Dense definiteness of domains of powers of subnormal weighted shifts on directed trees

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Weighted shifts on directed trees

Directed trees

- $\mathscr{T} = (V, E)$ – directed tree,
- $\mathsf{par}(u) = \{v \in V\colon (v, u) \in E\}$ – parent of $u \in V$,
- $\mathsf{Chi}(u) = \{v \in V\colon (u, v) \in E\}$ – children of $u \in V$,
- root – root of $\mathscr{T}$ (provided it exists),
- $V^\circ = V \setminus \{\mathsf{root}\}$,

Weighted shifts on directed trees

Let $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ} \subseteq \mathbb{C}$. Define $\Lambda_{\mathscr{T}}\colon \mathbb{C}^V \to \mathbb{C}^V$ by

$$(\Lambda_{\mathscr{T}} f)(v) = \begin{cases} \lambda_v \cdot f\big(\mathsf{par}(v)\big) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \mathsf{root}. \end{cases}$$

Define $\mathsf{S} = \mathsf{S}_{\boldsymbol{\lambda}}\colon \ell^2(V) \supseteq \mathcal{D}(\mathsf{S}) \to \ell^2(V)$ by

$$\begin{aligned} \mathcal{D}(\mathsf{S}) &= \{f \in \ell^2(V)\colon \Lambda_{\mathscr{T}} f \in \ell^2(V)\}, \\ \mathsf{S} f &= \Lambda_{\mathscr{T}} f, \quad f \in \mathcal{D}(\mathsf{S}). \end{aligned}$$

Weighted shifts on directed trees

- Weighted shifts on directed trees are generalizations of **classical weighted shifts**.
- Weighted shifts on directed trees are **concrete**.
- Weighted shifts on rootless and leafless directed trees are unitarily equivalent to **composition operators in ℓ^2-spaces**.
- Weighted shifts on directed trees have **interesting properties**.

Subnormality

S in $\mathcal{H}$ is **subnormal** if S is densely defined and there exists a Hilbert space $\mathcal{K}$ and a normal operator N in $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding) and $Sh = Nh$ for all $h \in \mathcal{D}(S)$.

N in $\mathcal{H}$ is said to be **normal** iff N is closed, densely defined and $NN^* = N^*N$.

Subnormality

Theorem [Lambert]

Let S be a bounded operator on $\mathcal{H}$. Then S is subnormal iff for every $f \in \mathcal{H}$ there exists a positive Borel measure ϑ_f on $\mathbb{R}_+$ such that

$$\|S^n f\|^2 = \int_{0\infty} t^n \, \vartheta_f(\mathrm{d}\, t), \quad n \in \mathbb{Z}_+. \tag{1}$$

Generating Stieltjes moment sequences

Let S be an operator in $\mathcal{H}$. Then S **generates Stieltjes moment sequences** iff (1) is satisfied for every $f \in \mathcal{D}^\infty(S) = \bigcap_{n \in \mathbb{N}} \mathcal{D}(S^n)$.

Subnormality

If S is a subnormal operator in a complex Hilbert space $\mathcal{H}$, then S generates Stieltjes moment sequences.

Theorem [Jabłoński-Jung-Stochel]

There exist a **non-hyponormal** weighted shift S on a directed tree $\mathscr{T} = (V, E)$ such that S generates Stieltjes moment sequences and $\mathcal{D}^\infty(\mathsf{S})$ is dense in $\ell^2(V)$.

Subnormality

- **Invariance of the domain** is assumed in most of the applicable **criteria for subnormality** (Albrecht-Vasilescu, Cichoń-Stochel-Szafraniec, Stochel-Szafraniec).
- There exist **closed symmetric** operators whose **squares** have **trivial domains** (Naimark, Chernoff, Schmüdgen).
- **Symmetric** weighted shifts on directed trees are automatically **normal** (B.-Jabłoński-Jung-Stochel).
- **Symmetric** operators are **subnormal**.

Subnormality

Problem

Does there exist a weighted shift S on a directed tree, which has the following properties:

- $\mathcal{D}(\mathsf{S}^2) = \{0\}$,
- S is subnormal?

Criterion

Theorem (B.-Dymek-Jabłoński-Stochel)

Let $\mathscr{T} = (V, E)$ be countably infinite and S be **densely defined** weighted shift on $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$. If there exist a family $\{\mu_v\}_{v \in V}$ of Borel probability measures on $\mathbb{R}_+$ and a family $\{\varepsilon_v\}_{v \in V}$ of nonnegative real numbers such that

$$\mu_u(\sigma) = \sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 \int_\sigma \frac{1}{t} \mu_v(\mathrm{d}\, t) + \varepsilon_u \delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\ u \in V,$$

then S is **subnormal**.

Criterion

- The criterion becomes a **characterization** in the bounded case.
- The criterion do not appeal to **density of the domain** of any **power** of S greater or equal to 2.
- The criterion has its origin in a criterion for subnormality of **unbounded composition operators**.

Criterion

Theorem (B.-Dymek-Jabłoński-Stochel)

Let $n \in \mathbb{N}$. Let $\mathscr{T} = (V, E)$ be countably infinite and S be **densely defined** weighted shift on $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$. If there exist a family $\{\mu_v\}_{v \in V}$ of Borel probability measures on $\mathbb{R}_+$ and a family $\{\varepsilon_v\}_{v \in V}$ of nonnegative real numbers such that

$$\mu_u(\sigma) = \sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 \int_\sigma \frac{1}{t} \mu_v(\mathrm{d}\, t) + \varepsilon_u \delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\, u \in V,$$

holds, then S^n is **densely defined** if and only if

$$\int_0^\infty s^n \,\mathrm{d}\, \mu_u(s) < \infty, \quad u \in V.$$

Example

Theorem (B.-Jabłoński-Jung-Stochel)

Let $n \in \mathbb{N}$. Then there exists a weighted shift S on a directed tree such that

(i) S is subnormal,

(ii) S^n is densely defined,

(iii) $\mathcal{D}(\mathsf{S}^{n+1}) = \{0\}$.

Example

- The proof is **constructive**.
- S acts on the **extremal** directed tree.
- The measures μ_v are **discrete**.
- The criterion is **indispensable**.
- There exists a **composition operator** with **analogous** properties.

References

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