Non Commutative Haar Condition

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An *operator system* in a C^*-algebra $\mathcal{A}$ is a subset S of $\mathcal{A}$ containing $1_{\mathcal{A}}$ such that $C^*(S) = \mathcal{A}$, where $C^*(S)$ denotes the C^*-algebra generated by $\mathcal{A}$. Arveson called *Korovkin* sets in $\mathcal{A}$ for completely positive maps as *hyperrigid sets*

A finite or countably infinite set G of generators of a C^*-algebra $\mathcal{A}$ is said to be *hyperrigid* if every faithful representation of $\mathcal{A} \subseteq B(H)$, H a separable Hilbert space, and for every sequence of identity preserving completely (UCP) positive maps

$\Phi_n : B(H) \to B(H), n = 1, 2, 3, \ldots,$

$$\lim_{n\to\infty} \|\Phi_n(g) - g\| = 0,\ \forall g \in G \Longrightarrow \lim_{n\to\infty} \|\Phi_n(a) - a\| = 0\ \forall a \in \mathcal{A}.$$

A subset S of a W^*-algebra $\mathcal{A}$ containing identity $1_{\mathcal{A}}$ is called weakly hyperrigid if
(i) $\mathcal{A}$ equals the W^*-algebra $W^*(\mathcal{S})$ generated by S, and
(ii) for every faithful representation of $\mathcal{A} \subseteq B(H)$, H a separable Hilbert space and for every net of contractive completely postive map $\Phi_\alpha : B(H) \to \mathcal{B}(H)$,

$$\lim_\alpha \Phi_\alpha(s) = s \text{ weakly } \forall s \in S \Longrightarrow \lim_\alpha \Phi_\alpha(a) = a \text{ weakly } \forall a \in \mathcal{A}.$$

The study of Čebyšev subspaces in the general operator algebra setting was initiated by A.G.Robertson [?] followed by Robertson and Yost [?] and then Pedersen [13].
In [?], Robertson gives a characterization of one dimensional Čebyšev subspaces of von Nuemann algebras. The result is as follows:

Theorem ([?], Theorem 1)

Let M be a von Neumann algebra. Let x be an operator in M. Then the one-dimensional subspace $\mathbb{C}x$ spanned by x is a Čebyšev subspace of M if and only if $\exists$ a projection p in the centre of M such that px is left invertible in pM and $(1-p)x$ is right invertible in $(1-p)M$.

The proof uses the existence of central projections in von Neumann algebras together with Hann-Banach and Krein-Milman theorems. Another important result of Robertson is regarding the non existence of higher dimensional Čebyšev subspaces of infinite dimensional von Neumann algebras which are also $*$-subalgebras.

Theorem ([?], Theorem 6)

Let M be an infinite dimensional von Neumann algebra. Let N be a finite dimensional *-subalgebra of M with dimension greater than one. Then N is not a Čebyšev subspace of M.

For the proof, Robertson uses the rich structural properties of von Nueman algebras. Attempts to prove the analogue of Haar's theorem [7] led to quite a few interesting results in the non-commutative C^*-algebra setting. A result in that direction by Robertson and Yost is the following.

Theorem ([?], Theorem 2.3)

Let A be a norm-closed two sided ideal in a von Neuman algebra, $x \in A$. Then $\mathbb{C}x$ is a Čebyšev subspace in A, if and only if there is no irreducible representation π of A for which 0 is an eigenvalue of both $\pi(x)$ and $\pi(x^*)$. When this happens, $x^*x + xx^*$ is strictly positive.

Theorem ([13], Theorem 2)

Let V be an n-dimensional subspace of a C^*-algebra $\mathcal{A}$. The following conditions are equivalent.

(i) *V is not a Čebyšev subspace;*

(ii) *There is a unitary operator u in $\tilde{\mathcal{A}}$, a non-zero element x_0 in V and an atomic space ϕ, which is a convex combination of m orthogonal pure states, such that $\phi(x_0^* x_0) = \phi(u x_0 x_0^* u^*) = 0$.*
If $m < n$, we further have $\phi(uV) = 0$.

In the following two theorems Pedersen characterizes the one-dimensional and two-dimensional Čebyšev subspaces of C^*-algebras in terms of irreducible representations, their eigen values and eigen vectors. These results can also be seen as the generalization of Haar's theorem to the first two dimensions. Pedersen remarks in the context of the theorem above that it seems to be the best one can do in generalizing Haar's theorem (Theorem **??**). However a recent work [**?**] generalises Pedersen's result for all finite dimensions.

Let $\mathcal{A}$ be a C^*-algebra with unit 1 and let $x_0 \in \mathcal{A}$ is not a multiple of 1. In this setting Pedersen [13] obtained the following results.

Theorem ([13], Theorem 3)

Let x_0 be a non-zero element in a C^*-algebra $\mathcal{A}$. The following conditions are equivalent.

(i) $\mathbb{C}x_0$ *is a Čebyšev subspace of* $\mathcal{A}$

(ii) $x_0^* x_0 + u x_0 x_0^*$ *is stricly positive in* $\mathcal{A}$

(iii) *In no irreducible representation* $(\pi, \mathcal{H})$ *of* $\mathcal{A}$ *do the operators* $\pi(x_0)$ *and* $\pi(x_0^*)$ *both have zero as an eigen value*

Further to the work by Pedersen in 1977 [13] in trying to extent the classical Haar condition to the non-commutative case, though with limited success (for dimensions one and two) nothing has been done in the last thirty to forty years till the work by Namboodiri, Pramod and Vijayarajan [**?**] emerged extending the result to all finite dimensions. This work crucially involves the notion of non-commutative Haar condition introduced in [**?**]. This work also establishes a still much to be explored relationship with Arveson' s notion of boundary representation.
The **non-commutative Haar condition** as follows.

Let $\mathcal{A}$ be a C^*-algebra with unit $1_{\mathcal{A}}$. For $x_1, x_2, ..., x_{n-1} \in \mathcal{A}$, let $\mathcal{V} = \mathbb{C}1_{\mathcal{A}} + \mathbb{C}x_1 + ...\mathbb{C}x_{n-1}$ be n dimensional. Then $\{1_{\mathcal{A}}, x_1, ...x_{n-1}\}$ is said to satisfy the non-commutative Haar condition if the following conditions are satisfied:

For a given $\lambda \in \mathbb{C}$,

(a) there are at most $n-1$ irreducible representations $(\pi_i, \mathcal{H}_i)$ (up to equivalence) and a non-zero vector $z_0 \in span\,(x_1, x_2..., x_{n-1})$ such that λ and $\overline{\lambda}$ are eigenvalues of $\pi_i(z_0)$ and $\pi_i(z_0^*)$ respectively $(i = 1, 2, ..., n-1)$.

(b) Assume that there are $m \leq n-1$ irreducible representations $(\pi_i, \mathcal{H}_i)$ (up to equivalence) and a non-zero vector $z_0 \in span\,(x_1, x_2..., x_{n-1})$ such that λ and $\overline{\lambda}$ are eigenvalues of $\pi_i(z_0)$ and $\pi_i(z_0^*)$ respectively $(i = 1, 2, ..., n-1)$. If n_i (respectively $\overline{n}_i$) $i = 1, 2, ..., m$ are the multiplicities of λ (respectively $\overline{\lambda}$) in $\mathcal{H}_i$, then
$$\sum_{i=1}^{m} n_i \leq n-1 \left(respectively \sum_{i=1}^{m} \overline{n_i} \leq n-1 \right).$$

Moreover, at least one eigenvector of $\tilde{\pi}_i(z_0^*)$ of the form $\tilde{\pi}_i(u)$ for some unitary $u \in \mathcal{A}$ which is not in $\mathcal{V}$ corresponding to $\overline{\lambda}$ is not orthogonal to $\tilde{\pi}_i(\mathcal{V})\tilde{\mathcal{H}}_i$ where $\left(\tilde{\pi}_i, \tilde{\mathcal{H}}_i\right)$ is the G.N.S representation corresponding to $(\mathcal{A}, \phi_i)$, ϕ_i is the pure state defined by $\phi_i(a) = \langle \pi_i(a)\xi_i, \xi_i \rangle$; $i = 1, 2, ..., m, a \in \mathcal{A}$ and ξ_i is an eigenvector of $\pi_i(z_0)$ corresponding to λ.

A vector $\omega_0 = \sum_{j=0}^{n-1} \beta_j (f_j \otimes a_j)$ in G satisfies condition (b) of the non-commutative Haar condition if and only if there exist at most m cyclic vectors $\xi_1, \xi_2, ..., \xi_m$ $(m \leq n-1)$ in $\mathbb{C}^N$, distinct points $x_1, x_2, ... x_m$ in X for the identity representation on C^N and unitary matrices $u_1, u_2, ... u_m$ in M_N such that

$$AB = \bar{\lambda} B \tag{1}$$

where

$$A = \begin{pmatrix} \bar{\beta}_1 \bar{f}_1(x_1) a_1^* & \bar{\beta}_2 \bar{f}_2(x_1) a_2^* & \ldots \bar{\beta}_{n-1} \bar{f}_{n-1}(x_1) a_{n-1}^* \\ \bar{\beta}_1 \bar{f}_1(x_2) a_1^* & \bar{\beta}_2 \bar{f}_2(x_2) a_2^* & \ldots \bar{\beta}_{n-1} \bar{f}_{n-1}(x_2) a_{n-1}^* \\ . & & \\ . & & \\ . & & \\ \bar{\beta}_1 \bar{f}_1(x_m) a_1^* & \bar{\beta}_2 \bar{f}_2(x_m) a_2^* & \ldots \bar{\beta}_{n-1} \bar{f}_{n-1}(x_m) a_{n-1}^* \end{pmatrix}$$

$B = diagonal(u_1(\xi_1), ..., u_m(\xi_m))$
and the diagonal matrix on the right side of (1) with non-zero diagonal entries is non-singular. Also the multiplicities n_i (respectively $\overline{n}_i$) of λ_0 (respectively $\bar{\lambda}_0$) satisfy the inequality
$\sum_{i=1}^{m} n_i \leq n-1 \left(respectively \sum_{i=1}^{m} \overline{n}_i \leq n-1 \right)$.

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Thank You!