COMUTATION THEOREMS & SPLITTING THEOREMS IN VON NEUMANN ALGEBRAS

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① VON NEUMANN ALGEBRA $1_H \in M \subset B(H)$ $*$-SUBALGEBRA, (WO)SO-CLOSED
CENTER $Z(M) = M \cap M'$; M FACTOR : $Z(M) = \mathbb{C} \cdot 1_M$ $\quad M'' = M$
EQUIVALENCE OF PROJECTIONS $e, f \in M$: $e \sim f$ $(e \prec f)$
$(\exists)\ v \in M$, $v^* v = e$, $vv^* = f$ $(vv^* \leq f)$
COMPARISON THEOREM : $(\exists)$ $p \in Z(M)$, projection
$ep \prec fp \quad , \quad f(1-p) \prec e(1-p)$
M FACTOR : EITHER $e \prec f$, OR $f \prec e$

J. VON NEUMANN'S REDUCTION THEORY $M = \int^{\oplus} M(t)\, d\mu(t)$
REQUIRES SEPARABILITY OF M_* (H) $\quad M(t)$ factors

ALGEBRAIC REDUCTION THEORY (NO SEPARABILITY ASSUMPTIONS)
GLIMM : Ω GELFAND SPECTRUM OF $Z(M)$
$\Omega \ni t \mapsto \mathfrak{M}_t \subset M$ CLOSED 2-SIDED IDEAL GEN. BY t

$\Omega \ni t \mapsto M_t = M / \mathfrak{M}_t$
SAKAI : M FINITE $\Rightarrow$ M_t FACTORS
S-Z (1971) : M SEMIFINITE $\Rightarrow$ SOME NATURAL COMPLETIONS $\widetilde{M_t}$ ARE FACTORS

② DIXMIER MAPS
$\mathcal{F}$ = convex hull $\{M \ni x \mapsto uxu^* ; u \in M \text{ unitary}\} \subset B(M)$
$\mathcal{T} = \overline{\mathcal{F}}^{w/p}$ w-point closure in $B(M)$

DIXMIER SETS
$\mathcal{K}(x) = \overline{\mathcal{F}(x)}^{NORM} \cap Z(M) \neq \phi$ (DIXMIER'S THEOREM)
$\widehat{\mathcal{C}}(x) = \overline{\mathcal{F}(x)}^{w} \cap Z(M) = \mathcal{T}(x) \cap Z(M)$

M FINITE FACTOR $\Rightarrow$ $\mathcal{K}(x) = \{\tau(x)\}$, τ trace, $\tau(1) = 1$

M INFINITE FACTOR: ESSENTIAL NUMERICAL RANGES (CONWAY, 1970)

$$\mathcal{K}(x) = \overline{V}_J(x) := \{\varphi(x) \,;\, \varphi \in S(M),\ \varphi|_J = 0\}$$

$$\mathcal{C}(x) = \overline{V}_I(x) := \{\varphi(x) \,;\, \varphi \in S(M),\ \varphi|_I = 0\}$$

I = CLOSED 2-SIDED IDEAL GENERATED BY FINITE PROJECTION

J = —"— PROJECTIONS $e \nsim 1$

M PROPERLY INFINITE VON NEUMANN ALGEBRA (SZ 1972)

$$\Sigma(M) = \{\Phi : M \to Z(M) \,;\, \text{COND. EXPECTATIONS}\}$$

$$\mathcal{K}(x) = \overline{V}_J(x) := \{\Phi(x) \,;\, \Phi \in \Sigma,\ \Phi|_J = 0\}$$

$$\mathcal{C}(x) = \overline{V}_I(x) := \{\Phi(x) \,;\, \Phi \in \Sigma,\ \Phi|_I = 0\}$$

M SEMIFINITE : <u>NO</u> $0 \neq \Phi \in \Sigma$, $\Phi|_I = 0$ IN NORMAL

M TYPE $\underline{III}$ i.e. $I = 0$: $(\exists)$ ENOUGH MANY NORMAL $\Phi \in \Sigma$

IMPLICIT CONSEQUENCE (EXPLICIT IN SZ 1972):

- <u>APPROXIMATION THEOREM</u>: M PROPERLY INFINITE $\Rightarrow$ ANY CONDITIONAL EXPECTATION $\Phi : M \to Z(M)$ s.t. $\Phi|_I = 0$ CAN BE W-POINT APPROXIMATED BY DIXMIER MAPS.

- M TYPE $\underline{III}$: <u>ANY</u> CONDITIONAL EXPECTATION $\Phi : M \to Z(M)$ CAN BE W-POINT APPROXIMATED WITH DIXMIER MAPS
$$x \longmapsto \sum_{k=1}^{n} \alpha_k u_k x u_k^*$$

- M FACTOR OF TYPE $\underline{III}$: ANY STATE $\varphi : M \to \mathbb{C}\cdot 1 \subset M$ CAN BE W-POINT APROXIMATED BY DIXMIER MAPS

THE KEY ROLE IS PLAYED HERE BY THE LESS USUAL DIXMIER SET $\mathcal{C}(x)$ SINCE

$\mathcal{C}(x) \rightsquigarrow$ finiteness of projections, while

$\mathcal{K}(x) \rightsquigarrow$ non equivalence to 1 OF PROJECTIONS

③ CLASSICAL ELEMENTARY RESULT: $(M_n \approx Mat_n(\mathbb{C}))$

$$M_n \subset M \Rightarrow M = Mat_n(M_n' \cap M) = M_n \otimes (M_n' \cap M)$$

CONNES - TAKESAKI (1975 CONSISTENT USE OF MODULAR THEORY)

M, R FACTORS, A ABELIAN, $M \otimes 1_A \subset R \subset M \otimes A \Rightarrow R = M \otimes 1_A$

THEOREM (GE-KADISON, INVENT. MATH 1996) M FACTOR

$\otimes$ $\quad M \otimes 1_H \subset R \subset M \bar{\otimes} B(H) \Longrightarrow R = M \bar{\otimes} N$ i.e. R IS SPLIT

OBVIOUSLY NOT TRUE FOR M A NON-FACTOR:

$$M = M_1 \oplus M_2, \; R = (M_1 \bar{\otimes} N_1) \oplus (M_2 \bar{\otimes} N_2), \; N_1 \neq N_2$$

QUESTIONS:

1°. WHAT REMAINS TRUE IF $Z(M)$ IS ARBITRARY?

2°. UNDER WHAT EXTRA CONDITIONS $\otimes$ REMAINS TRUE WITH $Z(M)$ NON TRIVIAL?

WHAT IS N IN $\otimes$?

$$N \equiv 1_M \otimes N = R \cap (Z(M) \bar{\otimes} B(H)) = R \cap (M \otimes 1_H)'$$

ANSWERS:

1° • THEOREM (S-Z) $M \otimes 1_H \subset R \subset M \bar{\otimes} B(H) \Longrightarrow$

$$\Longrightarrow R = (M \otimes 1_H) \vee (R \cap (M \otimes 1_H)') \qquad \text{„}R \text{ is SPLIT"}$$

ACTUALLY $R = (M \otimes 1_H) \underset{Z(M)}{\bar{\otimes}} (R \cap (M \otimes 1_H)')$ TENSOR PRODUCT OVER $Z(M)$

As $Z(M) \otimes 1 \subset R \cap (Z(M) \bar{\otimes} B(H)) \subset Z(M) \bar{\otimes} B(H)$ WE GET A PARTIAL ANSWER TO QUESTION 2°:

• COROLLARY If $M \otimes 1_H \subset R \subset M \bar{\otimes} B(H)$ THEN

$$R \text{ IS SPLIT} \Longleftrightarrow R \cap (Z(M) \otimes B(H)) \text{ IS SPLIT}$$

i.e. A COMPLETE REDUCTION OF QUESTION 2° TO THE COMMUTATIVE CASE FOR M.

2° $\mathcal{Z}\otimes 1_H \subset R \subset \mathcal{Z}\bar{\otimes} B(H) \overset{?}{\underset{\text{CONDITIONS}}{\Longrightarrow}} R = \mathcal{Z}\bar{\otimes} N$ ($\mathcal{Z}$ COMMUTATIVE)

NECESSARY CONDITION: $(\alpha\otimes id)(R)\subset R$, $(\forall)\, \alpha\in Aut(\mathcal{Z})$

NOT SUFFICIENT: ——— ..

A CERTAIN „HOMOGENEITY" IS NECESSARY

D. MAHARAM – PROC. NAS, 1942

- THEOREM (S-Z) IF $M\otimes 1_H \subset R \subset M\bar{\otimes} B(H)$ AND $\mathcal{Z}(M)$ IS LOCALLY HOMOGENEOUS, $(\alpha\bar{\otimes} id)(R\cap(\mathcal{Z}(M)\bar{\otimes} B(H))) \subset R\cap(\mathcal{Z}(M)\bar{\otimes} B(H))$, $(\forall)\, \alpha\in Aut(\mathcal{Z}(M))$ THEN

$$R = M\bar{\otimes} N \quad (R \text{ IS SPLIT})$$

THE BASIC RESULT USED FOR THE PROOF IS

- <u>PROPOSITION (S-Z)</u> FOR G A LOCALLY COMPACT GROUP:

$$\left.\begin{array}{l} L^\infty(G)\otimes 1_H \subset R \subset L^\infty(G)\bar{\otimes} B(H) \\ (Ad(\lambda_g)\otimes id)(R)\subset R,\ (\forall)\, g\in G \end{array}\right] \Rightarrow R = L^\infty(G)\bar{\otimes} N$$

PROOF. $\alpha : G \longrightarrow Aut(R)$, $\alpha_g = Ad(\lambda_g)\bar{\otimes} id$

$$v : L^\infty(G) \longrightarrow R \quad , \quad v(f) = f\otimes 1$$

$$\alpha_g(v(f)) = v(Ad(\lambda_g) f)$$

By the LANDSTAD TYPE THEOREM FOR CROSSED PRODUCTS BY GROUP DUALS (S-VOICULESCU-Z, 1976):

$$R = (L^\infty(G)\otimes 1_H) \vee R^\alpha \quad (R^\alpha \text{ fixed pts of } \alpha)$$

$$R^\alpha \subset (L^\infty(G)\bar{\otimes} B(H)) \cap \{\lambda_g\otimes 1_H \,;\, g\in G\}'$$
$$\subset 1_G\otimes B(H).$$

④ FOR THE PROOF CONSIDER FIRST THE GE-KADISON CASE:

M FACTOR, $M\otimes 1_H \subset R \subset M\bar{\otimes} B(H) \implies R = M\bar{\otimes} N$

- OUR FIRST REMARK: WE MAY ASSUME R OF TYPE III:

  $(\exists)$ TYPE III FACTOR, $P\bar{\otimes} M\otimes 1_H \subset P\bar{\otimes} R \subset P\bar{\otimes} M \bar{\otimes} B(H$

  $P\otimes M$ IS TYPE III: $\cancel{P}\otimes R = \cancel{P}\otimes M\otimes N$ $\left(\varphi\otimes id_{M\otimes B(H)}\right)$

- WHAT IS N? $N \equiv 1_M\otimes N = \{(\varphi\otimes id)(R);\ \varphi\in S(M)\text{ normal}\}''$

  CHECK THAT $R = M\bar{\otimes} N$:

$R\subset M\bar{\otimes} N \iff (M\otimes N)' \subset R' \iff M'\bar{\otimes} N' \subset R'$ BY TOMITA's COMMUTATION THM

$<$ $M'\otimes 1_H \subset R' \iff R\subset M\bar{\otimes} B(H)$ ASSUMED

$1_M\otimes N' \subset R'$ IS EASY: $(\forall)\, X\in R,\ y'\in N',\ \varphi\in M_*$:

$\cancel{(\varphi\otimes id)}_{\psi}(X(1\otimes y')) = (\varphi\otimes id)(X)\cdot(1\otimes y')$

$= (1\otimes y')(\varphi\otimes id)(X) = \cancel{(\varphi\otimes id)}_{\psi}((1\otimes y')X)$

$R\supset M\bar{\otimes} N$ $<$ $R\supset M\otimes 1_H$ ASSUMED

$R\supset 1_M\otimes N$ i.e. $R\supset(\varphi\otimes id)(R)$

$(\forall)\ \varphi\in S(M)$ NORMAL

INDEED, BY THE APPROXIMATION THEOREM

$\varphi = w_p$-limit of convex$(x\mapsto uxu^*)$, $u\in M$

$\varphi\otimes id = w_p$-limit of convex$(X\mapsto (u\otimes 1)X(u^*\otimes 1))$

$M\otimes 1\subset R$ HENCE $\underset{R}{\cap} \implies \underset{R}{\cap}$

In THE GENERAL CASE:

WE HAVE THE APPROXIMATION THEOREM

WE NEED A CERTAIN COMMUTATION THEOREM

⑤ USUAL TENSOR PRODUCT $M = M_1 \bar{\otimes} M_2$:

$M = M_1 \vee M_2$ AND $(\exists)$ N_0 TYPE I FACTOR $< \begin{matrix} M_1 \subset N_0 \\ M_2 \subset N_0' \end{matrix}$

EXISTENCE, UNIQUENESS... $N_0 = B(H_1) \otimes 1_{H_2}$, $N_0' = 1_{H_1} \otimes B(H_2)$

THEN THE TOMITA COMMUTATION THEOREM READS:

$$(M_1 \vee M_2)' = (M_1' \cap N_0) \vee (M_2' \cap N_0')$$

TENSOR PRODUCT OVER Z COMMUTATIVE:

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, hook] & M_1 \arrow[d, hook] \\ M_2 \arrow[r, hook] & M \end{tikzcd}$$

$M_1 \underset{Z}{\bar{\otimes}} M_2$: $M = M_1 \vee M_2$ AND

$(\exists)$ TYPE I vN ALGEBRA WITH $Z(N_0) = Z$ SUCH THAT

$$M_1 \subset N_0 \;,\; M_2 \subset N_0'$$

EXISTENCE (MODEL): $M_1 \subset B(K_0) \bar{\otimes} Z \otimes 1_{K_0} = N_0$

(REPRESENT Z MAXIMAL ABELIAN) $M_2 \subset 1_{K_0} \bar{\otimes} Z \otimes B(K_0) = N_0'$

$$M = M_1 \vee M_2$$

UNIQUENESS: π *-ISO $\Rightarrow \cdots \pi \otimes id$ SPATIAL ISO

$$Z(e) = Z(f) \Rightarrow \cdots \; e \otimes 1 \sim f \otimes 1$$

- THEOREM (COMMUTATION THEOREM) (S-Z)

$$(M_1 \vee M_2)' = (M_1' \cap N_0) \vee (M_2' \cap N_0')$$

ACTUALLY: $\underset{Z}{\otimes}$ $\qquad \underset{Z}{\otimes}$

WITH Z MAXIMAL ABELIAN ($Z = Z'$) THIS MEANS:

IF $1_{M_1} \otimes Z \subset M_1 \subset B(H_1) \bar{\otimes} Z$, $\quad M_2 \subset Z \bar{\otimes} B(H_2)$, THEN

- $[(M_1 \otimes 1_{H_2}) \vee (1_{H_1} \otimes M_2)]' = (M_1' \otimes 1_{H_2}) \vee (1_{H_1} \otimes M_2')$

THE PROOF OF THE COMMUTATION THEOREM:

- IN THE SEPARABLE CASE ($N_0 \subset B(H)$, H SEPARABLE) IT IS REDUCED TO THE TOMITA COMMUTATION THEOREM BY A ROUTINE APPLICATION OF VON NEUMANN REDUCTION THEORY

IN THE GENERAL CASE, A SIMILAR APPROACH USING OUR „ALGEBRAIC REDUCTION" FAILED

- WE PROVED IT BY EXTENDING THE RIEFFEL-VAN DAELE „REAL HILBERT SPACE" PROOF OF TOMITA'S THEOREM
... HILBERT MODULES OVER REAL PARTS OF OPERATOR ALGEBRAS ...

IT APPEARS THAT BOTH OUR $\begin{cases}\text{SPLITTING THEOREM} \\ \text{COMMUTATION THEOREM FOR } \overset{\otimes}{\scriptstyle Z}\end{cases}$

- ARE JUST PARTICULAR CASES OF A GENERAL COMMUTATION THEOREM

6 $N, N_1, N_2 \subset B(H)$ von NEUMANN ALGEBRAS $\leftarrow$ commute, same center, N_1, N_2 type I

$$M_1 \subset N \vee N_1 \quad , M_2 \subset N \vee N_2$$

(1) $(N \vee M_1 \vee M_2)' = (M_1' \cap N_1) \vee (M_2' \cap N_2) \vee (N \vee N_1 \vee N_2)'$

(2) $(M_1 \vee M_2)' \cap (N_1 \vee N_2) = (M_1' \cap N_1) \vee (M_2' \cap N_2)$

CONSEQUENCES

- 1°. $(N = Z)$ THE COMMUTATION THEOREM FOR $\underset{Z}{\otimes}$
- 2°. $(N_1 = N_0, M_1 = M, M_2 = N_2 = Z)$ DUAL SPLITTING THEOREM:

$N, N_0 \subset B(H)$ $\begin{cases}\text{COMMUTE} \\ \text{SAME CENTER } Z, \\ N_0 \text{ TYPE I}\end{cases}$ $N \subset M \subset N \vee N_0 \Rightarrow$

$$\Rightarrow M' = (M' \cap N_0) \vee (N \vee N_0)'$$

$$M' \cap N_0 = M' \cap (N \vee N_0) = (M \cap N_0)' \cap N_0$$

3°. (BY APPLYING 2° TO THE COMMUTANTS) SPLITTING THEOREM

SAME ASSUMPTIONS AS ABOVE $\Rightarrow$

$$M = N \vee (M \cap N_0)$$

$$M \cap N_0 = N' \cap M = \Phi(M) \quad (\Phi : N \vee N_0 \to N_0)$$

$$Z(M) = Z(M \cap N_0) \supset Z$$

ACTUALLY: $M = N \underset{Z}{\bar{\otimes}} (N' \cap M)$

⑦ INVARIANCE THEOREM FOR CONDITIONAL EXPECTATIONS

$$N, N_0 \subset B(H) \leftarrow \begin{array}{l} \text{commute} \\ \text{same center } Z \\ N_0 \text{ type I} \end{array}$$

(i) $\underset{N \vee N_0 \to N_0}{\Phi} \longmapsto \underset{N \to Z}{\Phi|_N}$ 1-1 CORESPONDENCE O NORMAL COND. EXPECTATIONS

(ii) $N \subset M \subset N \vee N_0 \Rightarrow \Phi(M) = M \cap N_0 = N' \cap M$

$(\forall)\ \Phi : N \vee N_0 \to N_0$ NORMAL C.E.

GENERAL SCHEME OF THE PROOF:

$$\begin{tikzcd}
\text{APPROXIMATION THEOREM} \arrow[d] & \\
\text{INVARIANCE THEOREM} \arrow[dr] & \text{COMMUTATION THEOREM (COMMUTATIVE CASE)} \arrow[d] \\
& \text{SPLITTING THEOREM} \arrow[d] \arrow[dd, bend left] \\
& \text{DUAL SPLITTING THEOREM} \arrow[d] \\
& \text{GENERAL COMMUTATION THEOREN}
\end{tikzcd}$$

8. TENSOR PRODUCT OVER SUBALGEBRAS

$$M = M_1 \underset{N}{\bar{\otimes}} M_2 \qquad \begin{array}{ccc} Z = Z(N) \subset N & \hookrightarrow & M_1 \\ \downarrow & & \downarrow \\ M_2 & \hookrightarrow & M \end{array}$$

MEANS:

1°. $M = M_1 \vee M_2$

2° $(\exists)\ N_1, N_2 \subset N'$ — COMMUTE, $Z(N_1) = Z(N_2) = Z$ s.t. $M_1 \subset N \vee N_1$, $M_2 \subset N \vee N_2$; N_1, N_2 TYPE I

BY THE SPLITTING THEOREM THIS IMPLIES

$$M_1 = N \underset{Z}{\bar{\otimes}} (N' \cap M_1) \qquad M_2 = N \underset{Z}{\bar{\otimes}} (N' \cap M_2)$$

$$M = N \vee (N' \cap M_1) \vee (N' \cap M_2)$$

IF N IS A FACTOR THIS MEANS

$$M_1 = N \bar{\otimes} P_1\ ,\ M_2 = N \bar{\otimes} P_2 \text{ AND } M_1 \underset{N}{\bar{\otimes}} M_2 = N \bar{\otimes} P_1 \bar{\otimes} P_2$$

PROPERLY INTERPRETED THE GENERAL COMMUTATION THEOREM SAYS:

- $$(R_1 \overline{\otimes}_{R} R_2)' = R_1' \overline{\otimes}_{R'} R_2'$$

NAMELY:

N, N_1, N_2 — COMMUTE, SAME CENTER, N_1, N_2 TYPE I

$$N \subset M_1 \subset N \vee N_1$$
$$N \subset M_2 \subset N \vee N_2$$

- $$(M_1 \overline{\otimes}_{N} M_2)' = (M_1 \vee N_2)' \overline{\otimes}_{(N \vee N_1 \vee N_2)'} (M_2 \vee N_1)'$$

MODEL:

$$N = 1_{H_1} \otimes R \otimes 1_{H_2}$$
$$N_1 = B(H_1) \overline{\otimes} Z(R) \otimes 1_{H_2} \qquad (N \vee N_1 \vee N_2)' = 1_{H_1} \otimes R' \otimes 1_{H_2}$$
$$N_2 = 1_{H_1} \otimes Z(R) \overline{\otimes} B(H_2)$$

$$1_{H_1} \otimes R \subset R_1 \subset B(H_1) \overline{\otimes} R, \quad M_1 = R_1 \otimes 1_{H_2}$$
$$R \otimes 1_{H_2} \subset R_2 \subset R \overline{\otimes} B(H_2), \quad M_2 = 1_{H_1} \otimes R_2$$

- $$[(R_1 \otimes 1_{H_2}) \vee (1_{H_1} \otimes R_2)]' = (R_1' \otimes 1_{H_2}) \vee (1_{H_1} \otimes R_2')$$

(9) • APPLICATION: PARTIAL ANSWER TO A PROBLEM OF S. POPA (1983)

$P_1 \subset P_2 \subset B(H)$, H SEPARABLE

P_1 MAXIMAL INJECTIVE IN P_2

1° $\Downarrow$ REDUCTION THEORY

$Z \otimes P_1$ MAXIMAL INJECTIVE IN $Z \otimes P_2$, $(\forall)$ Z COMMUTATIVE

2° $\Downarrow$ USING THE SPLITTING THEOREM

• $R \otimes P_1$ MAXIMAL INJECTIVE IN $R \otimes P_2$, $(\forall)$ R INJECTIVE

1° $Z \otimes B(H) = L^\infty(\Omega, \mu) \otimes B(H) \equiv L^\infty(\Omega, \mu; B(H))$

$Z \otimes 1_H \subset Z \bar{\otimes} P_1 \subset M \subset Z \bar{\otimes} P_2 \subset Z \otimes B(H)$ $\quad \omega \mapsto P_1 \subset M(\omega) \subset P_2 \subset B(H)$

M INJECTIVE $\xRightarrow{\text{CONNES}}$ $M(\omega)$ INJECTIVE μ-a.e.

....

2° $R \bar{\otimes} P_1 \subset M \subset R \otimes P_2 \xRightarrow{\text{SPLIT. THM}} M = (R \otimes 1_H) \vee (M \cap (Z(R) \otimes B(H)))$

$\Phi : R \to Z(R)$ $\quad (\Phi \bar{\otimes} id)(M)$ HENCE INJECTIVE

$$\begin{array}{ccccc} & & \shortparallel & & \\ Z(R) \otimes P_1 & \subset & M \cap (Z(R) \bar{\otimes} B(H)) & \subset & Z(R) \bar{\otimes} P_2 \\ \text{MAX. INJECTIVE} & = & & & \end{array}$$

THEREFORE $M = R \bar{\otimes} P_1$.

THE CASE R INJECTIVE <u>FACTOR</u> WAS CONSIDERED IN GE-KADISON

(10) EXTENSION FOR C^*-ALGEBRAS: A UNITAL SIMPLE NUCLEAR

$A \otimes 1 \subset B \subset A \otimes_{min} D \Rightarrow B = A \otimes_{min} C$

L. ZSIDÓ, J. ZACHARIAS INDEPENDENT DIFFERENT PROOFS