Rational and H^∞ dilation

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Some definitions

- $\mathbb{D}$ denotes the unit disk in the complex plane and $\overline{\mathbb{D}}$ its closure.
- The disk algebra, $\mathbb{A}(\mathbb{D})$, is the closure of analytic polynomials in $C(\overline{\mathbb{D}})$, the space of continuous functions on $\overline{\mathbb{D}}$ with the supremum norm.
- The *Neil algebra* is the subalgebra of the disk algebra given by

$$\mathcal{A} = \{f \in \mathbb{A}(\mathbb{D}) : f'(0) = 0\} = \mathbb{C} + z^2\mathbb{A}(\mathbb{D}).$$

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- Constrained algebras, of which $\mathcal{A}$ is one of the simplest examples, are of current interest as a venue for function theoretic operator theory.
- Given $0 < q < 1$, let $\mathbb{A}$ denotes the annulus $\{z \in \mathbb{C} : q < |z| < 1\}$.
- $A(\mathbb{A})$ consists of those functions continuous on the closure of $\mathbb{A}$ and analytic in $\mathbb{A}$ in the uniform norm.
- The annuli can be identified with the distinguished varieties in $\mathbb{C}^2$ (ie, intersecting the boundary of $\mathbb{D}^2$ only in the torus $\mathbb{T}^2$) determined by

$$z^2 = \frac{w^2 - t^2}{1 - t^2 w^2}$$

- The limiting case, $z^2 = w^2$ corresponds to two disks intersecting at the origin $(0,0) \in \mathbb{C}^2$.
- The *Neil parabola* is the distinguished variety given by $z^2 = w^3$.

The rational dilation problem

- The *Sz.-Nagy dilation theorem* states that on a Hilbert space, every contraction operator (ie, operator norm less than or equal to 1) dilates to a unitary operator.
- Unitary operators are normal operators with spectrum contained in the boundary of $\mathbb{D}$; that is, $\mathbb{T}$.
- A corollary of the Sz.-Nagy dilation theorem is the *von Neumann inequality*, which implies that T is a contraction if and only if $\|p(T)\| \leq \|p\|$ for every polynomial p, where $\|p\|$ is the again the norm of p in $C(\overline{\mathbb{D}})$.

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- Given a compact subset X of $\mathbb{C}^d$, let $R(X)$ denote the algebra of rational functions with poles off of X with the norm $\|r\|_X$ equal to the supremum of the values of $|r(x)|$ for $x \in X$.
- The set X is a *spectral set* for the commuting d-tuple T of operators on the Hilbert space H if the spectrum of T lies in X and $\|r(T)\| \leq \|r\|_X$ for each $r \in R(X)$ (that is, a version of the von Neumann inequality holds).
- If N is also a d-tuple of commuting operators with spectrum in X and acting on the Hilbert space K, then T *dilates* to N provided there is an isometry $V : H \to K$ such that $r(T) = V^* r(N) V$ for all $r \in R(X)$.

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- *Rational dilation problem*: If X is a spectral set for T does T dilate to a tuple N of commuting normal operators with spectrum in the Shilov boundary of X relative to the algebra $R(X)$?

Known positive cases

- The Sz.-Nagy dilation theorem is just the statement that rational dilation holds for the closed disk.
- Foias and Lebow extended this to more general simply connected planar domains.
- Jim Agler showed that rational dilation holds for annuli.
- Andô's theorem is a two variable version of the Sz.-Nagy dilation theorem. Hence rational dilation holds on the bidisk $\mathbb{D}^2$ in $\mathbb{C}^2$.
- Jim Agler and Nicholas Young showed that rational dilation holds for the symmetrized bidisk.

Known negative cases

- An example due to Parrott shows that rational dilation fails for $\mathbb{D}^d$ for $d > 2$.
- Using computer algebra methods, Agler, Harland and Rafael showed that rational dilation fails for the unit disk with two particular smaller disks removed (a triply connected domain).
- Dritschel and McCullough showed that it fails for all compact triply connected regions with smooth boundary components.
- Pickering extended this to compact planar regions with higher connectivity as long as the Schottky double is hyperelliptic (automatic in the triply connected case).
- Sourav Pal has shown that it also fails for the tetrablock.

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- A tuple T acting on the Hilbert space H with spectrum in X determines a unital representation of π_T of $R(X)$ on H via $\pi_T(r) = r(T)$.
- The condition that X is a spectral set for T is equivalent to the condition that this representation is *contractive*.
- A representation π of $R(X)$ is *completely contractive* if for all n and all $F \in M_n(R(X))$, $\pi^{(n)}(F) := (\pi(F_{i,j}))$ is contractive, the norm of F being given by $\|F\|_\infty = \sup\{\|F(x)\| : x \in X\}$ with $\|F(x)\|$ the operator norm of $F(x)$.

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- Arveson proved that T dilates to a tuple N of normal operators with spectrum in the (Shilov) boundary of X (with respect to $R(X)$) if and only if π_T is completely contractive.
- *Reformulated rational dilation problem:* Is every contractive representation of $R(X)$ completely contractive?

The Schur-Agler class over $\mathbb{D}^3$

- As a special example, consider the tridisk $\mathbb{D}^3$. Define the *admissible kernels* to be $\mathcal{K} = \{k \geq 0 : (1 - z_j z_j^*) * k \geq 0\}$.
- The *Schur-Agler algebra* $H^\infty(\mathcal{K}, \mathcal{H})$ consists of those $\mathcal{L}(\mathcal{H})$-valued functions φ on $\mathbb{D}^3$ for which there is a $c > 0$ such that

$$(c1_{\mathcal{L}(\mathcal{H})} - \varphi\varphi^*) * (k \otimes 1_{\mathcal{L}(\mathcal{H})}) \geq 0.$$

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- We write $H^\infty(\mathcal{K})$ when $\mathcal{H} = \mathbb{C}$.
- The algebra $H^\infty(\mathcal{K})$ is an *operator algebra* when the operator space structure is defined as above. Write $A(\mathcal{K})$ for the subalgebra of those functions extending continuously to $\mathbb{T}^3$.
- It can be shown that any contractive representation of $A(\mathcal{K})$ is completely contractive.
- Any representation mapping the coordinate functions to irreducible commuting unitaries is an example of a *boundary representation*. These correspond to irreducible completely contractive representations with the property that they can only be dilated by direct sums.

- Consider the Parrott representation π with

$$\pi(z_1) = \begin{pmatrix} 0 & I_2 \\ 0 & 0 \end{pmatrix}, \quad \pi(z_2) = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}, \quad \pi(z_1) = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix},$$

$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

The Parrott representation and the Schur-Agler class over $\mathbb{D}^3$

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- This is irreducible and a straightforward calculation shows not only does it not have a unitary dilation, but that the only contractive dilations are by means of direct sums.
- Hence it is a boundary representation for $H^\infty(\mathcal{K})$.
- Kalyuzhnyi-Verbovetskii showed that this representation is *not* contractive for $H^\infty(\mathbb{D}^3, M_2(\mathbb{C}))$.

The Neil algebra and the Neil parabola

- Recall that $W = \{(z,w) \in \mathbb{D}^2 : z^2 = w^3\}$ in $\mathbb{C}^2$ is called the *Neil parabola*. It is a distinguished variety, and is a manifold except near the origin, where it has a cusp.
- Write $R(W)$ for the algebra of rational functions in two variables with poles off of W.
- The mapping from $R(W)$ to the Neil algebra $\mathcal{A}$ sending $p(z,w)$ to $p(t^2,t^3)$ is a (complete) isometry.

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- Hence any (completely) contractive representation of $\mathcal{A}$ induces a (completely) contractive representation of $R(W)$, and dilations translate as well.
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- Hence any (completely) contractive representation of $\mathcal{A}$ induces a (completely) contractive representation of $R(W)$, and dilations translate as well.
- So if we can solve the reformulated rational dilation problem for $\mathcal{A}$, this will lead to a solution of the rational dilation problem for $R(W)$.
- Studying the rational dilation problem on such varieties helps us to begin to understand more generally why rational dilation holds for some sets and not for others.

Completely contractive representations of $\mathcal{A}$

- As a (unital) Banach algebra, $\mathcal{A}$ is generated by the functions z^2 and z^3.
- Hence any bounded unital representation is determined by its values on these two functions. If $\pi : \mathcal{A} \to B(H)$ is a bounded representation, $X = \pi(z^2)$ and $Y = \pi(z^3)$, then X, Y are commuting operators which satisfy $X^3 = Y^2$.
- If we further insist that π is contractive, then X and Y are contractions

Theorem 1 (Broshinski).

A representation $\pi : \mathcal{A} \to B(H)$ is completely contractive if and only if there is a Hilbert space $K \supset H$ and a unitary operator $U \in B(K)$ such that for all $n \geq 0$, $n \neq 1$,

$$\pi(z^n) = P_H U^n|_H. \tag{1}$$

Example

- Let K be a separable Hilbert space with orthonormal basis $\{e_j\}_{j\in\mathbb{Z}}$, and let U be the bilateral shift.
- Let $H \subset K$ be defined as $H = e_0 \vee \bigvee_{n=2}^{\infty} e_n$.
- H is invariant for U^2 and U^3, and so by the above theorem, π given by $\pi(z^n) = P_H U^n|_H = U^n|_H$, $n \geq 0$, $n \neq 1$, is a completely contractive representation of $\mathcal{A}$.
- Suppose there were some $T \in B(H)$ with $T^2 = \pi(z^2)$ and $T^3 = \pi(z^3)$.
- Then $e_3 = U^3 e_0 = \pi(z^3)e_0 = \pi(z^2)Te_0$.
- However, $\left\langle \pi(z^2)e_n, e_3 \right\rangle = \left\langle U^2 e_n, e_3 \right\rangle = 0$ for $n \geq 0$, $n \neq 1$, and hence e_3 is orthogonal to the range of $\pi(z^2)$.
- Hence there is no way to define Te_0 so that $e_3 = \pi(z^2)Te_0$, and so there can be no such T.

A set of test functions for $\mathcal{A}$

- For $\lambda \in \mathbb{D}$, let
$$\varphi_\lambda(z) = \frac{z-\lambda}{1-\lambda^* z},$$
and
$$\psi_\lambda(z) = z^2 \varphi_\lambda(z).$$
Write ∞ for the point at infinity in the one point compactification $\mathbb{D}_\infty$ of $\mathbb{D}$ and set $\psi_\infty = z^2$.
- The set
$$\Psi = \{\psi_\lambda : \lambda \in \mathbb{D}_\infty\},$$
with the topology and Borel structure inherited from $\mathbb{D}_\infty$ is called a set of *test functions*.
- That is, for any $x \in \mathbb{D}$, the $\sup_{\psi \in \Psi} |\psi(x)| < 1$ and the elements of Ψ separate the points of $\mathbb{D}$.

Kernels

- For a set X and C^*-algebra $\mathcal{A}$, a function $k : X \times X \to \mathcal{A}$ is called a *kernel*. It is a *positive kernel* if for every finite subset $\{x_1, \ldots, x_n\}$ of X, $(k(x_i, x_j)) \in M_n(\mathcal{A})$ is positive semidefinite.

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- Let $M(\Psi)$ be the space of finite Borel measures on the set of test functions. Given $S \subseteq \mathbb{D}$, denote by $M^+(S) = \{\mu : S \times S \to M(\Psi)\}$ the collection of positive kernels on $S \times S$ into $M(\Psi)$. Write μ_{xy} for the value of μ at the pair (x, y).
- By μ being positive, we mean that for all finite sets $\mathcal{G} \subset S$ and all Borel sets $\omega \subset \Psi$, the matrix
$$(\mu_{x,y}(\omega))_{x,y \in \mathcal{G}}$$
is positive semidefinite.
- For example, μ could be identically equal to a fixed positive measure ν, or more generally be of the form $\mu_{xy} = f(x)f(y)^*\nu$ for a fixed positive measure ν and bounded measurable function $f : \mathbb{C} \to \mathbb{D}$, or more generally still be a finite sum of such terms.

Theorem 2 (Dritschel, Pickering).

An analytic function f in the disk belongs to $\mathcal{A}$ and satisfies $\|f\|_\infty \leq 1$ if and only if there is a positive kernel $\mu \in M^+(\mathbb{D})$ such that

$$1 - f(x)f(y)^* = \int_\Psi (1 - \psi(x)\psi(y)^*)\, d\mu_{xy}(\psi). \tag{2}$$

for all $x, y \in \mathbb{D}$. Furthermore, Ψ is minimal, in the sense that there is no proper closed subset of $E \subset \Psi$ such that for each such f, there exists a μ such that

$$1 - f(x)f(y)^* = \int_E (1 - \psi(x)\psi(y)^*)\, d\mu_{xy}(\psi). \tag{3}$$

A Kaiser-Varopoulos type example

- Recall that Kaiser and Varopoulos first showed that there exist three commuting contractions (T_1, T_2, T_3) such that the unital representation π of $R(\mathbb{D}^3)$ given by $\pi(z_j) = T_j$ is not contractive.

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- Coming back to the last theorem, for $E \subset \Psi$ a closed subset, let $C_{1,E}$ denote the cone consisting of the kernels

$$\left(\int_E (1 - \psi(x)\psi(y)^*)\, d\mu_{x,y}(\psi) \right)_{x,y \in \mathbb{D}}. \tag{4}$$

- In particular, if we choose $E = \{z^2, z^3\}$, it follows from this theorem that there exists a function $f \in \mathcal{A}$ with $\|f\|_\infty \leq 1$ such that $1 - f(x)f(y)^* \notin C_{1,E}$.

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We therefore have

Corollary 3.

There exists a pair of commuting contractive matrices X, Y with $X^3 = Y^2$, but such that the representation of $\mathcal{A}$ determined by $\pi(z^2) = X$, $\pi(z^3) = Y$ is not contractive.

How to disprove rational dilation

- Given $F \in M_2(\mathcal{A})$, let $\Sigma_{F,\mathcal{F}}$ denote the kernel

$$\Sigma_{F,\mathcal{F}} = (1 - F(x)F(y)^*)_{x,y\in\mathcal{F}}.$$

- Let I denote the ideal of functions in $\mathcal{A}$ which vanish on $\mathcal{F}$. Write $q : \mathcal{A} \to \mathcal{A}/I$ for the canonical projection, which is completely contractive.

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Lemma 4.

If $F \in M_2(\mathcal{A})$, but $\Sigma_{F,\mathcal{F}} \notin C_{2,\mathcal{F}}$, then there exists a Hilbert space H and representation $\tau : \mathcal{A}/I \to B(H)$ such that

(i) *For $a \in \mathcal{A}$, $\sigma(\tau(a)) \subset a(\mathcal{F})$;*

(ii) *For $a \in \mathcal{A}$ with $\|a\| \leq 1$, $\|\tau(q(a))\| \leq 1$; but*

(iii) $\|\tau^{(2)}(q(F^t))\| > 1$.

Therefore if it is the case that $\|F\| \leq 1$, then the representation $\tau \circ q$ is contractive, but not 2-contractive, and hence not completely contractive.

The counterexample for the Neil algebra

- Arveson's theorem tells us that for rational dilation to hold, contractive representations must be completely contractive.
- Roughly speaking, this will imply that when rational dilation holds, any matrix valued function on which the von Neumann inequality holds must "diagonalise", thus reducing the matrix case back to the scalar case.

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- Roughly speaking, this will imply that when rational dilation holds, any matrix valued function on which the von Neumann inequality holds must "diagonalise", thus reducing the matrix case back to the scalar case.
- For the Neil algebra, set

  $$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1 & 0 \\ 0 & 1 \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ 0 & \varphi_2 \end{pmatrix},$$

  where $\frac{1}{\sqrt{2}} U$ is a 2×2 unitary matrix with all non-zero entries. To be concrete, choose

  $$U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

- Then Φ is a 2×2 matrix inner function with $\det \Phi(\lambda) = 0$ at precisely the two nonzero points λ_1 and λ_2.
- The function

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  is in $M_2(\mathcal{A})$ and is a rational inner function, so $\|F\|_\infty = 1$.
- Ultimately we identify a set of six points $\mathcal{F}$ which is a set of uniqueness for F and show that $\Sigma_{F,\mathcal{F}}$ is not in a certain cone.
- We do this by showing that if $\Sigma_{F,\mathcal{F}}$ were in this cone, then it it would have to diagonalise, which is impossible with the choice of U we made in defining Φ.

Annuli

- Recall that given $0 < q < 1$, let $\mathbb{A}$ denotes the annulus $\{z \in \mathbb{C} : r < |z| < 1\}$, and that these can be identified with the distinguished varieties in $\mathbb{C}^2$ determined by
$$z^2 = \frac{w^2 - t^2}{1 - t^2 w^2}$$
- The limiting case, $V := \{(z, w) \in \mathbb{D}^2 : z^2 = w^2\}$ corresponds to two disks intersecting at the origin $(0, 0) \in \mathbb{C}^2$.
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- Agler first showed that rational dilation holds for annuli.
- A diagonalisation argument can be used to prove Agler's theorem.
- *Is there a geometric characterization of distinguished varieties of the bidisk where rational dilation holds/fails?*

H^∞ dilation

- As noted, by the Sz-Nagy dilation theorem, every contractive representation of the disk algebra $A(\mathbb{D})$ is completely contractive.
- It then follows that if π is a contractive, *weak-$*$ continuous* representation of H^∞, then π is completely contractive.
- In this case, the representation has the form $\pi(f) = f(T)$ for some completely nonunitary contraction T (or rather the operator is $T \oplus U$, where T is c.n.u. and U is a unitary with absolutely continuous spectrum).

H^∞ dilation

- As noted, by the Sz-Nagy dilation theorem, every contractive representation of the disk algebra $A(\mathbb{D})$ is completely contractive.
- It then follows that if π is a contractive, *weak-$*$ continuous* representation of H^∞, then π is completely contractive.
- In this case, the representation has the form $\pi(f) = f(T)$ for some completely nonunitary contraction T (or rather the operator is $T \oplus U$, where T is c.n.u. and U is a unitary with absolutely continuous spectrum).
- Because of the complicated nature of the maximal ideal space of $H^\infty(\mathbb{D})$, there are contractive unital representations which are *not* weak-$*$ continuous.

Basics of the maximal ideal space of $H^\infty(\mathbb{D})$

- Write $\mathfrak{M}$ for the maximal ideal space of H^∞ equipped with the hull-kernel topology.
- View $f \in H^\infty$ as a continuous function on $\mathfrak{M}$ via $\widehat{f}(\phi) = \phi(f)$.
- Embed $\mathbb{D}$ in $\mathfrak{M}$ by sending $\lambda \in \mathbb{D}$ to the ideal $\{f \in H^\infty : f(\lambda) = 0\}$.

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- Embed $\mathbb{D}$ in $\mathfrak{M}$ by sending $\lambda \in \mathbb{D}$ to the ideal $\{f \in H^\infty : f(\lambda) = 0\}$.
- Everything else in $\mathfrak{M}$ is fibered over the circle: for $\alpha \in \mathbb{T}$, the fiber $\mathfrak{M}_\alpha$ consists of all those ϕ for which $\phi(z) = \alpha$.
- The Shilov boundary of H^∞ lies entirely in the fibers and is canonically identified with the maximal ideal space of L^∞.

Gleason parts

- Each point $m \in \mathfrak{M}$ lies in a unique *Gleason part* $P(m)$.
- For $m \in \mathbb{D}$, $P(m) = \mathbb{D}$. Otherwise, $P(m)$ is either a one-point part or an *analytic disk*. Such disks are the images of 1-1 mappings $L : \mathscr{D} \to \mathfrak{M}$, where $\mathscr{D}$ is another copy of $\mathbb{D}$ used to parameterize mappings.

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- The map L is analytic in the sense that for all $f \in H^\infty$, the function $\widehat{f} \circ L$ is analytic in $\mathscr{D}$.
- If m lies in an analytic disk, then there is an interpolating sequence $S = \{\alpha_n\} \subset \mathbb{D}$ such that m lies in the closure of S; moreover there is a net $\alpha_{n(i)}$ such that the maps $L_\alpha : \mathscr{D} \to \mathfrak{M}$

$$L_\alpha(z) = \frac{z + \alpha}{1 + \alpha^* z} \tag{5}$$

  converge pointwise to an analytic map L_m taking $\mathscr{D}$ bijectively to the Gleason part $P(m)$.

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  converge pointwise to an analytic map L_m taking $\mathscr{D}$ bijectively to the Gleason part $P(m)$.
- Note that there need *not* exist an H^∞ function f with $f \circ L_m(z) = z$; this happens if and only if the map L_m is a homeomorphism, which is not always the case.

Basic representations

- We consider representations of H^∞ having finite rank; that is $\pi : H^\infty \to M_n$. More precisely:
- A representation π will be called *basic* if there exists a finite set $\mathfrak{F} \subset \mathfrak{M}$ and a positive kernel $k : \mathfrak{F} \times \mathfrak{F} \to \mathbb{C}$ such that

$$\pi(f)^* k_m = \widehat{f}(m)^* k_m, \tag{6}$$

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- In this case, say π is *supported in $\mathfrak{F}$*.
- Note that $\ker \pi = \bigcap_{m \in \mathfrak{F}} m$. Let $I_{\mathfrak{F}} \subset H^\infty$ denote the ideal of functions vanishing on $\mathfrak{F}$.
- A basic representation factors through the quotient map $q_{\mathfrak{F}} : H^\infty \to H^\infty / I_{\mathfrak{F}}$.
- A basic representation with kernel k is contractive if and only if the matrices

$$(1 - \widehat{f}(m_i)\widehat{f}(m_j)^*)k(m_i, m_j)$$

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  are positive semidefinite for all $f \in H^\infty$.
- We can show that for a contractive basic representation, if $m_1, m_2 \in \mathfrak{F}$ lie in distinct Gleason parts of $\mathfrak{M}$, then $k(m_1, m_2) = 0$, and so any contractive basic representation splits as a direct sum of contractive basic representations supported in distinct Gleason parts.

Contractive implies completely contractive for basic representations

Theorem 5.
Every contractive basic representation of $H^\infty(\mathbb{D})$ is completely contractive.

- Idea of the proof: A contractive basic representation splits into direct sums over distinct Gleason parts. Three cases:

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- If one part is $\mathbb{D}$, the direct summand is weak-$*$ continuous.
- If the Gleason part consists of one point, the direct summand is scalar valued and thus completely contractive.
- Finally, if the Gleason part is an analytic disk $P(m) \subset \mathfrak{M} \setminus \mathbb{D}$, we pull back the kernel over these points to $\mathscr{D}$ via L_m, and the induced representation is completely contractive. Then the direct summand is a composition of completely contractive maps, and so is completely contractive.

Motivation and questions

- Let $\mathcal{A}_F$ denote the (operator) subalgebra of $H^\infty(\mathbb{D})$ consisting of those functions f which extend continuously to $\mathbb{T}$ at the points in F and satisfy $f(z) = f(0)$ for all $z \in F$.
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- This is sort of like the Neil algebra.
- However, unlike the Neil algebra, we discovered that every contractive representation of $\mathcal{A}_F$ is completely contractive!
- There are various directions one can pursue in trying to extend the result on contractive basic representations to all contractive representations:
- Some sort of compactness argument?
- Use Blaschke products associated to interpolating sequences as test functions and ...?


That's all Folks!