Multipliers on Weighted Semigroups and Associated Beurling Banach algebras

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This is joint work with S. J. Bhatt, H. V. Dedania and Manish Pandey.

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For $s \in S$, define $\gamma_s : S \to S$ as $\gamma_s(t) = st$ $(t \in S)$. Then $\gamma_s \in M(S)$, and $\gamma_s \gamma_t = \gamma_{st}$ $(s, t \in S)$.

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A semigroup S is *faithful* if for $s, t \in S$, $su = tu$ for all $u \in S$ implies $s = t$.
Thus S is identified with an ideal of $M(S)$ via $s \mapsto \gamma_s$.

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Then $\omega(Ts) \leq \widetilde{\omega}(T)\omega(s)$ for every $T \in M_\omega(S)$ and $s \in S$.

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3. *For any weight ω on S, $M_\omega(S)$ is a subsemigroup of $M(S)$ and S is an ideal in $M_\omega(S)$.*
4. *If S is involutive and ω is symmetric, then each of $M(S)$ and $M_\omega(S)$ are involutive and S is a $*$-ideal.*

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3. An involutive semigroup S is $*$-*separating* if $s = t$ whenever $s^*s = t^*t = s^*t$ and $s, t \in S$.

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4. *S is separating iff $M_\omega(S)$ is separating.*
5. *There exists a semigroup S such that both S and $M(S)$ are separating; but the quotient $M(S)/S$ fails to be separating.*

Definition 1.5

A weight ω a *uniform weight* (respectively a C^*-*weight* for an involutive S) if $\omega(s^2) = \omega(s)^2$ $(s \in S)$ (respectively $\omega(s^* s) = \omega(s)^2$ $(s \in S)$).

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For examples, $\omega(n) = e^n$ $(n \in \mathbb{N})$ is a uniform weight on $\mathbb{N}$; and $\omega(m + \lambda n) = e^{-m-n}$ is a C^*-weight on $S := \{m + \lambda n : m, n \in \mathbb{N}\}$, $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ fixed, having involution $(m + \lambda n)^* = n + \lambda m$. In the present case, a uniform weight is a C^*-weight for the trivial involution $s^* = s$ on S.

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If S is involutive and ω is a C^*-weight on S, then ω is symmetric as well as a uniform weight.

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3. $\widetilde{\omega}(T) = \inf\{K > 0 : \omega(Ts) \leq K\omega(s)\ (s \in S)\} \quad (T \in M_\omega(S))$.

The following classes of weights arise in the study of associated Beurling algebras.

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Indeed, let $S = ([2,\infty), +)$, and let $\omega(n) = e^{\frac{n}{\log n}}$ $(n \in S)$. Then ω is a GRS-weight but it is not a Beurling-Domar weight.

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4. *Let ω be semisimple. Then $\nu_\omega(s) := \lim_{n\to\infty} \omega(s^n)^{\frac{1}{n}}$ $(s \in S)$ is a uniform weight, and it is the largest uniform weight dominated by ω.*

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4. *Let ω be semisimple. Then $\nu_\omega(s) := \lim_{n\to\infty} \omega(s^n)^{\frac{1}{n}}$ $(s \in S)$ is a uniform weight, and it is the largest uniform weight dominated by ω.*
5. *Let ω be semisimple. Then $\mu_\omega(s) = \nu_\omega(s^* s)^{\frac{1}{2}}$ $(s \in S)$ is a C^*-weight, and it is the largest C^*-weight dominated by ω.*

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It is a Banach space.It is a commutative Banach algebra with the *convolution* multiplication

$$(f \star g)(s) = \sum_{uv=s} f(u)g(v)$$

and $(f \star g)(s) = 0$ if $uv = s$ has no solution.
The following exhibits the relationship between the Beurling algebras $\ell^1(S, \omega)$ and $\ell^1(M_\omega(S), \widetilde{\omega})$.

Let $\omega_0 = \inf\{\omega(s) : s \in S\}$.

Theorem 2.1

Let ω be weakly regular with $\omega_0 > 0$. Then $\ell^1(S,\omega)$ is a closed ideal of $\ell^1(M_\omega(S),\widetilde{\omega})$ and the quotient algebra $\ell^1\left(M_\omega(S),\widetilde{\omega}\right)/\ell^1(S,\omega)$ is isomorphic to the Beurling algebra of the Rees quotient semigroup $M_\omega(S)/S$ with the quotient weight $\widetilde{\omega}_q$.

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Let $\Phi_{\omega s}(S)$ denote the set of all ω-bounded generalized semicharacters on S with the point open topology.

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Our next three results contain the semigroup multiplier analogues of a couple of results on multipliers on commutative Banach algebras [22, Theorems 1.4.1, 1.4.2, Corollary 1.4.1].

Theorem 2.2

If $\alpha \in \Phi_{\omega s}(S)$, then there exits unique $\widetilde{\alpha} \in \Phi_{\widetilde{\omega} s}(M_\omega(S))$ such that $\widetilde{\alpha}(\gamma_s) = \alpha(s)$ for all $s \in S$. If $\beta \in \Phi_{\widetilde{\omega} s}(M_\omega(S))$, then either $\beta(\gamma_s) = 0$ for all $s \in S$ or there is $\widetilde{\alpha} \in \Phi_{\widetilde{\omega} s}(M_\omega(S))$ such that $\beta = \widetilde{\alpha}$.

Let $\widetilde{\Phi}_{\widetilde{\omega} s}(S) = \{\widetilde{\alpha} \in \Phi_{\widetilde{\omega} s}(M_\omega(S)) : \widetilde{\alpha}(\gamma_s) \neq 0 \text{ for some } s \in S\}$, and let $h_{\omega s}(S) = \{\widetilde{\alpha} \in \Phi_{\widetilde{\omega} s}(M_\omega(S)) : \widetilde{\alpha}(\gamma_s) = 0 \, (s \in S)\}$.

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Corollary 2.3

Let ω be a weight on a semigroup S.

1. *Then* $\Phi_{\widetilde{\omega}s}(M_\omega(S)) = \widetilde{\Phi}_{\omega s}(S) \cup h_{\omega s}(S)$.
2. *Let S be involutive, and let ω be symmetric. Let* $h^*_{\omega s}(S) = \{\alpha \in \Psi_{\omega s}(M_\omega(S)) : \alpha(S) = \{0\}\}$. *Then* $\Psi_{\widetilde{\omega}s}(M_\omega(S)) = \widetilde{\Psi}_{\omega s}(S) \cup h^*_{\omega s}(S)$.

The following corresponds to the result that for a weighted locally compact abelian group (G, ω), the Gel'fand space $\Delta(L^1(G, \omega))$ is identified with the space of ω-bounded generalized characters on G. We omit the straightforward proof.

Corollary 2.4

1. $\Delta(\ell^1(S, \omega)) \cong \Phi_{\omega s}(S)$, *topologically as well.*
2. $\Delta(\ell^1(M_\omega(S), \widetilde{\omega})) \cong \widetilde{\Phi}_{\omega s}(S) \cup h_{\omega s}(S)$.
3. *Let* ω *be weakly regular. Then* $\Delta\left(\ell^1\left(M_\omega(S)/S, \widetilde{\omega}_q\right)\right) \cong h_{\omega s}(S)$.

Semisimplicity of a Beurling algebra is an important problem. For a locally compact group G, $L^1(G,\omega)$ is semisimple if G is abelian [7]; for non-abelian G, it is not known whether $L^1(G,\omega)$ is semisimple or not [14, Page-175]. For an abelian semigroup S, $\ell^1(S,\omega)$ is semisimple iff S is separating and ω is semisimple [13, Prop. 4.8]. This quickly gives the following.

Theorem 2.5

The Banach algebra $\ell^1(S,\omega)$ is semisimple iff $\ell^1(M_\omega(S),\widetilde{\omega})$ is semisimple. The quotient $\ell^1(M_\omega(S),\widetilde{\omega})/\ell^1(S,\omega)$ may fail to be semisimple.

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The UUNP turns out to be closely related with regularity [8, 23] and have applications to abelian Harmonic Analysis [5, 6, 11].A Banach algebra $\mathcal{A}$ is *regular* if in the Gel'fand space $\Delta(\mathcal{A})$, a point and a closed set can be separated by a Gel'fand transform [23]. For an abelian G, the algebra $\ell^1(G)$ is regular; and for a weighted group G, $\ell^1(G, \omega)$ is regular iff $\ell^1(G, \omega)$ has UUNP iff ω is a Beurling-Domar weight [8]. It would be interesting to search for a weighted semigroup (S, ω) such that $\ell^1(S, \omega)$ has UUNP but is not regular.

Theorem 2.6

1. *If $\ell^1(M_\omega(S), \widetilde{\omega})$ has UUNP, then $\ell^1(S, \omega)$ has UUNP.*
2. *If $\ell^1(M_\omega(S), \widetilde{\omega})$ is regular, then $\ell^1(S, \omega)$ is regular.*
3. *Let S be an inverse semigroup. Let ω be a Beurling-Domar weight on S. Then $\ell^1(S, \omega)$ is regular.*

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A Banach $*$-algebra $(\mathcal{B}, \|\cdot\|)$ has *Unique C^*-Norm Property (UC^*NP)* [2] if $\mathcal{B}$ admits exactly one C^*-norm.
A commutative Banach $*$-algebra $\mathcal{B}$ is *$*$-regular* [2] if given $F \subset \widetilde{\Delta}(\mathcal{B})$ closed and $\varphi \notin F$, there exists $x \in \mathcal{B}$ such that $\widehat{x}(\varphi) \neq 0$ and $\widehat{x}(F) = \{0\}$. In fact, UC^*NP and $*$-regularity (appropriately defined) acquires much greater significance in non-commutative Banach $*$-algebras [2]. Their role in commutative Banach $*$-algebras is discussed in [2, Section 2], [8, 11]. By [17], for a weighted compactly generated (not necessarily abelian) group (G, ω), $L^1(G, \omega)$ is symmetric iff ω is a GRS-weight. By [2], a commutative Banach $*$- algebra is regular iff it is $*$-regular and symmetric.

Theorem 2.7

Let S be be involutive, and let ω be symmetric.

1. *If $\ell^1(M_\omega(S), \widetilde{\omega})$ has UC^*NP, then $\ell^1(S, \omega)$ has UC^*NP.*
2. *If $\ell^1(M_\omega(S), \widetilde{\omega})$ is $*$-regular, then $\ell^1(S, \omega)$ is $*$-regular.*

The *multiplier Banach algebra* $M(\mathcal{A})$ of a commutative Banach algebra $(\mathcal{A}, \|\cdot\|)$ is the unital Banach algebra consisting of all $T : \mathcal{A} \to \mathcal{A}$ satisfying $T(ab) = aTb = (Ta)b$ $(a, b \in \mathcal{A})$ with the operator norm $\|T\| = \sup\{\|Ta\| : a \in \mathcal{A},\ \|a\| \leq 1\}$.

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Our question is:

The *multiplier Banach algebra* $M(\mathcal{A})$ of a commutative Banach algebra $(\mathcal{A}, \|\cdot\|)$ is the unital Banach algebra consisting of all $T : \mathcal{A} \to \mathcal{A}$ satisfying $T(ab) = aTb = (Ta)b$ $(a, b \in \mathcal{A})$ with the operator norm $\|T\| = \sup\{\|Ta\| : a \in \mathcal{A},\ \|a\| \leq 1\}$.Multipliers, either at the level of semigroups or at the level of algebras, constitute a kind of maximal unitization.
Our question is: **When is** $M(\ell^1(S, \omega)) = \ell^1(M_\omega(S), \widetilde{\omega})$**?**

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A semigroup S is *cancellative* if whenever for $s, t, u \in S$, $su = tu$ implies $s = t$.

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A semigroup S is *cancellative* if whenever for $s, t, u \in S$, $su = tu$ implies $s = t$.
Cancellative semigroups are precisely the subsemigroups of groups.

Lemma 1

Let S be an abelian faithful semigroup. Then the natural homomorphism $s \mapsto \gamma_s$ of S into $M_\omega(S)$ induces a homomorphism of $\ell^1(S, \omega)$ into $\ell^1(M_\omega(S), \widetilde{\omega})$ which is one-one if and only if $s \mapsto \gamma_s$ is one-one and onto if and only if $s \mapsto \gamma_s$ is onto.

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Lemma 2

Let ω be a weight on an abelian semigroup S, and let $\mu \in \ell^1(M_\omega(S),\widetilde{\omega})$. Then the map $T_\mu : \ell^1(S,\omega) \to \ell^1(S,\omega)$ defined by $T_\mu(f) = \mu \star f$ is a multiplier of $\ell^1(S,\omega)$. The map $\mu \mapsto T_\mu$ of $\ell^1(M_\omega(S),\widetilde{\omega})$ into $M(\ell^1(S,\omega))$ is a norm-decreasing homomorphism.

Lemma 3

Let S be an abelian semigroup with the property: Given $\alpha \in M_\omega(S)$, there exists $s_\alpha \in S$ such that for any $\beta \in M_\omega(S)$, $\alpha(s_\alpha) = \beta(s_\alpha)$ implies $\alpha = \beta$ (This holds in particular when S is cancellative). Then the map $\mu \mapsto T_\mu$ from $\ell^1(M_\omega(S), \widetilde{\omega})$ to $M(\ell^1(S, \omega))$ is one-one.

Let S be a cancellative semigroup. Then S, $M_\omega(S)$ and $M(S)$ can be embedded in a group $Q(S)$, called *the group of the semigroup* S, which has the property that $M(S) = \{\alpha \in Q(S) : \alpha S \subset S\}$.

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$$[s,t] = \{(u,v) \in S \times S : (u,v) \sim (s,t)\}.$$

Then $Q(S) = (S \times S)/\sim$ is a group with the binary operation

$$[s,t][u,v] = [su,tv] \quad ([s,t],[u,v] \in Q(S)).$$

The semigroup S is embedded in $Q(S)$ via the map $s \mapsto [su,u]$.

Let ω be a weight on S. Define $\omega_Q : Q(S) \to (0, \infty)$ as

$$\omega_Q([s,t]) = \sup \left\{ \frac{\widetilde{\omega}(su)}{\widetilde{\omega}(tu)} : u \in M_\omega(S) \right\}.$$

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Let $[s,t],[u,v] \in Q(S)$. By definition $\omega_Q([s,t]) > 0$. Let $x \in M_\omega(S)$. Then

$$\frac{\widetilde{\omega}(sux)}{\widetilde{\omega}(tvx)} = \frac{\widetilde{\omega}(sux)}{\widetilde{\omega}(tux)} \frac{\widetilde{\omega}(utx)}{\widetilde{\omega}(vtx)} \leq \omega_Q([s,t])\omega_Q([u,v]).$$

Therefore

$$\omega_Q([s,t][u,v]) = \omega_Q([su,tv]) \leq \omega_Q([s,t])\omega_Q([u,v]).$$

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Note that $\omega_Q([su,u]) = \sup\{\frac{\widetilde{\omega}(suv)}{\widetilde{\omega}(uv)} : v \in M_\omega(S)\} \leq \widetilde{\omega}(s)$ $(s \in M_\omega(S))$. Since $\widetilde{\omega}(\gamma_s) \leq \omega(s)$, it follows that $\omega_Q([su,u]) \leq \omega(s)$ $(s \in S)$.

Let ω be a weight on S. Define $\omega_Q : Q(S) \to (0, \infty)$ as

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Note that $\omega_Q([su,u]) = \sup\{\frac{\widetilde{\omega}(suv)}{\widetilde{\omega}(uv)} : v \in M_\omega(S)\} \le \widetilde{\omega}(s)$ $(s \in M_\omega(S))$. Since $\widetilde{\omega}(\gamma_s) \le \omega(s)$, it follows that $\omega_Q([su,u]) \le \omega(s)$ $(s \in S)$.Thus given a weight ω on a cancellative semigroup S, there exists a natural weight ω_Q on $Q(S)$ whose restriction on S is dominated by ω.

Lemma 4

Let (S, ω) be a cancellative, abelian weighted semigroup, and let $Q(S)$ be the group of the semigroup S. Then

$$M_\omega(S) = \{g \in Q(S) : gS \subset S,\ \omega(gs) \leq K_g \omega(s)\ (s \in S)\}.$$

Lemma 4

Let (S, ω) be a cancellative, abelian weighted semigroup, and let $Q(S)$ be the group of the semigroup S. Then

$$M_\omega(S) = \{g \in Q(S) : gS \subset S,\ \omega(gs) \leq K_g \omega(s)\ (s \in S)\}.$$

Lemma 5

Let S be cancellative, abelian semigroup. Then both $\ell^1(S, \omega)$ and $\ell^1(M_\omega(S), \widetilde{\omega})$ are subalgebras of $\ell^1(Q(S), \omega_Q)$.

Theorem 3.1

Let S be cancellative. Then $M(\ell^1(S,\omega))$ is homeomorphically isomorphic to $\ell^1(M_\omega(S),\widetilde{\omega})$.

The *annihilator* S_ω° of S with a zero element 0 (i.e. $0 \in S$ such that $0s = s0 = 0$ for all $s \in S$ [19]) in $M_\omega(S)$ is a semigroup ideal of $M_\omega(S)$ given by

$$S_\omega^\circ = \{\alpha \in M_\omega(S) : \alpha\gamma_s = 0 \text{ for all } s \in S\},$$

and it contains γ_0.

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and it contains γ_0.Analogously, the *annihilator* $\ell^1(S,\omega)^\circ$ of $\ell^1(S,\omega)$ in $\ell^1(M_\omega(S),\widetilde{\omega})$ is a closed algebra ideal of $\ell^1(M_\omega(S),\widetilde{\omega})$ given by

$$\ell^1(S,\omega)^\circ = \{\mu \in \ell^1(M_\omega(S),\widetilde{\omega}) : \mu \star f = 0 \ (f \in \ell^1(S,\omega))\}.$$

When S is a semigroup with zero element 0, $M_\omega(S)$ is also a semigroup having zero element γ_0. Also, $\alpha(0) = 0$ for all $\alpha \in M_\omega(S)$. When S has a zero element, we define

$$\ell^1(S, \omega) = \{f : S \to \mathbb{C} : f(0) = 0, \sum_{s \in S} |f(s)|\omega(s) < \infty\}$$

Theorem 3.2

Let S be a semigroup with zero element. Let $\widetilde{\omega}$ (in particular, ω) be bounded away from 0. Then $\ell^1(S,\omega)^\circ = \ell^1(S_\omega^\circ, \widetilde{\omega})$ and $\ell^1(M_\omega(S), \widetilde{\omega})/\ell^1(S_\omega^\circ, \widetilde{\omega})$ is isomorphic to the Beurling algebra $\ell^1(M_\omega(S)/S_\omega^\circ, \widetilde{\omega}_q)$.

Theorem 3.3

Let S be separating and ω be semisimple, and let $\widetilde{\omega}$ be bounded away from 0. Then the following holds.

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1. *The map $f \mapsto f + \ell^1(S, \omega)^\circ$ from $\ell^1(S, \omega)$ into $\ell^1(M_\omega(S), \widetilde{\omega})/\ell^1(S, \omega)^\circ$ is one-one and $\ell^1(M_\omega(S), \widetilde{\omega})/\ell^1(S, \omega)^\circ$ is semisimple.*

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2. *If $\ell^1(S,\omega)$ has a bounded approximate identity, then the map $\mu + \ell^1(S,\omega)^\circ \mapsto T_\mu$ is a homeomorphic isomorphism from $\ell^1(M_\omega(S),\widetilde{\omega})/\ell^1(S,\omega)^\circ$ onto $M(\ell^1(S,\omega))$.*

Let $\mathcal{A}$ be a commutative Banach algebra with identity. The following theorem shows the relationship between the Beurling algebras $\ell^1(S, \omega, \mathcal{A})$ and $\ell^1(M_\omega(S), \widetilde{\omega}, \mathcal{A})$.

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Theorem 4.1

Let ω be weakly regular with $\omega_0 > 0$. Then $\ell^1(S, \omega, \mathcal{A})$ is a closed ideal of $\ell^1(M_\omega(S), \widetilde{\omega}, \mathcal{A})$ and the quotient algebra $\ell^1(M_\omega(S), \widetilde{\omega}, \mathcal{A})/\ell^1(S, \omega, \mathcal{A})$ is isomorphic to $\ell^1(M_\omega(S)/S, \widetilde{\omega}_q, \mathcal{A})$.

Theorem 4.2

Let $\mathcal{A}$ be a commutative Banach algebra with identity, and ω be weakly regular with $\omega_0 > 0$. Then $\ell^1(S, \omega, \mathcal{A})$ is a closed ideal in $\ell^1(M_\omega(S), \widetilde{\omega}, \mathcal{A})$.

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Let $\mathcal{A}$ be a commutative Banach algebra with identity, and ω be weakly regular with $\omega_0 > 0$. Then $\ell^1(S, \omega, \mathcal{A})$ is a closed ideal in $\ell^1(M_\omega(S), \widetilde{\omega}, \mathcal{A})$.

Lemma 4.3

Let ω be a weight on an abelian semigroup S, $\mathcal{A}$ be a commutative Banach algebra with identity, and let $\mu \in \ell^1(M_\omega(S), \widetilde{\omega}, \mathcal{A})$. Then the map $T_\mu : \ell^1(S, \omega, \mathcal{A}) \to \ell^1(S, \omega, \mathcal{A})$ defined by $T_\mu(f) = \mu * f$ is a multiplier of $\ell^1(S, \omega, \mathcal{A})$. The map $\mu \to T_\mu$ of $\ell^1(M_\omega(S), \widetilde{\omega}, \mathcal{A})$ into $M(\ell^1(S, \omega, \mathcal{A}))$ is a norm decreasing homomorphism.

Theorem 4.4

Let S be cancellative abelian semigroup and $\mathcal{A}$ be a commutative Banach algebra with identity. Then $M(\ell^1(S,\omega,\mathcal{A}))$ is homeomorphically isomorphic to $\ell^1(M_\omega(S),\widetilde{\omega},\mathcal{A})$.

Theorem 4.5

Let S be separating, ω semisimple and $\mathcal{A}$ be a semisimple commutative Banach algebra with identity and let $\widetilde{\omega}$ be bounded away from zero. Then the following holds

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Theorem 4.5

Let S be separating, ω semisimple and $\mathcal{A}$ be a semisimple commutative Banach algebra with identity and let $\widetilde{\omega}$ be bounded away from zero. Then the following holds

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2. *If $\ell^1(S, \omega, \mathcal{A})$ has a bounded approximate identity, then the map $\mu + \ell^1(S, \omega, \mathcal{A})^0 \mapsto T_\mu$ is a homeomorphic isomorphism from $\ell^1(M_\omega(S), \widetilde{\omega}, \mathcal{A})/\ell^1(S, \omega, \mathcal{A})^0$ onto $M(\ell^1(S, \omega, \mathcal{A}))$.*

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THANK YOU