Unbounded Quasinormal Operators

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Quasinormal operators ($AA^*A = A^*AA$), which were introduced by A. Brown (1953), form a class of operators which is properly larger than that of normal operators ($A^*A = AA^*$), and properly smaller than that of subnormal operators (A has a normal extension).

W. E. Kaufman's paper (1983)

- A is a *pure contraction*, if $\|Ax\| < \|x\|$ for all $x \in \mathcal{H} \setminus \{0\}$.
- Let $\Gamma(A) = A(I - A^*A)^{-1/2}$.
- Γ maps the set of all pure contraction one-to-one onto the set of all closed and densely defined operators in $\mathcal{H}$.
- (Kaufman's definition - unbounded case) A is quasinormal, if A is closed and densely defined, and $(AA^*A = A^*AA)$.
- Γ and Γ^{-1} preserve quasinormality.

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Stochel and Szafraniec's definition (1989)

- If N is a normal operator, then its spectral measure is denoted by E_N.
- A closed and densely defined operator A in $\mathcal{H}$ is *quasinormal*, if A commutes with $E_{|A|}$, i.e., $E_{|A|}A \subseteq AE_{|A|}$.
- (Stochel and Szafraniec, 1989) A closed densely defined operator A in $\mathcal{H}$ is quasinormal if and only if $U|A| \subseteq |A|U$, where $A = U|A|$ is the polar decomposition of A.

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Equivalence of the definitions

Theorem

Let A be a closed densely defined operator in $\mathcal{H}$. Then the following conditions are equivalent:

(i) $AA^*A \subseteq A^*AA$,

(ii) $AA^*A = A^*AA$,

(iii) $(I + A^*A)^{-1}A \subseteq A(I + A^*A)^{-1}$,

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Two lemmata

Lemma

If A is a closed densely defined operator in $\mathcal{H}$ and $n \in \mathbb{Z}_+$, then

(i) $(A^*A)^n \subseteq A^{*n}A^n$ *if and only if* $(A^*A)^n = A^{*n}A^n$,

(ii) $(A^*A)^n \subseteq (A^n)^*A^n$ *if and only if* $(A^*A)^n = (A^n)^*A^n$ *provided* A^n *is densely defined,*

(iii) *if* $(A^*A)^n = A^{*n}A^n$, *then* $(A^*A)^n = (A^n)^*A^n$.

Lemma

If A is a quasinormal operator in $\mathcal{H}$, then

$$(A^*A)^n = A^{*n}A^n = (A^n)^*A^n, \quad n \in \mathbb{Z}_+.$$

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If A is closed and densely defined then TCAE:

(i) *A is quasinormal,*

(ii) *$A^{*n}A^n = (A^*A)^n$ for every $n \in \mathbb{Z}_+$,*

(iii) *$\exists$ a (unique) spec. Borel meas. E on $\mathbb{R}_+$ s.t. for $n \in \mathbb{Z}_+$*

$$A^{*n}A^n = \int_{\mathbb{R}_+} x^n E(\mathrm{d}x), \tag{1}$$

(iv) *$\exists$ a (unique) spec. Borel meas. E on $\mathbb{R}_+$ s.t. (1) holds for $n \in \{1, 2, 3\}$,*

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Corollary and open problem

Corollary

If A is a quasinormal operator, then for every $n \in \mathbb{Z}_+$, A^n is a quasinormal operator.

Question

Is it true that, if A is a closed operator in $\mathcal{H}$ such that A^3 is densely defined and $(A^*A)^n = (A^n)^*A^n$ for $n \in \{2, 3\}$, then A is quasinormal?

1. There exists a quasinormal operator A such that $A^{*n} \subsetneq (A^n)^*$ for every $n \geqslant 2$.

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An operator A in $\mathcal{H}$ is said to be *paranormal* if $\|Af\|^2 \leq \|A^2 f\|\|f\|$ for all $f \in \mathcal{D}(A^2)$.

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Let A be a closed densely defined operator in $\mathcal{H}$ such that

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Then the following assertions hold:

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Quasinormality by means of absolute continuity

Theorem

Let A be a closed densely defined operator in $\mathcal{H}$ and E be the spectral measure of $|A|$. Then FCAE:

(i) $E_{|A|}A \subseteq AE_{|A|}$ *(A is quasinormal),*

(ii) $\langle E(\sigma)Af, Af\rangle = \langle E(\sigma)|A|f, |A|f\rangle$ *for all* $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ *and* $f \in \mathcal{D}(A)$,

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Weakly quasinormal operators

- We say that a closed densely defined operator A in a complex Hilbert space $\mathcal{H}$ is *weakly quasinormal* if there exists $c \in \mathbb{R}_+$ such that

$$\langle E(\sigma)|A|f, |A|f\rangle \leqslant c\langle E(\sigma)Af, Af\rangle, \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\ f \in \mathcal{D}(A),$$

  or equivalently: for every $f \in \mathcal{D}(A)$, $\langle E(\cdot)|A|f, |A|f\rangle \ll \langle E(\cdot)Af, Af\rangle$ and $\mathrm{d}\langle E(\cdot)|A|f, |A|f\rangle / \mathrm{d}\langle E(\cdot)Af, Af\rangle \leqslant c$ almost everywhere with respect to $\langle E(\cdot)Af, Af\rangle$).
- The smallest such c will be denoted by $\mathfrak{c}_A$.
- The constant $\mathfrak{c}_A$ is always greater than or equal to 1 whenever the operator A is nonzero.

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Characterization

Theorem

Let A be a closed densely defined operator in $\mathcal{H}$ and let $c \in \mathbb{R}_+$. Then the following two conditions are equivalent:

(i) *A is weakly quasinormal with $\mathfrak{c}_A \leqslant c$,*

(ii) *there exists $T \in \boldsymbol{B}(\mathcal{H})$ such that*

$$TA = |A|,\ T|A| \subseteq |A|T \text{ and } \|T\| \leqslant \sqrt{c}.$$

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(ii) *A is weakly quasinormal with $\mathfrak{c}_A = 1$.*

A weighted shifts on a directed trees

- Let $\mathscr{T} = (V, E)$ be a directed tree.
- Let $\ell^2(V)$ be the space of all square summable function on V with a scalar products

$$\langle f, g \rangle = \sum_{u \in V} f(u)\overline{g(u)}, \quad f, g \in \ell^2(V).$$

- For $u \in V$, let us define $e_u \in \ell^2(V)$ by

$$e_u(v) = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases}$$

- $\{e_u\}_{u \in V}$ is an orthonormal basis in $\ell^2(V)$.

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- For a family $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$ let us define an operator $S_{\boldsymbol{\lambda}}$ in $\ell^2(V)$ by

$$\mathcal{D}(S_{\boldsymbol{\lambda}}) = \{f \in \ell^2(V) \colon \Lambda_{\mathscr{T}} f \in \ell^2(V)\},$$
$$S_{\boldsymbol{\lambda}} f = \Lambda_{\mathscr{T}} f, \quad f \in \mathcal{D}(S_{\boldsymbol{\lambda}}),$$

- where $\Lambda_{\mathscr{T}}$ is define on functions $f \colon V \to \mathbb{C}$ by

$$(\Lambda_{\mathscr{T}} f)(v) = \begin{cases} \lambda_v \cdot f(\operatorname{par}(v)) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \operatorname{root}. \end{cases}$$

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Characterization

Theorem

Let $S_{\boldsymbol\lambda}$ be a densely defined weigh. shift on a dir. tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$ and E be the spec. meas. of $|S_{\boldsymbol\lambda}|$. Then

(i) *For any $c \in \mathbb{R}_+$, $S_{\boldsymbol\lambda}$ is weakly quasinormal with $\mathfrak{c}_{S_{\boldsymbol\lambda}} \leqslant c$ iff*

$$\|S_{\boldsymbol\lambda} e_u\|^2 \leqslant c \sum_{v\in \mathrm{Chi}_{\mathrm{e}}(u)} |\lambda_v|^2, \quad u \in V,$$

where $\mathrm{Chi}_{\mathrm{e}}(u) := \{v \in \mathrm{Chi}(u)\colon \|S_{\boldsymbol\lambda} e_v\| = \|S_{\boldsymbol\lambda} e_u\|\}$.

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Quasinormal weighted shifts on directed trees

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Moreover, if $V^\circ \neq \varnothing$ and $\lambda_v \neq 0$ for all $v \in V^\circ$, then $S_{\boldsymbol\lambda}$ is quasinormal if and only if $\|S_{\boldsymbol\lambda}\|^{-1} S_{\boldsymbol\lambda}$ is an isometry.

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Examples

(i) There exists an injective quasinormal weighted shift on a directed binary tree whose restriction to $\ell^2(\mathrm{Des}(u))$ is unbounded for every $u \in V$,

(ii) for every $c \in (1, \infty)$, there exists an injective weighted shift S_λ on a directed tree such that $\mathfrak{c}_{S_\lambda} = c$

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Remarks

- Let q be a positive real number. Following S. Ota, we say that a closed densely defined operator A in a complex Hilbert space $\mathcal{H}$ is *q-quasinormal* if $U|A| \subseteq \sqrt{q}\,|A|U$, where $A = U|A|$ is the polar decomposition of A (or equivalently if and only if $U|A| = \sqrt{q}\,|A|U$).
- It was proved by Ota, that a closed densely defined operator A in $\mathcal{H}$ is q-quasinormal if and only if

$$UE(\sigma) = E(\psi_q^{-1}(\sigma))U, \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

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A generalization

Let $E(\varphi^{-1}(\cdot))$ stands for the spectral measure
$\mathfrak{B}(\mathbb{R}_+) \ni \sigma \longmapsto E(\varphi^{-1}(\sigma)) \in \boldsymbol{B}(\mathcal{H})$.

Proposition

Let A be a closed densely defined operator in $\mathcal{H}$, $A = U|A|$ be its polar decomposition and E be the spectral measure of $|A|$. Suppose ϕ and ψ are Borel functions from $\mathbb{R}_+$ to $\mathbb{R}_+$. Then the following conditions are equivalent:

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Composition operators

- Let μ be a σ-finite measure and let $L^2(\mu)$ be the Hilbert space of all square integrable complex functions on X with the standard inner product.
- Let $\phi \colon X \to X$ be an $\mathscr{A}$-*measurable* transformation of X, i.e., $\phi^{-1}(\Delta) \in \mathscr{A}$ for all $\Delta \in \mathscr{A}$.
- Denote by $\mu \circ \phi^{-1}$ the measure on $\mathscr{A}$ given by $\mu \circ \phi^{-1}(\Delta) = \mu(\phi^{-1}(\Delta))$ for $\Delta \in \mathscr{A}$.
- We say that ϕ is *nonsingular* if $\mu \circ \phi^{-1}$ is absolutely continuous with respect to μ.

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If ϕ is a nonsingular transformation of X, then the map $C_\phi\colon L^2(\mu) \supseteq \mathcal{D}(C_\phi) \to L^2(\mu)$ given by

$$\mathcal{D}(C_\phi) = \{f \in L^2(\mu)\colon f \circ \phi \in L^2(\mu)\}$$
$$C_\phi f = f \circ \phi \text{ for } f \in \mathcal{D}(C_\phi),$$

is well-defined. We call such C_ϕ a *composition operator*.

Composition operators

1. If ϕ is nonsingular, then by the Radon-Nikodym theorem there exists a unique (up to sets of measure zero) $\mathscr{A}$-measurable function $\mathsf{h}_\phi \colon X \to \overline{\mathbb{R}}_+$ such that

$$\mu \circ \phi^{-1}(\varDelta) = \int_\varDelta \mathsf{h}_\phi \mathrm{d}\mu, \quad \varDelta \in \mathscr{A}.$$

2. Given $n \in \mathbb{N}$, we denote by ϕ^n the n-fold composition of ϕ with itself; ϕ^0 is the identity transformation of X.
3. If ϕ is nonsingular and $n \in \mathbb{Z}_+$, then ϕ^n is nonsingular and thus h_{ϕ^n} makes sense.

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A conditional expectation

- Suppose that $\phi\colon X \to X$ is a nonsingular transformation such that $\mathsf{h}_\phi < \infty$ a.e. $[\mu]$. Then the measure $\mu|_{\phi^{-1}(\mathscr{A})}$ is σ-finite.
- Hence, by the Radon-Nikodym theorem, for every $\mathscr{A}$-measurable function $f\colon X \to \overline{\mathbb{R}}_+$ there exists a unique (up to sets of measure zero) $\phi^{-1}(\mathscr{A})$-measurable function $\mathsf{E}(f)\colon X \to \overline{\mathbb{R}}_+$ such that

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We call $\mathsf{E}(f)$ the *conditional expectation* of f with respect to $\phi^{-1}(\mathscr{A})$.

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