A RESULT ON NIJENHUIS OPERATOR

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Some History

- In 1984 Magri and Morosi worked on Poisson Nijenhuis manifolds via deformation of Hamiltonian system, Which was again studied by Kosmann and Magri in the year 1990.
- In 1990 T.J. Courant introduced Dirac structures[2]and around at same time I. Dorfman, one student of Gelfand independently studied Dirac structure in a different set up[3].
- Dorfman studied Nijenhuis operator via deformation of Lie algebra[4]. Introduction of Dirac structures by her gave new interpretations to the already existing Nijenhuis set ups.
- In 2004 Gallardo and Nunes da Costa introduced Dirac Nijenhuis structures[1].
- Guang and Kang separately developed Dirac Nijenhuis manifolds in 2004 [11].
- In 2011 Kosmann-Schwarzbach studied Dirac Nijenhuis structures on Courant algebroid[8].

Aim of this talk is

- To construct Nijenhuis operator on $\Gamma(TM \oplus T^*M)$ in the same sense Irene Dorfman has constructed Nijenhuis operator on $\Gamma(TM)$.
- To study the deformation of Dirac structures on $\Gamma(TM \oplus T^*M)$.
- To define Nijenhuis relation on the new set up.

Some pre-requirements

- Let M be a differentiable manifold. TM be its Tangent bundle.
- $\Gamma(TM)$, the space of section of TM is endowed with a bilinear operation on it, i.e. $[,] : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ defined by $[X, Y] = XY - YX, \quad X, Y \in \Gamma(TM)$, which is known as Lie bracket (Named in the honour of Sophus Lie).
- $(\Gamma(TM), [,])$ is a Lie algebra with the bracket operation on it.
- A vector space G with a bilinear operation on it $[,] : G \times G \to G$ is said to be a Lie algebra if the bracket satisfies two properties, i.e.
  1. $[X, Y]$ is skew symmetric.
  2. $[X, Y]$ satisfies Jacobi identity property, i.e.

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

Dorfman's Construction

Dorfman has started her calculation choosing a deformed bracket on $\Gamma(TM)$ with a parameter μ, i.e. $[X,Y]_\mu = [X,Y] + \mu\omega(X,Y)$, where ω is a bilinear map on $\Gamma(TM)$, i.e.

$$\omega : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM).$$

The given Lie algebra has a deformation w.r.t. ω if $[X,Y]_\mu$ has a Lie bracket structures. To have a Lie bracket structure we must have

1. ω is skew symmtric, i.e. $\omega(X,Y) = -\omega(Y,X)$.
2. ω must satisfy Jacobi identity, i.e.

$$\omega(\omega(X,Y),Z) + c.p. = 0.$$

Let $N : \Gamma(TM) \to \Gamma(TM)$ be an endomorphism on $\Gamma(TM)$.

For a fixed $N : \Gamma(TM) \to \Gamma(TM)$ the deformation of Lie algebra is said to be trivial if for $T_\mu = id + \mu N$ the following condition hold:

$$T_\mu[X,Y]_\mu = [T_\mu X, T_\mu Y]. \tag{1}$$

Now expanding L.H.S. and R.H.S. both and comparing them we get

$$\omega(X,Y) = [X, N(Y)] + [N(X), Y] - N[X,Y] \tag{2}$$

$$N\omega(X,Y) = [N(X), N(Y)] \tag{3}$$

From the above two equations $(2), (3)$ we have

$$[NX, NY] - N[NX, Y] - N[X, NY] + (N)^2[X,Y] = 0. \tag{4}$$

A linear operator N satisfying (4) is called a **Nijenhuis Operator**.

Theorem ([4])

Let $N : \Gamma(TM) \to \Gamma(TM)$ *be a Nijenhuis Operator. Then a trivial deformation of* $\Gamma(TM)$ *can be obtained by putting* $\omega(X, Y) = [Na, b] + [a, Nb] - N[a, b]$.

The space $\Gamma(TM \oplus T^*M)$

Let us consider $TM \oplus T^*M$ on M. $\Gamma(TM \oplus T^*M)$, be the space of sections of $TM \oplus T^*M$ defined by $\Gamma(TM \oplus T^*M) = \Gamma(TM) \oplus \Gamma(T^*M) = \{(X, \xi) | X \in \Gamma(TM), \xi \in \Gamma(T^*M)\}$. $\Gamma(TM \oplus T^*M)$ is naturally endowed with one symmetric and skew symmetric pairing:

$$\langle (X, \alpha), (Y, \beta) \rangle_{\pm} = \frac{1}{2} i_Y \alpha \pm i_X \beta.$$

And this is a algebra with a bracket operation on it, which is known as Courant bracket.

Definition (Courant Bracket)

For differentiable manifold M the Courant bracket$[X, Y]_c$ on $\Gamma(TM \oplus T^*M)$ is a bilinear operation defined by $[X + \xi, Y + \eta]_c = ([X, Y], L_Y \xi - L_X \eta - d\langle X + \xi, Y + \eta \rangle_+)$, where $[X + \xi, Y + \eta] \in \Gamma(TM) \oplus T^*M)$ and L_X is the Lie derivative and $\langle X + \xi, Y + \eta \rangle_+$ is the symmetric pairing on $\Gamma(TM \oplus T^*M)$.

Nijenhuis Operator on $TM \oplus T^*M$

Let us assume a λ-parametrised family of brackets on $\Gamma(TM \oplus T^*M)$, i.e. $[Y_1, Y_2]_\lambda = [Y_1, Y_2]_c + \lambda\varphi(Y_1, Y_2)$, where φ is a bilinear operator on $\Gamma(TM \oplus T^*M)$, $Y_1, Y_2 \in \Gamma(TM \oplus T^*M)$ and $[Y_1, Y_2]_c$ is the Courant bracket on $\Gamma(TM \oplus T^*M)$.Now we have to check the Courant bracket structure of $\varphi(Y_1, Y_2)$. And to have a Courant bracket structure $\varphi(Y_1, Y_2)$ must satisfy:

1. Skew symmetric property.
2. Jacobi Anomaly.

Let $\mathfrak{N} : \Gamma(TM \oplus T^*M) \to \Gamma(TM \oplus T^*M)$ be a linear map and define $\mathfrak{T}_\lambda = id + \lambda\mathfrak{N}$ on $\Gamma(TM \oplus T^*M)$. For $\mathfrak{T}_\lambda$ this deformation is said to be trivial if $\mathfrak{T}_\lambda[X, Y]_\lambda = [\mathfrak{T}_\lambda X, \mathfrak{T}_\lambda Y]$.

Comparing both sides of the above equation we have

$$\varphi(X,Y) = [X,\mathfrak{N}Y]_c + [\mathfrak{N}X,Y]_c - \mathfrak{N}[X,Y]_c \tag{5}$$

$$\mathfrak{N}\varphi(X,Y) = [\mathfrak{N}X,\mathfrak{N}Y]_c \tag{6}$$

From the equation (6) we can conclude that

$$\begin{aligned} &\mathfrak{N}\varphi(X,Y) - [\mathfrak{N}X,\mathfrak{N}Y]_c = 0 \\ &\mathfrak{N}([X,\mathfrak{N}Y]_c + [\mathfrak{N}X,Y]_c - \mathfrak{N}[X,Y]_c) - [\mathfrak{N}X,\mathfrak{N}Y]_c = 0 \end{aligned} \tag{7}$$

With the equation (7), $\mathfrak{N}$ is called a Nijenhuis Operator on $\Gamma(TM \oplus T^*M)$.

We know that $\mathfrak{N}$ is a linear mapping from $\Gamma(TM \oplus T^*M)$ to itself. And It is skew symmetric if $\mathfrak{N} = -(\mathfrak{N})^t$ because $\mathfrak{N}$ can be written as

$$\begin{pmatrix} N & \pi \\ \omega & N^* \end{pmatrix},$$

where $N : \Gamma(TM) \to \Gamma(TM), N^* : \Gamma(T^*M) \to \Gamma(T^*M), \pi : \Gamma(T^*M) \to \Gamma(TM)$ and $\omega : \Gamma(TM) \to \Gamma(T^*M)$. Here $N^2 = -Id_M$. Kosmann-Schwarzbach in her article [8] has shown that $\Gamma(TM \oplus T^*M)$ has a weak deformation with respect to a Nijenhuis Operator $\mathfrak{N}$ only if $\mathfrak{N}$ is equivalent to a almost complex structure, i.e. $\mathfrak{N}^2 = -Id_M$. $\mathfrak{N}$ satisfies almost complex structure iff π, ω vanishes on $\Gamma(TM \oplus T^*M)$.

Dirac Structure

Here my aim is not to study the deformation of $\Gamma(TM \oplus T^*M)$. It is to study the deformation of Dirac structure on it.

1. Consider $TM \oplus T^*M$, a bundle on M.
2. $L \subset TM \oplus T^*M$, a subbundle of $TM \oplus T^*M$ is said to be a Dirac structure on M if
   - $L = L^{\perp}$.
   - $[\chi_1, \chi_2]_c = \chi_3$, where $\chi_1, \chi_2, \chi_3 \in \Gamma(L)$(Integrability condition)
3. It is also called a maximally isotropic subbundle of $TM \oplus T^*M$, as the symmetric pairing on $TM \oplus T^*M$ vanishes on L.
4. On $TM \oplus T^*M$, Courant bracket does not satisfy Jacobi identity , it satisfies an anomaly known as Jacobi Anomaly,which gives rise to a non vanishing three tensor $T(Z_1, Z_2, Z_3)$ on $TM \oplus T^*M$.
5. $T(Z_1, Z_2, Z_3) = \langle [Z_1, Z_2], Z_3 \rangle_+ + c.p.$.

Some authors like Gallardo, Nunes da costa[1], Longguang, Baokang[11] have studied independently on deformation of Dirac structure on 2004. I have tried to give a new approach following Dorfman's construction.

Theorem

Let $\mathfrak{N} : \Gamma(TM \oplus T^*M) \to \Gamma(TM \oplus T^*M)$ *be a Nijenhuis Operator, then the deformation of* $\Gamma(TM \oplus T^*M)$ *is not a weak deformation if and only if both* $\mathfrak{N}$ *and* $\varphi(X, Y)$ *are restricted to Dirac structures, otherwise the deformation is weak deformation on the whole space.*

Proof.

We have $\varphi(X,Y) = [X, \mathfrak{N}Y]_c + [\mathfrak{N}X, Y]_c - \mathfrak{N}[X,Y]_c$. φ is skew symmetric as Courant bracket is skew symmetric.
$\delta\varphi(X,Y,Z) = [X, \varphi(Y,Z)] + \varphi([X,Y],Z) + c.p. \neq 0$ as Courant bracket does not satisfy Jacobi identity property. Therefore on $\Gamma(TM \oplus T^*M)$ deformation is not trivial. Suppose both $\mathfrak{N}, \varphi(X,Y)$ are restricted to the section of Dirac structure L on $\Gamma(TM \oplus T^*M)$. That means $\mathfrak{N} : \Gamma(L) \to \Gamma(L)$ and $\varphi : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$.
As we know that Dirac Structure L is a maximally isotropic subbundle of $(TM \oplus T^*M)$, the natural symmetric pairing on it vanishes. Courant bracket is restricted to the Dirac structures satisfies Jacobi Identity. Therefore on L, $\delta\varphi = 0$ as $\delta\varphi$ is the representation of Jacobi identity on the given space. □

The weak deformation of $\Gamma(TM \oplus T^*M)$ has been studied by Kosmann-Schwarzbach in [8].

Nijenhuis Relation on $\Gamma(TM)$:

- Let $A \subset \Gamma(TM) \oplus \Gamma(TM)$ and $A^* \subset \Gamma(T^*M) \oplus \Gamma(T^*M)$.
- Let us take $a_1 \oplus a_2 \in \Gamma(TM) \oplus \Gamma(TM)$ and $\zeta_1 \oplus \zeta_2 \in \Gamma(T^*M) \oplus \Gamma(T^*M)$.
- Choose $\zeta_1 \oplus \zeta_2 \in \Gamma(T^*M) \oplus \Gamma(T^*M)$ such that $(\zeta_1, a_2) = (\zeta_2, a_1)$ for arbitrary $a_1 \oplus a_2 \in A$.
- A relation $A \subset \Gamma(TM) \oplus \Gamma(TM)$ is said to be a Nijenhuis relation for arbitrary $a_1, a_2, b_1, b_2 \in \Gamma(TM)$ and $\zeta_1, \zeta_2, \zeta_3 \in \Gamma(T^*M)$ satisfying $a_1 \oplus a_2, b_1 \oplus b_2 \in A$ and $\zeta_1 \oplus \zeta_2, \zeta_2 \oplus \zeta_3 \in A_*$ if the following holds:

$$(\zeta_1, [a_2, b_2]) - (\zeta_2, [a_2, b_1] + [a_1, b_2]) + (\zeta_3, [a_1, b_1]) = 0.$$

- Proposition:(Dorfman) The graph of a Nijenhuis Operator $N : \Gamma(TM) \to \Gamma(TM)$ is a Nijenhuis relation. Conversely if a graph of some operator $N : \Gamma(TM) \to \Gamma(TM)$ is a Nijenhuis relation, then N is a Nijenhuis Operator.

Dirac pairs in previous set up:

- Two Dirac structures $L, M \subset (TM \oplus T^*M)$ are said to be a pair of Dirac structures, if the set

$$A_{L,M} = \{a_1 \oplus a_2 : \exists \zeta \in \Gamma(T^*M), a_1 \oplus \zeta \in M, a_2 \oplus \zeta \in L\}$$

is a Nijenhuis relation.

Nijenhuis relation on $TM \oplus T^*M$:

- $\mathfrak{A} \subset \Gamma(TM \oplus T^*M) \oplus \Gamma(TM \oplus T^*M)$ and $A^* \subset \Gamma(T^*M) \oplus \Gamma(T^*M)$.
- Let us take $\alpha_1 \oplus \alpha_2 \in \Gamma(TM \oplus T^*M) \oplus \Gamma(TM \oplus T^*M)$ and $\eta_1 \oplus \eta_2 \in \Gamma(T^*M) \oplus \Gamma(T^*M)$
- As per the above calculation here the pairing is defined as

$$\begin{aligned} &(\eta_1, \alpha_2) = (\eta_2, \alpha_1) \\ &\Rightarrow (\eta_1, X_2 + \xi_2) = (\eta_2, X_1 + \xi_1) \\ &\Rightarrow i_{X_2}\eta_1 + \eta_1 \wedge \xi_2 = i_{X_1}\eta_2 + \eta_2 \wedge \xi_1. \end{aligned}$$

- A relation $\mathfrak{A} \subset \Gamma(TM \oplus T^*M) \oplus \Gamma(TM \oplus T^*M)$ is said to be a Nijenhuis relation for arbitrary $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma(TM \oplus T^*M)$ and $\eta_1, \eta_2, \eta_3 \in \Gamma(T^*M)$ satisfying $\alpha_1 \oplus \alpha_2, \beta_1 \oplus \beta_2 \in A$ and $\eta_1 \oplus \eta_2, \eta_2 \oplus \eta_3 \in A_*$ if the following holds:

$$(\eta_1, [\alpha_2, \beta_2]) - (\eta_2, [\alpha_2, \beta_1] + [\alpha_1, \beta_2]) + (\eta_3, [\alpha_1, \beta_1]) = 0.$$

Future work

- I am now trying to associate a pair of some geometric structures like Dirac structures with this above defined Nijenhuis relation.
- Deformation of Kahler manifold with respect to Nijenhuis Operator may be seen.
- Hamiltonian pairs and Symplectic pairs can be associated to the Nijenhuis Operator on $\Gamma(TM \oplus T^*M)$.
- One can study the deformation of the space $\Gamma(\Lambda(TM) \oplus \Lambda(T^*M))$ through Nijenhuis Operators.

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$$Thank \quad You$$