c_i} d_j \langle U(\lambda_1(x_i)), U(\lambda_1(y_j))\rangle_{\mathcal{H}_2} \\
&= \sum_{i,j} \overline{c_i} d_j \langle \lambda_2(x_i), \lambda_2(y_j)\rangle_{\mathcal{H}_2} \\
&= \sum_{i,j} \overline{c_i} d_j k(x_i, y_j) \\
&= \sum_{i,j} \overline{c_i} d_j \langle \lambda_1(x_i), \lambda_1(y_j)\rangle_{\mathcal{H}_1} \\
&= \langle \sum_{i=1}^n c_i\lambda_1(x_i), \sum_{j=1}^m d_j\lambda_1(y_j)\rangle_{\mathcal{H}_1}.
\end{aligned}$$

Thus U is an isometry from a dense subspace of $\mathcal{H}_1$ to a dense subspace of $\mathcal{H}_2$. Hence U extends to an isomorphism of the Hilbert spaces. That proves the uniqueness. ■

Obtaining a Hilbert space $\mathcal{H}$ from a positive definite kernel k is sometimes referred to as the Gelfand-Naimark-Segal (GNS) construction. The space $\mathcal{H}$ is called the GNS space and the pair $(\mathcal{H}, \lambda)$ together is called the GNS pair. The only word of caution is that in general, the GNS space $\mathcal{H}$ need not be separable.

We close this section by showing how an innocuous looking map between two sets with kernels preserving the kernel structures in a suitable way is loaded enough to imply that the GNS space of the first kernel in contained in that of the other. Suppose there are two positive definite kernels k_1 and k_2 on two sets X_1 and X_2 respectively with their respective GNS pairs $(\mathcal{H}_i, \lambda_i)$ for $i = 1, 2$. Suppose $\varphi : X_1 \to X_2$ is a map. Now φ may not have any relation with the given kernels. However, it may be the case that φ satisfies

$$k_2(\varphi(x), \varphi(y)) = k_1(x, y), \qquad x, y \in X.$$

Then, using the GNS pair, we immediately have

$$\langle \lambda_2(\varphi(x)), \lambda_2(\varphi(y))\rangle_{H(k_2)} = k_2(\varphi(x), \varphi(y)) = k_1(x, y) = \langle \lambda_1(x), \lambda_1(y)\rangle_{H(k_1)},$$

holding for all $x, y \in X_1$. Define $U_\varphi : H(k_1) \to H(k_2)$ by defining it first on $\{\lambda_1(x) : x \in X\}$ by

$$U_\varphi \lambda_1(x) = \lambda_2(\varphi(x))$$

and then extending it to linear combinations. The equality above shows that U_φ is an inner product preserving linear map defined on a dense subset of $\mathcal{H}(k_1)$. Hence we get a unique linear isometry from $\mathcal{H}(k_1)$ into $\mathcal{H}(k_2)$ such that

$$U_\varphi(\lambda_1(x)) = \lambda_2(\varphi(x)), \qquad x \in X_1. \tag{1.1.4}$$

As expected, it is not difficult to verify that $U_{\varphi_1}U_{\varphi_2} = U_{\varphi_1 \circ \varphi_2}$ holds.

Good references for the material presented above is a recent historical survey article [**2**] by Arveson and the Section 15 of the book [**8**] by Parthasarathy.

2. Dilation of a Single Contraction

All the Hilbert spaces in this book are over the complex field and are separable. If there is an isometry $V : \mathcal{H} \to \widehat{\mathcal{H}}$, any bounded operator T on $\mathcal{H}$ is then identified with the bounded operator VTV^* on Ran V. The projection from $\widehat{\mathcal{H}}$ onto Ran V (equivalently $\mathcal{H}$) will be denoted by $P_{\mathcal{H}}$.

Definition 2.1. *Let $\mathcal{H}$ be a Hilbert space. Suppose T is bounded operator on $\mathcal{H}$. If $\widehat{\mathcal{H}}$ is a Hilbert space and A is a bounded operator on $\widehat{\mathcal{H}}$ satisfying $\mathcal{H} \subset \widehat{\mathcal{H}}$ and*

(1.2.1) $\quad T^n h = P_{\mathcal{H}} A^n h$ *for all* $h \in \mathcal{H}$ *and all non-negative integers* n

then $\widehat{\mathcal{H}}$ is called a dilation space and the operator A is called a dilation for the operator T. A dilation A of T is called minimal if

$$\overline{span}\{A^n h : h \in \mathcal{H}, n = 0, 1, 2, \ldots\} = \hat{\mathcal{H}}.$$

An isometric (respectively unitary) dilation of T is a dilation A which is an isometry (respectively unitary).

Let A_1 and A_2 be two dilations of the same operator T on $\mathcal{H}$. Let the dilation spaces be $\widehat{\mathcal{H}}_1$ and $\widehat{\mathcal{H}}_2$ respectively and let the embedding isometries be $V_1 : \mathcal{H} \to \widehat{\mathcal{H}}_1$ and $V_2 : \mathcal{H} \to \widehat{\mathcal{H}}_2$ respectively. Then the dilations $(\widehat{\mathcal{H}}_1, A_1)$ and $(\widehat{\mathcal{H}}_2, A_2)$ are called unitarily equivalent *if there is a unitary $U : \widehat{\mathcal{H}}_1 \to \widehat{\mathcal{H}}_2$ which satisfies $UA_1U^* = A_2$ and $UV_1h = V_2h$ for all $h \in \mathcal{H}$.*

Given any dilation space $\mathcal{K}$ and an isometric dilation A of an operator T on $\mathcal{H}$, one can always consider the subspace $\widehat{\mathcal{H}} \stackrel{\text{def}}{=} \overline{span}\{A^n h : h \in \mathcal{H}, n = 0, 1, 2, \ldots\}$. This is a reducing subspace for A. It is a dilation space for T and $A|_{\widehat{\mathcal{H}}}$ is a minimal isometric dilation for T. Moreover, note that if $\mathcal{H}$ is separable, then $\widehat{\mathcal{H}}$ is the span closure of elements of the form $A^n h$ where n is a non-negative integer and h comes from a countable dense subset of $\mathcal{H}$. Thus $\widehat{\mathcal{H}}$ is separable.

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded operators on the Hilbert space $\mathcal{H}$ and let $\|\cdot\|$ denote the operator norm on $\mathcal{B}(\mathcal{H})$. An element T of $\mathcal{B}(\mathcal{H})$ is called a contraction if $\|T\| \le 1$. Clearly, if T has an isometric dilation, then T is a contraction. Conversely, given a contraction T acting on a Hilbert space $\mathcal{H}$, Sz.-Nagy showed that T always has a minimal isometric dilation. We shall give three proofs of the theorem.

THEOREM 2.2. *Given a contraction T on a Hilbert space $\mathcal{H}$, there is a minimal isometric dilation which is unique up to unitary equivalence.*

There are a few comments to make before we embark on the proof.

- First note that if a minimal dilation A exists, then its uniqueness up to unitary equivalence is straightforward because $\langle A^n h, A^m h' \rangle$ does not depend on a particular minimal dilation A. Indeed,

$$\langle A^n h, A^m h' \rangle = \begin{cases} \langle A^{n-m} h, h' \rangle = \langle T^{n-m} h, h' \rangle \text{ if } n \geq m \geq 0, \\ \langle h, A^{m-n} h' \rangle = \langle h, T^{m-n} h' \rangle \text{ if } m \geq n \geq 0. \end{cases} \tag{1.2.2}$$

  So given two minimal dilations A_1 and A_2 on the spaces $\widehat{\mathcal{H}}_1$ and $\widehat{\mathcal{H}}_2$ respectively, define U on $\operatorname{span}\{A_2^n h : h \in \mathcal{H}, n = 0, 1, 2, \ldots\}$ by

$$U(\sum_{n=0}^{N} A_2^n h_n) = \sum_{n=0}^{N} A_1^n h_n.$$

  This is a well-defined isometric linear transformation from a dense subspace of $\widehat{\mathcal{H}}_2$ onto a dense subspace of $\widehat{\mathcal{H}}_1$. Thus U extends to a unitary operator from $\widehat{\mathcal{H}}_2$ to $\widehat{\mathcal{H}}_1$. Moreover, note that for $h \in \mathcal{H}$, we have $Uh = U(A_2^0)h = A_1^0 h = h$, thus the isometric embedding of $\mathcal{H}$ into the dilation spaces is respected by U. Of course, U has been so constructed that $UA_2U^* = A_1$.
- If A is a bounded operator on $\widehat{\mathcal{H}} = \mathcal{H} \oplus (\widehat{\mathcal{H}} \ominus \mathcal{H})$ such that the operator matrix of A with respect to the decomposition above is

$$A = \begin{pmatrix} T & 0 \\ * & * \end{pmatrix}$$

  then it is clear from matrix multiplication that

$$A^n = \begin{pmatrix} T^n & 0 \\ * & * \end{pmatrix}$$

  for any positive integer n. Thus A is a dilation of T. Since A^* leaves $\mathcal{H}$ invariant, $\mathcal{H}$ is called a *is a co-invariant subspace for* A. Of course, this is equivalent to saying that $A^*|_{\mathcal{H}}$ has to be T^*.
- The minimal isometric dilation A that we shall construct has the property that $\mathcal{H}$ is a co-invariant subspace for A. Now note that if this is true for one minimal isometric dilation A, then it is true for any minimal isometric dilation. Indeed, let $(\widehat{\mathcal{H}}_1, A_1)$ and $(\widehat{\mathcal{H}}_2, A_2)$ be two minimal isometric dilations. Suppose that $\mathcal{H}$ is a co-invariant subspace for A_1. We know by minimality that there is a unitary operator $U : \widehat{\mathcal{H}}_1 \to \widehat{\mathcal{H}}_2$ such that $Uh = h$ for all $h \in \mathcal{H}$ and $A_2 = UA_1U^*$. With respect to the decompositions $\widehat{\mathcal{H}}_i = \mathcal{H} \oplus (\widehat{\mathcal{H}}_i \ominus \mathcal{H})$ for $i = 1, 2$, we have

$$A_1 = \begin{pmatrix} T & 0 \\ * & * \end{pmatrix} \text{ and } U = \begin{pmatrix} I & 0 \\ 0 & * \end{pmatrix}.$$

Obviously, then $A_2 = UA_1U^*$ also has the operator matrix form

$$\begin{pmatrix} T & 0 \\ * & * \end{pmatrix}$$

which means that $\mathcal{H}$ is a co-invariant subspace for A_2. Thus any minimal isometric dilation leaves $\mathcal{H}$ co-invariant.

- Here are some basic notations and terminology which will be used throughout.
  Defect operators: For a contraction T, the operators $1 - TT^*$ and $1 - T^*T$ are called the *defect operators*. Let $D_{T^*} = (1 - TT^*)^{1/2}$ and $D_T = (1 - T^*T)^{1/2}$.
  Defect spaces: $\mathcal{D}_{T^*} = \overline{\mathrm{Ran}} D_{T^*}$ and $\mathcal{D}_T = \overline{\mathrm{Ran}} D_T$ are called the defect spaces.

3. First proof of existence - an abstract method

We start with a lemma which will be useful in this proof as well as in many other places.

LEMMA 3.1. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two Hilbert spaces, let $B \in \mathcal{B}(\mathcal{K}_1)$ be a positive operator and let $A \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$. Then*

$$\begin{pmatrix} B & A^* \\ A & I \end{pmatrix} : \mathcal{K}_1 \oplus \mathcal{K}_2 \to \mathcal{K}_1 \oplus \mathcal{K}_2 \tag{1.3.1}$$

is a positive definite operator if and only if $A^*A \leq B$.

PROOF Let $k_i \in \mathcal{K}_I$ for $i = 1, 2$. Then

$$\left\langle \begin{pmatrix} B & A^* \\ A & I \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \right\rangle = \langle Bk_1, k_2 \rangle + \langle A^*k_2, k_1 \rangle + \langle Ak_1, k_2 \rangle + \langle k_2 k_2 \rangle. \tag{1.3.2}$$

Now given that $A^*A \leq B$, we have $\|Ak_1\| \leq \|B^{1/2}k_1\|$ for all $k_1 \in \mathcal{K}_1$. So from (1.3.2), we have

$$\begin{aligned} \left\langle \begin{pmatrix} B & A^* \\ A & I \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \right\rangle &= \langle Bk_1, k_2 \rangle + 2 \text{ Re } \langle Ak_1, k_2 \rangle + \langle k_2, k_2 \rangle \\ &\geq \|B^{1/2}k_1\|^2 + \|k_2\|^2 - 2|\langle Ak_1, k_2 \rangle| \\ &\geq \|B^{1/2}k_1\|^2 + \|k_2\|^2 - 2\|Ak_1\| \, \|k_2\| \\ &\geq \|B^{1/2}k_1\|^2 + \|k_2\|^2 - 2\|B^{1/2}k_1\|\|k_2\| \geq 0. \end{aligned}$$

Conversely, suppose the operator matrix in (1.3.1) defines a positive operator. Then the right hand side of (1.3.2) is non-negative for every k_1 and k_2. We put $k_2 = -Ak_1$ to get the result. ■

With this background, we embark on the proof. Taking cue from 1.2.2, we define the set

$$X = \{(n, g) : n \in \mathbb{Z}_+, g \in \mathcal{H}\}$$

and $k : X \times X \to \mathbb{C}$ by

$$k((m,g),(n,h)) = \begin{cases} \langle g, (T^*)^{m-n}h\rangle & \text{if } m \geq n, \\ \langle g, T^{n-m}h\rangle & \text{if } n \geq m. \end{cases}$$

Now we claim that k is a positive definite kernel. A moment's thought shows that positive definiteness of k follows if we show that the operator matrix

$$X_n := \begin{pmatrix} I & T & T^2 & \cdots & T^n \\ T^* & I & T & \cdots & T^{n-1} \\ (T^*)^2 & T^* & I & \cdots & T^{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ (T^*)^n & (T^*)^{n-1} & (T^*)^{n-2} & \cdots & I \end{pmatrix} \tag{1.3.3}$$

is a positive operator on $\mathcal{H}^{n+1}$ for every n. To prove that, we apply induction on n. For $n = 1$, we have $X_1 = \begin{pmatrix} I & T \\ T^* & I \end{pmatrix}$ which is positive by Lemma 3.1 because T is a contraction.

Suppose then that $n \geq 2$ is a positive integer and X_{n-1} is a positive operator on $\mathcal{H}_n$. Note that

$$X_n = \begin{pmatrix} X_{n-1} & A \\ A^* & I \end{pmatrix} \text{ with } A = \begin{pmatrix} T^n \\ T^{n-1} \\ \vdots \\ T \end{pmatrix}.$$

Hence by Lemma 3.1, we need to show that $AA^* \leq X_{n-1}$. For $k = 0, 1, 2, \ldots, n$, denote the operator

$$\begin{array}{cccc} (T^k, T^{k-1}, \ldots, T, I, 0, \ldots, 0) & : & \mathcal{H}^{n+1} & \to \mathcal{H} \\ & & \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} & \to T^k h_1 + \cdots + Th_k + h_{k+1} \end{array}$$

by L_k. Then a straightforward, albeit a little cumbersome, computation shows that

$$X_{n-1} - AA^* = \sum_{k=0}^{n} L_k^* D_{T^*}^2 L_k.$$

Hence the positive definiteness of the operator matrix (1.3.3) follows. So k is a positive definite kernel.

Let $(\widehat{\mathcal{H}}, \lambda)$ be the GNS pair for the positive definite kernel k. Note that $\langle \lambda(0,g), \lambda(0,h)\rangle_{\widehat{\mathcal{H}}} = k((0,g),(0,h)) = \langle g, h\rangle_{\mathcal{H}}$, so that we can embed $\mathcal{H}$ isometrically in $\widehat{\mathcal{H}}$ by identifying h in $\mathcal{H}$ with $\lambda(0,h)$ in $\widehat{\mathcal{H}}$. Define A on the total set of vectors by

$$A\lambda(m,h) = \lambda(m+1,h), \text{ where } (m,h) \in X$$

and then extend it linearly to finite linear combinations of these vectors. Let $m_1 \leq m_2 \leq \cdots \leq m_k$ be non-negative integers and let $h_1, h_2, \ldots, h_k$ be vectors in $\mathcal{H}$. Then using the definition of the inner product in terms of the kernel we get

$$\begin{aligned}\|\sum_{i=1}^{k} c_i\lambda(m_i,h_i)\|^2 &= \sum_{i=1}^{k}\sum_{j=1}^{k}\overline{c_i}c_j\langle\lambda(m_i,h_i),\lambda(m_j,h_j)\rangle\\ &= \sum_{i>j}\overline{c_i}c_j\langle h_i,(T^*)^{m_i-m_j}h_j\rangle+\sum_{i=1}^{k}|c_i|^2\|h_i\|^2+\sum_{i<j}\overline{c_i}c_j\langle h_i,T^{m_j-m_i}h_j\rangle\\ &= \sum_{i=1}^{k}\sum_{j=1}^{k}\overline{c_i}c_j\langle\lambda(m_i+1,h_i),\lambda(m_j+1,h_j)\rangle\\ &= \|\sum_{i=1}^{k} c_i\lambda(m_i+1,h_i)\|^2=\|A(\sum_{i=1}^{k} c_i\lambda(m_i,h_i))\|^2.\end{aligned}$$

Thus A is an isometry on this dense subspace and hence extends to the whole space as an isometry. It is a straightforward computation to see that $A^*|_{\mathcal{H}}=T^*$.

4. Second proof of existence - Schaffer construction

In this section, we give a concrete construction for the minimal isometric dilation. This elegant construction appears in a one page paper by Schaffer [**10**]. The dilation space is constructed by setting

$$\widehat{\mathcal{H}}=\mathcal{H}\oplus\mathcal{D}_T\oplus\mathcal{D}_T\cdots.$$

The space $\mathcal{H}$ is identified with the subspace of $\widehat{\mathcal{H}}$ consisting of elements of the form $(h,0,0,\dots)$ where $h\in\mathcal{H}$. This is the way $\mathcal{H}$ is isometrically embedded into $\widehat{\mathcal{H}}$. Define the operator A on $\widehat{\mathcal{H}}$ by

$$A(h_0,h_1,h_2,\dots)=(Th_0,D_Th_0,h_1,h_2,\dots).$$

For every $h\in\mathcal{H}$, we have $\|Th\|^2+\|D_Th\|^2=\|h\|^2$, so that the operator A defined above is an isometry. Now for $h_0\in\mathcal{H}$ and $(k_0,k_1,k_2,\dots)\in\widehat{\mathcal{H}}$, we have

$$\begin{aligned}< A^*h_0,(k_0,k_1,k_2,\dots)> &= < h_0,A(k_0,k_1,k_2,\dots)>\\ &= <(h_0,0,0,\dots),(Tk_0,D_Tk_0,k_1,k_2,\dots)>\\ &= < h_0,Tk_0>\\ &= < T^*h_0,k_0>=< T^*h_0,(k_0,k_1,k_2,\dots)>.\end{aligned}$$

Thus $A^*h=T^*h$ for all $h\in\mathcal{H}$. The isometry A on $\widehat{\mathcal{H}}$ has the property that $\mathcal{H}$ is is left invariant by A^*. So A is an isometric dilation of T. Minimality becomes clear by observing that

$$A^n(h_0,h_1,h_2,\dots)=(T^nh_0,D_TT^{n-1}h_0,\dots,D_Th_0,h_1,h_2,\dots)$$

from which it follows that

$$A^n(h_0,0,0,\dots)=(T^nh_0,D_TT^{n-1}h_0,\dots,D_Th_0,0,0,\dots). \tag{1.4.1}$$

Thus

$$\overline{span}\{A^n(h,0,0,\dots) : h \in \mathcal{H}, n = 0,1,2,\dots\} = \mathcal{H} \oplus \mathcal{D}_T \oplus \mathcal{D}_T \cdots = \widehat{\mathcal{H}}.$$

The Schaffer construction, apart from being direct and explicit, emphasizes a very basic ingredient in all of operator theory. To detect it, note that with respect to the decomposition

$$\widehat{\mathcal{H}} = \mathcal{H} \oplus (\mathcal{D}_T \oplus \mathcal{D}_T \oplus \mathcal{D}_T \oplus \mathcal{D}_T \oplus \cdots)$$

of the dilation space, the block matrix of the dilation operator is

$$\begin{pmatrix} T & | & 0 & 0 & 0 & 0 & \dots \\ - & - & - & - & - & - & - \\ D_T & | & 0 & 0 & 0 & 0 & \dots \\ 0 & | & 1 & 0 & 0 & 0 & \dots \\ 0 & | & 0 & 1 & 0 & 0 & \dots \\ 0 & | & 0 & 0 & 1 & 0 & \dots \\ \vdots & | & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{1.4.2}$$

The bottom right hand corner of the block matrix above is an example of a *unilateral shift.* This is a class of operators which plays a fundamental role.

Definition 4.1. *An isometry S on a Hilbert space $\mathcal{M}$ is called a unilateral shift if there is a subspace $\mathcal{L}$ of $\mathcal{M}$ satisfying*

(i) $S^n\mathcal{L} \perp \mathcal{L}$ for all $n = 1,2,\dots$ and

(ii) $\mathcal{L} \oplus S\mathcal{L} \oplus S^2\mathcal{L} \cdots = \mathcal{M}$.

The subspace $\mathcal{L}$ is called the generating subspace for S and $\dim \mathcal{L}$ *is called the multiplicity of S.*

The bottom right corner of (1.4.2) has a unilateral shift with the full space

$$\mathcal{M} = \mathcal{D}_T \oplus \mathcal{D}_T \oplus \mathcal{D}_T \oplus \mathcal{D}_T \oplus \cdots$$

the generating subspace

$$\mathcal{L} = \mathcal{D}_T \oplus 0 \oplus 0 \oplus \cdots$$

and

$$S(h_1, h_2, h_3, \dots) = (0, h_1, h_2, \dots).$$

Its multiplicity is dim $\mathcal{D}_T$. Thus the block operator matrix of the dilation A constructed in this section is

$$\left(\begin{array}{c|ccc} T & 0 & 0 & 0 \\ \hline D_T & & & \\ 0 & & S & \\ 0 & & & \end{array}\right).$$

A unilateral shift has a unique generating subspace and is determined up to *unitary equivalence* by its multiplicity, i.e., if S_1 and S_2 are two unilateral shifts on Hilbert spaces $\mathcal{L}_1$ and $\mathcal{L}_2$

respectively with the same multiplicity, then there is a unitary $U : \mathcal{L}_1 \to \mathcal{L}_2$ such that $US_1U^* = S_2$. The proofs and other facts about unilateral shifts can be found in, for example [**11**].

5. The unitary dilation

Definition 5.1. *Let $\mathcal{H} \subset \mathcal{K}$ be two Hilbert spaces. Suppose V and U are bounded operators on $\mathcal{H}$ and $\mathcal{K}$ respectively such that*

$$U^n h = V^n h \text{ for all } h \in \mathcal{H}.$$

Then U is called an extension of V. A unitary extension is an extension which is also a unitary operator.

Note that the block operator matrix of the operator U with respect to the decomposition $\mathcal{K} = \mathcal{H} \oplus (\mathcal{K} \ominus \mathcal{H})$ is

$$U = \begin{pmatrix} V & * \\ 0 & * \end{pmatrix}.$$

An extension U of a bounded operator V is also a dilation of V because $P_{\mathcal{H}}U^n h = P_{\mathcal{H}}V^n h = V^n h$ for any $h \in \mathcal{H}$ and $n \geq 1$. It moreover has the property that $\mathcal{H}$ is an invariant subspace for U. It is this second property that makes it clear that in general, contractions could not have isometric extensions.

Remark 5.2. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Suppose V is a dilation of T on a Hilbert space $\mathcal{K}_1 \supset \mathcal{H}$ and U is an extension of V on a Hilbert space $\mathcal{K}_2 \supset \mathcal{K}_1$. Then U is a dilation of T. Indeed,*

$$\begin{aligned} P_{\mathcal{H}}U^n h &= P_{\mathcal{H}}V^n h \text{ because } U \text{ is an extension of } V \text{ and } h \in \mathcal{H} \subset \mathcal{K}_1 \\ &= T^n h \text{ because } V \text{ is a dilation of } T. \end{aligned}$$

Definition 5.3. *A unitary operator U on a Hilbert space $\mathcal{M}$ is called a bilateral shift if there is a subspace $\mathcal{L}$ of $\mathcal{M}$ satisfying*

(i) $U^n\mathcal{L} \perp \mathcal{L}$ for all integers $n \neq 0$ and

(ii) $\oplus_{n=-\infty}^{\infty} U^n\mathcal{L} = \mathcal{M}$.
The subspace $\mathcal{L}$ is called a generating subspace for U and $\dim \mathcal{L}$ is called the multiplicity of U.

Lemma 5.4. *A unilateral shift V on $\mathcal{M}$ always has an extension to a bilateral shift. Moreover, the extension preserves multiplicity.*

Proof: The generating subspace of V is $\mathcal{L} = \mathcal{M} \ominus V\mathcal{M}$. Define $\mathcal{K} = \oplus_{-\infty}^{\infty}\mathcal{L}_n$ where each $\mathcal{L}_n$ is the same as $\mathcal{L}$. For an element $(\dots, l_{-2}, l_{-1}, l_0, l_1, l_2, \dots)$ of $\mathcal{K}$ with $l_n \in \mathcal{L}_n$ for every $n \in \mathbb{Z}$, define

$$U(\ldots, l_{-2}, l_{-1}, l_0, l_1, l_2, \ldots) = (\ldots, l'_{-2}, l'_{-1}, l'_0, l'_1, l'_2, \ldots),$$

where now $l'_n \in \mathcal{L}_n$ and $l'_n = l_{n-1}$ for all $n \in \mathbb{Z}$. Clearly, U is unitary and $\{(\ldots, 0, 0, l_0, 0, 0, \ldots) : l_0 \in \mathcal{L}_0\}$ is a generating subspace for U. This subspace has the same dimension as that of $\mathcal{L}$. An element $\sum_0^\infty V^n l_n$ of $\mathcal{H}$ is identified with the element $(\ldots, 0, 0, l_0, l_1, l_2, \ldots)$ of $\mathcal{K}$. This is an isometric embedding.

Now for $h = \sum_0^\infty V^n l_n$, we have

$$Uh = U(\ldots, 0, 0, l_0, l_1, l_2, \ldots) = (\ldots, 0, 0, 0, l_0, l_1, \ldots) = \sum_0^\infty V^n l_{n-1} = Vh.$$

That completes the proof. ■

An immediate corollary is the following.

Corollary 5.5. *An isometry V on $\mathcal{H}$ always has an extension to a unitary.*

Proof: By Wold decomposition ([**11**], page 3), we have $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ where $\mathcal{H}_0$ and $\mathcal{H}_1$ are reducing subspaces of V and $V = V_0 \oplus V_1$ where $V_0 = V|_{\mathcal{H}_0}$ is a unitary and $V_1 = V|_{\mathcal{H}_1}$ is a unilateral shift. By the lemma above, V_1 can be extended to a bilateral shift, say U_1. Now $V_0 \oplus U_1$ is a unitary extension of V. ■

Theorem 5.6. *For every contraction T on a Hilbert space $\mathcal{H}$, there is a minimal unitary dilation which is unique up to unitary equivalence.*

Proof: Obtaining a unitary dilation is immediate from the above discussions. We take an isometric dilation and then its unitary extension, say U_0. This is a unitary dilation, although may not be minimal. Let

$$\mathcal{K} = \overline{span}\{U_0^n h : h \in \mathcal{H} \text{ and } n = 0, 1, 2, \ldots\}.$$

This is a reducing subspace for U_0 and the restriction U of U_0 to $\mathcal{K}$ is a minimal unitary dilation.

The uniqueness (up to unitary equivalence) proof is exactly on the same lines as the proof of uniqueness of isometric dilation. ■

The unitary dilation of a contraction gives a quick proof of von Neumann's inequality. In its original proof [**13**], von Neumann first proved it for Mobius functions and then used the fact that the space of absolutely convergent sums of finite Blashke products is isometrically isomorphic to the disk algebra, the algebra of all functions which are analytic in the interior and continuous on the closure of the unit disk. See Drury [**5**] or Pisier [**9**] for the details of this proof. The following proof using the dilation is due to Halmos [**6**].

THEOREM 5.7. (**von Neumann's inequality**): *For every polynomial $p(z) = a_0 + a_1 z + \cdots + a_m z^m$, let*

$$\|p\| = \sup\{|p(z)| : |z| \leq 1\}.$$

If T is a contraction and p is a polynomial, then

$$\|p(T)\| \leq \|p\|.$$

PROOF: First note that by spetcral theory, if U is a unitary operator and p is a polynomial, then $\sigma(p(U)) = \{p(z) : z \in \sigma(U)\}$.

Since U is unitary, $\sigma(U) \subset \mathbb{T}$ where $\mathbb{T}$ is the unit circle. Thus $\|p(U)\| = \sup\{|p(z)| : z \in \sigma(U)\} \leq \sup\{|p(z)| : z \in \mathbb{T}\} = \|p\|$. Now by the unitary dilation theorem, $p(T) = P_{\mathcal{H}} p(U)|_{\mathcal{H}}$ which gives

$$\|p(T)\| = \|P_{\mathcal{H}} p(U)|_{\mathcal{H}}\| \leq \|p(U)\| \leq \|p\|.$$

■

The survey article by Drury [**5**] is an excellent source for more discussions on von Neumann's inequality.

6. Tuples of Commuting Contractions

Ando gave a beautiful generalization of Sz.-Nagy and Foias's theorem for two commuting contractions. The concept of dilation for a tuple of operators is similar to Definition 2.1.

DEFINITION 6.1. *Let $\mathcal{H} \subset \mathcal{K}$ be two Hilbert spaces. Suppose $\underline{T} = (T_1, T_2, \ldots, T_n)$ and $\underline{V} = (V_1, V_2, \ldots, V_n)$ are tuples of bounded operators acting on $\mathcal{H}$ and $\mathcal{K}$ respectively, i.e., $T_i \in \mathcal{B}(\mathcal{H})$ and $V_i \in \mathcal{B}(\mathcal{K})$. The operator tuple $\underline{V}$ is called a dilation of the operator tuple $\underline{T}$ if*

$$T_{i_1} T_{i_2} \ldots T_{i_k} h = P_{\mathcal{H}} V_{i_1} V_{i_2} \ldots V_{i_k} h \text{ for all } h \in \mathcal{H}, \quad k \geq 1 \text{ and all } 1 \leq i_1, i_2, \ldots, i_k \leq n.$$

If V_i are isometries with orthogonal ranges, i.e., $V_i^* V_j = \delta_{ij}$ for $1 \leq i, j \leq n$, then $\underline{V}$ is called an isometric dilation. A dilation $\underline{V}$ of T is called minimal if $\overline{span}\{V_{i_1} V_{i_2} \ldots V_{i_k} h : h \in H, k \geq 0$ and $1 \leq i_1, i_2, \ldots, i_k \leq n\} = \mathcal{K}$.

THEOREM 6.2. *For a pair $\underline{T} = (T_1, T_2)$ of commuting contractions on a Hilbert space $\mathcal{H}$, there is a commuting isometric dilation $\underline{V} = (V_1, V_2)$.*

PROOF: Let $\mathcal{H}_+ = \mathcal{H} \oplus \mathcal{H} \oplus \cdots$ be the direct sum of infinitely many copies of $\mathcal{H}$. Define two isometries W_1 and W_2 on $\mathcal{H}_+$ as follows. For $h = (h_0, h_1, h_2, \ldots) \in \mathcal{H}_+$, set

$$W_i h = (T_i h_0, D_{T_i} h_0, 0, h_1, h_2, \ldots), \text{ for } i = 1, 2.$$

Here D_{T_i} are the defect operators as defined in the proof of Theorem 2.2. Clearly, W_1 and W_2 are isometries. However, they need not commute. We shall modify them to get commuting isometries.

Let $\mathcal{H}_4 = \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$ and let v be a unitary operator on $\mathcal{H}_4$. We shall specify v later. Identify $\mathcal{H}_+$ and $\mathcal{H} \oplus \mathcal{H}_4 \oplus \mathcal{H}_4 \oplus \cdots$ by the following identification:

$$h = (h_0, h_1, h_2, \ldots) \to (h_0, \{h_1, h_2, h_3, h_4\}, \{h_5, h_6, h_7, h_8\}, \ldots).$$

Now define a unitary operator $W : \mathcal{H}_+ \to \mathcal{H}_+$ by

$$Wh = (h_0, v(h_1, h_2, h_3, h_4), v(h_5, h_6, h_7, h_8), \ldots),$$

where $h = (h_0, h_1, h_2, \ldots)$. The unitarity of W is clear since v is a unitary and

$$W^* h = W^{-1} h = (h_0, v^{-1}(h_1, h_2, h_3, h_4), v^{-1}(h_5, h_6, h_7, h_8), \ldots).$$

We define $V_1 = WW_1$ and $V_2 = W_2 W^{-1}$. These are isometries because they are products of isometries. These act on $\mathcal{H}_+$ and

$$V_i^*(h_0, 0, \ldots) = (T_i^* h_0, 0, \ldots), \text{ for } i = 1, 2.$$

Now we shall see that v can be chosen so that V_1 and V_2 commute.

To choose such a v, we first compute $V_1 V_2$ and $V_2 V_1$.

$$\begin{aligned}
& V_1 V_2(h_0, h_1, \ldots) = WW_1 W_2 v^{-1}(h_0, h_1, \ldots) \\
& = WW_1 W_2((h_0, v^{-1}(h_1, h_2, h_3, h_4), v^{-1}(h_5, h_6, h_7, h_8), \ldots) \\
& = WW_1(T_2 h_0, D_{T_2} h_0, 0, v^{-1}(h_1, h_2, h_3, h_4), v^{-1}(h_5, h_6, h_7, h_8), \ldots) \\
& = W(T_1 T_2 h_0, D_{T_1} T_2 h_0, 0, D_{T_2} h_0, 0, v^{-1}(h_1, h_2, h_3, h_4), \\
& \qquad\qquad v^{-1}(h_5, h_6, h_7, h_8), \ldots) \\
& = (T_1 T_2 h_0, v(D_{T_1} T_2 h_0, 0, D_{T_2} h_0, 0), (h_1, h_2, h_3, h_4), \\
& \qquad\qquad (h_5, h_6, h_7, h_8), \ldots)
\end{aligned}$$

and

$$\begin{aligned}
V_2 V_1(h_0, h_1, \ldots) &= W_2 W_1(h_0, h_1, \ldots) \\
&= W_2(T_1 h_0, D_{T_1} h_0, 0, h_1, h_2, \ldots) \\
&= (T_2 T_1 h_0, D_{T_2} T_1 h_0, 0, D_{T_1} h_0, 0, h_1, h_2, \ldots).
\end{aligned}$$

Since T_1 and T_2 commute, $V_1 V_2$ will be equal to $V_2 V_1$ if

$$v(D_{T_1} T_2 h, 0, D_{T_2} h, 0) = (D_{T_2} T_1 h, 0, D_{T_1} h, 0) \tag{1.6.1}$$

for all $h \in \mathcal{H}$. Now a simple calculation shows that

$$\|(D_{T_1} T_2 h, 0, D_{T_2} h, 0)\| = \|(D_{T_2} T_1 h, 0, D_{T_1} h, 0)\|, \text{ for all } h \in \mathcal{H}.$$

Hence one can define an isometry v by (1.6.1) from

$$\mathcal{L}_1 \stackrel{\text{def}}{=} \overline{\text{span}}\{(D_{T_1} T_2 h, 0, D_{T_2} h, 0) : h \in \mathcal{H}\}$$

onto $\mathcal{L}_2 \stackrel{\text{def}}{=} \overline{\text{span}}\{(D_{T_2}T_1 h, 0, D_{T_1}h, 0) : h \in \mathcal{H}\}$. To extend v to the whole of $\mathcal{H}_4$ as a unitary operator, i.e., an isometry of $\mathcal{H}_4$ onto itself, one just needs to check that $\mathcal{H}_4 \ominus \mathcal{L}_1$ and $\mathcal{H}_4 \ominus \mathcal{L}_2$ have the same dimension. If $\mathcal{H}$ is finite dimensional, this is obvious because $\mathcal{L}_1$ and $\mathcal{L}_2$ are isometric. When $\mathcal{H}$ is infinite dimensional, note that $\mathcal{L}_1^\perp$ and $\mathcal{L}_2^\perp$ have dimension at least as large as dim$\mathcal{H}$ because each of $\mathcal{L}_1^\perp$ and $\mathcal{L}_2^\perp$ contain a subspace isomorphic to $\mathcal{H}$, for example the subspace $\{(0, h, 0, 0) : h \in \mathcal{H}\}$. Thus

$$\dim \mathcal{H}_4 \geq \dim(\mathcal{H}_4 \ominus \mathcal{L}_i) \geq \dim \mathcal{H} = \dim \mathcal{H}_4 \text{ for } i = 1, 2.$$

So they have the same dimension. This completes the proof that v can be so defined that V_1 and V_2 commute. ■

THEOREM 6.3. *Let V_1 and V_2 be two commuting isometries on a Hilbert space $\mathcal{H}$. Then there is a Hilbert space $\mathcal{K}$ and two commuting unitaries U_1 and U_2 on $\mathcal{K}$ such that*

$$U_1 h = V_1 h \text{ and } U_2 h = V_2 h \text{ for all } h \in \mathcal{H}.$$

In other words, two commuting isometries can be extended to two commuting unitaries.

We remark here that simple modifications of the proof of this theorem yield that the same is true for any number (finite or infinite) of commuting isometries.

PROOF: Using Corollary 5.5, we first find a unitary extension U_1 on a Hilbert space $\tilde{\mathcal{H}}$ of the isometry V_1. Without loss of generality we may assume that this extension is *minimal*, i.e.,

$$\tilde{\mathcal{H}} = \overline{\text{span}}\{U_1^n h : n \in \mathbb{Z} \text{ and } h \in \mathcal{H}\}.$$

We want to define an isometric extension $\tilde{V}_2$ on $\tilde{\mathcal{H}}$ of V_2 which

(1) would commute with U_1,
(2) would be a unitary on $\tilde{\mathcal{H}}$ if V_2 already is a unitary.

Assume for a moment that this has been accomplished, i.e., we have found a $\tilde{V}_2$ satisfying the above conditions. Then if V_2 happens to be unitary, we are done. If not, then just repeat the construction by applying Corollary 5.5 again, this time to the isometry $\tilde{V}_2$ instead of V_1. Since U_1 is already a unitary, the resulting extensions that we shall get will be commuting unitaries.

Now we get down to finding a $\tilde{V}_2$ satisfying (1) and (2). To that end, note that

$$\begin{aligned}
&\| \sum_{n=-\infty}^{\infty} U_1^n V_2 h_n \|^2 \\
&= \sum_{n,m} \langle U_1^n V_2 h_n, U_1^m V_2 h_m \rangle \\
&= \sum_{n \geq m} \langle U_1^{n-m} V_2 h_n, V_2 h_m \rangle + \sum_{n<m} \langle V_2 h_n, U_1^{m-n} V_2 h_m \rangle \\
&= \sum_{n \geq m} \langle V_1^{n-m} V_2 h_n, V_2 h_m \rangle + \sum_{n<m} \langle V_2 h_n, V_1^{m-n} V_2 h_m \rangle \text{ as } U_1 \text{ extends } V_1 \\
&= \sum_{n \geq m} \langle V_2 V_1^{n-m} h_n, V_2 h_m \rangle + \sum_{n<m} \langle V_2 h_n, V_2 V_1^{m-n} h_m \rangle \text{ as } V_2,\ V_1 \text{ commute} \\
&= \sum_{n \geq m} \langle V_1^{n-m} h_n, h_m \rangle + \sum_{n<m} \langle h_n, V_1^{m-n} h_m \rangle \text{ as } V_2 \text{ is an isometry} \\
&= \| \sum_{n=-\infty}^{\infty} U_1^n h_n \|^2 \text{ tracing the steps back with } V_2 = \mathbf{1}_{\mathcal{H}}.
\end{aligned}$$

Thus on the dense subspace $\text{span}\{U_1^n h : n \in \mathbb{Z}, h \in \mathcal{H}\}$, one can unambiguously define an isometry $\tilde{V}_2$ by

$$\tilde{V}_2 \left(\sum_{n=-\infty}^{\infty} U_1^n h_n \right) = \sum_{n=-\infty}^{\infty} U_1^n V_2 h_n.$$

Of course, $\tilde{V}_2$ extends V_2 and commutes with U_1. Finally if V_2 is a unitary, i.e., a surjective isometry, then it follows from the definition of $\tilde{V}_2$ that it has a dense range. An isometry with a dense range is surjective and hence unitary. We have finished the proof. ∎

This theorem of course immediately produces the unitary dilation theorem for two commuting contractions:

THEOREM 6.4. *Given two commuting contractions T_1 and T_2 on a Hilbert space $\mathcal{H}$, there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and two commuting unitaries U_1 and U_2 on $\mathcal{K}$ such that*

$$T_1^m T_2^n h = P_{\mathcal{H}} U_1^m U_2^n h, \quad \textit{for all } h \in \mathcal{H} \textit{ and } m, n = 0, 1, 2, \ldots. \tag{1.6.2}$$

PROOF First get a commuting isometric dilation $\underline{V} = (V_1, V_2)$ on a space $\mathcal{H}_+$ by Theorem 6.2. Then get a commuting unitary extension $\underline{U} = (U_1, U_2)$ of $\underline{V}$ by Theorem 6.4. Now $\underline{U}$ is the required dilation because U_1 and U_2 commute and for any $h \in \mathcal{H}$ and $m, n = 0, 1, 2, \ldots$, we have

$$\begin{aligned}
P_{\mathcal{H}} U_1^m U_2^n h &= P_{\mathcal{H}} V_1^m V_2^n h \text{ because } \underline{U} \text{ extends } \underline{V} \text{ and } h \in \mathcal{H} \subset \mathcal{H}_+ \\
&= T_1^m T_2^n h \text{ because } \underline{V} \text{ dilates } \underline{T}.
\end{aligned}$$

■

The simultaneous unitary dilation theorem immediately produces a von Neumann's inequality.

Corollary 6.5. (**von Neumann's inequality**): *Let T_1 and T_2 be two commuting contractions acting on a Hilbert space $\mathcal{H}$. Suppose $p(z_1, z_2)$ is any polynomial in two variables. Then*

$$\|p(T_1, T_2)\| \leq \sup\{|p(z_1, z_2)| : |z_1| \leq 1, |z_2| \leq 1\}.$$

Proof Let $\mathcal{K} \supset \mathcal{H}$ be a unitary dilation space for $\underline{T}$ and U_1 and U_2 be two commuting unitaries on $\mathcal{K}$ as obtained from Theorem 6.4. Since U_1 and U_2 are commuting unitaries, all four of U_1, U_2, U_1^* and U_2^* commmute. Thus the C^*-algebra $\mathcal{C}$ generated by U_1 and U_2 is commutative. So by Gelfand theory ([**4**], Chapter I), $\mathcal{C}$ is isometrically $*$-isomorphic to $C(\mathcal{M}_\mathcal{C})$ where $\mathcal{M}_\mathcal{C}$ is the set of all multiplicative linear functionals χ on $\mathcal{C}$. Such functionals satisfy $\|\chi\| = \chi(1) = 1$.

Thus $|\chi(U_i)|^2 = \chi(U_i)\chi(U_i^*) = \chi(U_iU_i^*) = \chi(\mathbf{1}_\mathcal{K}) = 1$. So for any polynomial $p(z_1, z_2)$, we have

$$\begin{aligned} \|p(U_1, U_2)\| &= \sup_{\chi \in \mathcal{M}_\mathcal{C}} |\chi(p(U_1, U_2)| \\ &= \sup_{\chi \in \mathcal{M}_\mathcal{C}} |p(\chi(U_1), \chi(U_2))| \text{ as } \chi \text{ is multiplicative \& linear} \\ &\leq \sup_{|z_1|=1, |z_2|=1} |p(z_1, z_2)|. \end{aligned}$$

Hence

$$\|p(T_1, T_2)\| = \|P_\mathcal{H} p(U_1, U_2)|_\mathcal{H}\| \leq \|p(U_1, U_2)\| \leq \sup_{|z_1|=1, |z_2|=1} |p(z_1, z_2)|.$$

■

The simultaneous unitary dilation of a pair of contractions is due to Ando [**1**] and the proofs given here are essentially the same as his original ones. We shall end this section with the rather striking fact that Ando's theorem does not generalize to more than two commuting contractions. The unitary extension theorem of isometries holds good, as remarked above, for any number of commuting isometries. It is the isometric dilation of contractions which fails for more than a pair of contractions.

Perhaps the easiest way to see it is to construct a triple of commuting contractions which do not have a commuting unitary dilation. To that end, let $\mathcal{L}$ be a Hilbert space and let A_1, A_2, A_3 be three unitary operators on $\mathcal{L}$ such that

$$A_1 A_2^{-1} A_3 \neq A_3 A_2^{-1} A_1.$$

(For example, $A_2 = \mathbf{1}$ and A_1, A_3 two non-commuting unitaries will do.) Let $\mathcal{H} = \mathcal{L} \oplus \mathcal{L}$ and let $T_i \in \mathcal{B}(\mathcal{H})$ for $i = 1, 2, 3$ be defined as

$$T_i(h_1, h_2) = (0, A_i h_1) \text{ where } h_1, h_2 \in \mathcal{L}.$$

Clearly $\|T_i\| = \|A_i\| = 1$ for $i = 1, 2, 3$ and $T_iT_j = T_jT_i = 0$ for $i, j = 1, 2, 3$. SO (T_1, T_2, T_3) is a commuting triple of contractions on $\mathcal{H}$. Suppose there exist commuting unitary operators U_1, U_2, U_3 on some Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that $T_i = P_{\mathcal{H}}U_i|_{\mathcal{H}}$ for $i = 1, 2, 3$. Then

$$P_{\mathcal{H}}U_i(h, 0) = T_i(h, 0) = (0, A_ih), h \in \mathcal{H} \;\; i = 1, 2, 3.$$

Note that $\|U_i(h, 0)\| = \|h\|$ and $\|(0, A_ih)\| = \|A_ih\| = \|h\|$. So $U_i(h, 0) = (0, A_ih)$. Hence

$$\begin{aligned} U_kU_j^{-1}U_i(h, 0) &= U_kU_j^{-1}(0, A_ih) = U_kU_j^{-1}(0,\ A_j(A_j^{-1}A_ih)) \\ &= U_kU_j^{-1}U_j(0, A_j^{-1}A_ih) = U_k(0, A_j^{-1}A_ih) = (0, A_kA_j^{-1}A_ih). \end{aligned}$$

Since the U_i commute, $U_kU_j^{-1}U_i = U_iU_j^{-1}U_k$ for all $i, j = 1, 2, 3$. So $A_kA_j^{-1}A_i = A_iA_j^{-1}A_k$ for all $i, j = 1, 2, 3$. That is a contradiction. So there is no commuting dilation.

Note that von Nemann's inequality is an immediate corollary of unitary dilation. So one way to show non-existence of dilation is to show that von Neumann's inequality is violated. This is what Crabb and Davie did with a triple of operators acting on an eight-dimensional space [**3**]. The literature over the years is full of a lot of discussions and considerations of many aspects of the issue originating from this spectacular failure of von Neumann's inequality. The survey article of Drury [**5**] is very insightful, so is the monograph by Pisier [**9**]. The reader is referred to Varopoulos [**12**] for his probabilistic arguments and establishing connection with Grothedieck's inequality, and Parrott [**7**] who gave an example which satisfies von Neumann's inequality but does not have a unitary dilation.

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