The Bergman kernel and the Bergman metric

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kernel functions

Let $\mathcal{D}$ be a domain in $\mathbb{C}^d$, V be a normed linear space and $K : \mathcal{D} \times \mathcal{D} \to V$ be a function, which is holomorphic in the first variable and anti-holomorphic in the second.

For two functions of the form $K(\cdot, w_i)\zeta_i$, ζ_i in V ($i = 1, 2$), define their inner product by the reproducing property, that is,

$$\langle K(\cdot, w_1)\zeta_1, K(\cdot, w_2)\zeta_2\rangle = \langle K(w_2, w_1)\zeta_1, \zeta_2\rangle.$$

This extends to an inner product on the linear span of the vectors

$$\mathcal{H}_0 = \{ \sum K(\cdot, w)\zeta_i | \zeta_1, \ldots, \zeta_n \in V; w_1, \ldots, w_n \in \mathcal{D} \text{ and } n \in \mathbb{N}\}$$

if and only if K is positive definite in the sense that

$$\begin{aligned} \sum_{j,k=1}^{n} \langle K(z_j, z_k)\zeta_k, \zeta_j\rangle &= \sum_{k=1}^{n} \langle K(\cdot, z_k)\zeta_k, \sum_{j=1}^{n} K(\cdot, z_j)\zeta_j\rangle \\ &= \| \sum_{k=1}^{n} \langle K(\cdot, z_k)\zeta_k \|^2 > 0. \end{aligned}$$

Gram matrix

The completion $\mathcal{H}$ of the linear space $\mathcal{H}_0$ is a Hilbert space with respect to the inner product induced by $K,$ or equivalently,

$$\langle f, K(\cdot, w)\zeta\rangle_{\mathcal{H}} = \langle f(w), \zeta\rangle_V, w \in \mathcal{D}, \zeta \in V.$$

Let $G : \mathcal{D} \times \mathcal{D} \to V$ be the Grammian $G(z, w) = \big(\!\big(\langle u_j(w), u_k(z)\rangle\big)\!\big)_{j,k}$ of a set of $r(:= \dim V)$ anti-holomorphic functions $u_\ell : \mathcal{D} \to \mathcal{H},$ $1 \leq \ell \leq r,$ taking values in some Hilbert space $\mathcal{H}.$ We have

$$\begin{aligned}
\sum_{p,q=1}^{n} \langle G(z_p, z_q)^\sharp \zeta_q, \zeta_p\rangle_V &= \sum_{j,k=1}^{r} \sum_{pq=1}^{n} G(z_p, z_q)_{j,k} \zeta_q(j)\overline{\zeta_p(k)} \\
&= \sum_{j,k=1}^{r} \Big(\sum_{pq=1}^{n} \langle u_j(z_q), u_k(z_p)\rangle \zeta_q(j)\overline{\zeta_p(k)} \Big) \\
&= \| \sum_{jk} \zeta_q(j) u_q(z_q) \|^2 > 0.
\end{aligned}$$

We therefore conclude that $G(z, w)^\sharp$ defines a positive definite kernel on $\mathcal{D}.$

orthonormal basis

Let $\{e_\ell : \mathcal{D} \xrightarrow{\text{hol}} V,\ \ell \in \mathbb{N}\}$ be an orthonormal basis in the Hilbert space $\mathcal{H}$. Given $\zeta \in V$, let $\zeta^\sharp$ be the function $\eta \to \langle \eta, \zeta \rangle_V$. Thus $\zeta^\sharp$ defines an element in V^*. Assume that $f \to f(w), w \in \mathcal{D}$ is uniformly locally bounded. Then the sum $\sum_\ell e_\ell(z) e_\ell(w)^\sharp$, is convergent on compact subsets of $\mathcal{D}$. It also has the reproducing property:

$$\begin{aligned}
\Big\langle f(\cdot), \sum_\ell e_\ell(\cdot) e_\ell(w)^\sharp \zeta \Big\rangle &= \Big\langle f(\cdot), \sum_\ell e_\ell(\cdot) \langle \zeta, e_\ell(w) \rangle \Big\rangle \\
&= \sum_\ell \langle e_\ell(w), \zeta \rangle \, \langle f(\cdot), e_\ell(\cdot) \rangle \\
&= \langle f(w), \zeta \rangle,\ \zeta \in V.
\end{aligned}$$

Since K is uniquely determined by the reproducing property, we have

$$K(z, w) = \sum_\ell e_\ell(z) e_\ell(w)^\sharp.$$

example

For $\zeta \in V$, let $\zeta^\dagger$ be the linear map $\xi \to \langle \xi, \zeta \rangle_V$. For any domain $\mathcal{D}$ in V, the function $K : \mathcal{D} \times \mathcal{D} \to \mathrm{Hom}(V, V)$ defined by the formula $K(z, w) = zw^\sharp$ is positive definite, whereas $K(z, w)^\sharp$ is not!

For the Bergman space $\mathbb{A}^2(\mathbb{D}^m)$, of the polydisc $\mathbb{D}^m$, the orthonormal basis is $\{\sqrt{\prod_{i=1}^m (n_i + 1)} z^I : I = (i_1, \ldots, i_m)\}$. Clearly, we have

$$B_{\mathbb{D}^m}(z, w) = \sum_{|I|=0}^{\infty} \Big(\prod_{i=1}^{m} (n_i + 1)\Big) z^I \bar{w}^I = \prod_{i=1}^{m} (1 - z_i \bar{w}_i)^{-2}.$$

Similarly, for the Bergman space of the ball $\mathbb{A}^2(\mathbb{B}^m)$, the orthonormal basis is $\{\sqrt{\binom{-m-1}{|I|}\binom{|I|}{I}} z^I : I = (i_1, \ldots, i_m)\}$. Again, it follows that

$$B_{\mathbb{B}^m}(z, w) = \sum_{|I|=0}^{\infty} \binom{-m-1}{\ell} \Big(\sum_{|I|=\ell} \binom{|I|}{I} z^I \bar{w}^I\Big) = (1 - \langle z, w \rangle)^{-m-1}.$$

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new from old

Let W be a second finite dimensional inner product space and $T : \mathcal{H} \to \mathrm{Hol}(\mathcal{D}, W)$ be a linear map for which the evaluation at $z \in \mathcal{D},$ namely, $f \to (Tf)(z), f \in \mathcal{H},$ is continuous. Transplant the inner product from $\mathcal{H}/\ker T$ to the linear space $T\mathcal{H}.$ In consequence, $T_{(z)}K(z,w)T^{\sharp}_{(w)} : W \to W$ is the reproducing kernel of $T\mathcal{H}$:

$$TK(z,w)\zeta := \big(T_{(z)}K_w\zeta\big)(z) = \sum_\ell \langle \zeta, e_\ell(w)\rangle (Te_\ell)(z).$$

Linearity in ζ implies that $TK(z,w)$ is in $\mathrm{Hom}(V, T\mathcal{H}).$ We have

$$T_{(z)}K(z,w) = \sum_\ell (Te_\ell(z))e_\ell(w)^\sharp$$

and

$$K(z,w)T^\sharp := \big(T_{(w)}K(w,z)\big)^\sharp = \sum_\ell e_\ell(z)(Te_\ell(w))^\sharp$$

(For fixed $w,$ $\{Te_\ell(w)^\sharp\zeta\}$ is in ℓ^2 for all $\zeta.$) Applying T to this we have

$$TK(z,w)T^\sharp = \sum_\ell (Te_\ell)(z)\big(Te_\ell(w)^\sharp\big).$$

the inclusion map

Suppose $\mathcal{H} \subseteq \mathrm{Hol}(\mathcal{D}, V)$ is a Hilbert space possessing a reproducing kernel K and $T : \mathcal{H} \to \mathrm{Hol}(\mathcal{D}, W)$ is a linear map such that $f \to (Tf)(z), f \in \mathcal{H},$ is continuous. Let $\mathcal{H}' \subseteq \mathrm{Hol}(\mathcal{D}, W)$ be another Hilbert space with reproducing kernel $K' : \mathcal{D} \times \mathcal{D} \to \mathrm{Hom}(W, W)$.

Lemma

If $TK(z,w)T^\sharp \prec CK'(z,w),$ then the image of T is contained in $\mathcal{H}'$ and as an operator from $\mathcal{H}$ to $\mathcal{H}',$ it is bounded by $C.$

Proof. Without loss of generality, may assume $C = 1.$ If $\mathcal{H}_i,\ i = 1, 2$ are two Hilbert spaces with reproducing kernels $K_i,\ i = 1, 2,$ then their sum is the reproducing kernel of the Hilbert space

$$\{g | g = f_1 + f_2 \text{ for some } f_1 \in \mathcal{H}_1 \textit{ and } f_2 \in \mathcal{H}_2\}$$

equipped with the norm $\|g\|^2 = \inf\{\|f_1\|^2 + \|f_2\|^2 | g = f_1 + f_2\}.$

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the proof, continued

Apply this with $\mathcal{H}_1 := T\mathcal{H}, \quad K_1 := TKT^\sharp.$ Set $\mathcal{H}_2$ to be the Hilbert space corresponding to the kernel function $K_2 := K' - K_1,$ which is positive definite by assumption. For f in $\mathcal{H},$ write $f = f_1 + f_2,$ where $f_1 = Tf$ and $f_2 = 0.$ Then we have

$$\|Tf\|^2_{\mathcal{H}'} \leq \|Tf\|^2_{\mathcal{H}_1} = \|Tf\|^2_{T\mathcal{H}} \leq \|f\|^2_{\mathcal{H}}.$$

quasi-invariance of B

Any bi-holomorphic map $\varphi : \mathcal{D} \to \tilde{\mathcal{D}}$ induces a unitary operator $U_\varphi : \mathbb{A}^2(\tilde{\mathcal{D}}) \to \mathbb{A}^2(\mathcal{D})$ defined by the formula

$$(U_\varphi f)(z) = (J(\varphi, z)\big(f \circ \varphi\big)(z),\ f \in \mathbb{A}^2(\tilde{\mathcal{D}}),\ z \in \mathcal{D}.$$

This is an immediate consequence of the change of variable formula for the volume measure on $\mathbb{C}^n$.

Consequently, if $\{\tilde{e}_n\}_{n\geq 0}$ is any orthonormal basis for $\mathbb{A}^2(\tilde{\mathcal{D}})$, then $\{e_n\}_{n\geq 0}$, where $\tilde{e}_n = J(\varphi, \cdot)(\tilde{e}_n \circ \varphi)$ is an orthonormal basis for the Bergman space $\mathbb{A}^2(\tilde{\mathcal{D}})$.

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Expressing the Bergman kernel $B_{\mathcal{D}}$ of the domains $\mathcal{D}$ as the infinite sum $\sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}$ using the orthonormal basis in $\mathbb{A}^2(\mathcal{D})$, we see that the Bergman Kernel B is *quasi-invariant*, that is, If $\varphi : \mathcal{D} \to \widetilde{\mathcal{D}}$ is holomorphic then we have the transformation rule

$$J(\varphi, z)B_{\tilde{\mathcal{D}}}(\varphi(z), \varphi(w))\overline{J(\varphi, w)} = B_{\mathcal{D}}(z, w),$$

where $J(\varphi, w)$ is the Jacobian determinant of the map φ at w.

If $\mathcal{D}$ admits a transitive group of bi-holomorphic automorphisms, then this transformation rule gives an effective way of computing the Bergman kernel. Thus

$$B_{\mathcal{D}}(z, z) = |J(\varphi_z, z)|^2 B_{\mathcal{D}}(0, 0), \; z \in \mathcal{D},$$

where φ_z is the automorphism of $\mathcal{D}$ with the property $\varphi_z(z) = 0$.

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the multiplier

Consider the special case, where $\varphi : \mathcal{D} \to \mathcal{D}$ is an automorphism. Clearly, in this case, U_φ is unitary on $\mathbb{A}^2(\mathcal{D})$ for all $\varphi \in \mathrm{Aut}(\mathcal{D})$.

The map $J : \mathrm{Aut}(\mathcal{D}) \times \mathcal{D} \to \mathbb{C}$ satisfies the cocycle property, namely

$$J(\psi\varphi, z) = J(\varphi, \psi(z))J(\psi, z),\ \varphi, \psi \in \mathrm{Aut}(\mathcal{D}),\ z \in \mathcal{D}.$$

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more representations

Exploit the quasi-invariance of the Bergman kernel to construct unitary representations of the automorphism group $\mathrm{Aut}(\mathcal{D})$. Let $B^{\lambda}(z,w)$ be the polarization of the function $B(w,w)^{\lambda}$, $w \in \mathcal{D}$, $\lambda > 0$.

Now, as before,

$$J_{\varphi}(z)^{\lambda} B^{\lambda}(\varphi(z),\varphi(w)) \overline{J_{\varphi}(w)}^{\lambda} = B^{\lambda}(z,w),\ \varphi \in \mathrm{Aut}(\mathcal{D}),\ z,w \in \mathcal{D}.$$

Let $\mathcal{O}(\mathcal{D})$ be the ring of holomorphic functions on $\mathcal{D}$. Define

$$U^{(\lambda)} : \mathrm{Aut}(\mathcal{D}) \to \mathrm{End}(\mathcal{O}(\mathcal{D}))$$

by the formula

$$(U^{(\lambda)}_{\varphi} f)(z) = \big(J_{\varphi^{-1}}(z)\big)^{\lambda} (f \circ \varphi^{-1})(z)$$

and note that $\varphi \mapsto U_{\varphi}$ is a homomorphism.

When is it unitarizable?

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new kernels?

Let K be a complex valued positive definite kernel on $\mathcal{D}$. For w in $\mathcal{D}$, and p in the set $\{1, \ldots, d\}$, let $e_p : \Omega \to \mathcal{H}$ be the antiholomorphic function:

$$e_p(w) := K_w(\cdot) \otimes \frac{\partial}{\partial \bar{w}_p} K_w(\cdot) - \frac{\partial}{\partial \bar{w}_p} K_w(\cdot) \otimes K_w(\cdot).$$

Setting $G(z,w)_{p,q} = \langle e_p(w), e_q(z) \rangle$, we have

$$\frac{1}{2} G(z,w)_{p,q}{}^{\sharp} = K(z,w) \frac{\partial^2}{\partial z_q \partial \bar{w}_p} K(z,w) - \frac{\partial}{\partial \bar{w}_p} K(z,w) \frac{\partial}{\partial z_q} K(z,w)).$$

The curvature K of the metric K is given by the $(1,1)$ - form $\sum \frac{\partial^2}{\partial w_q \partial \bar{w}_p} \log K(w,w) dw_q \wedge d\bar{w}_p$. Set

$$\mathcal{K}_K(z,w) := \left(\!\!\left(\frac{\partial^2}{\partial z_q \partial \bar{w}_p} \log K(z,w) \right)\!\!\right)_{qp}.$$

We note that $K(z,w)^2 \mathcal{K}(z,w) = \frac{1}{2} G(z,w)^{\sharp}$. Hence $K(z,w)^2 \mathcal{K}(z,w)$ defines a positive definite kernel on $\mathcal{D}$ taking values in $\mathrm{Hom}(V,V)$.

transformation rule

Let $\varphi : \mathcal{D} \to \mathcal{D}$ be a holomorphic map. Applying the change of variable formula twice to the function $\log K(\varphi(z), \varphi(w))$, we have

$$\left(\!\!\left(\frac{\partial^2}{\partial z_i \partial \bar{w}_j} \log K(\varphi(z), \varphi(w)) \right)\!\!\right)_{ij} = \left(\!\!\left(\frac{\partial \varphi_\ell}{\partial z_i} \right)\!\!\right)_{i\ell} \left(\!\!\left(\left(\frac{\partial^2}{\partial z_\ell \partial \bar{w}_k} \log K \right)(\varphi(z), \varphi(w)) \right)\!\!\right)_{\ell k} \left(\!\!\left(\frac{\partial \bar{\varphi}_k}{\partial \bar{z}_j} \right)\!\!\right)_{kj}.$$

Now, we set $K(w,w) = B_{\mathcal{D}}(w,w)$, the Bergman kernel of $\mathcal{D}$, which transforms according to the rule:

$$\det{}_{\mathbb{C}} D\varphi(w) B_{\mathcal{D}}(\varphi(w), \varphi(w)) \overline{\det{}_{\mathbb{C}} D\varphi(w)} = B_{\mathcal{D}}(w,w),$$

Thus $\mathcal{K}_{B_{\mathcal{D}} \circ (\varphi, \varphi)}(w,w)$ equals $\mathcal{K}_{B_{\mathcal{D}}}(w,w)$. Hence we conclude that $\mathcal{K} := \mathcal{K}_{B_{\mathcal{D}}}$ is invariant under the automorphisms φ of $\mathcal{D}$ in the sense that

$$D\varphi(w)^{\sharp} \mathcal{K}(\varphi(w), \varphi(w)) \overline{D\varphi(w)} = \mathcal{K}(w,w), \;\; w \in \mathcal{D}.$$

rewrite the transformation rule

Or equivalently,

$$\begin{aligned}\mathcal{K}(\varphi(z),\varphi(w)) &= D\varphi(z)^{\sharp^{-1}}\mathcal{K}(z,w)\overline{D\varphi(z)}^{-1}\\ &= D\varphi(z)^{\sharp^{-1}}\mathcal{K}(z,w)\left(D\varphi(w)^{\sharp^{-1}}\right)^{*}\\ &= m_0(\varphi,z)\mathcal{K}(z,w)m_0(\varphi,w)^{*},\end{aligned}$$

where $m_0(\varphi,z)=D\varphi(z)^{\sharp^{-1}}$ and multiplying both sides by K^2, we have

$$K(\varphi(z),\varphi(w))^2\mathcal{K}(\varphi(z),\varphi(w)) = m_2(\varphi,z)K(z,w)^2\mathcal{K}(z,w)m_2(\varphi,w)^{*},$$

where $m_2(\varphi,z)=\left(\det_{\mathbb{C}} D\varphi(w)^2 D\varphi(z)^{\sharp}\right)^{-1}$ is a multiplier. Of course, we now have that

(i) $K^{2+\lambda}(z,w)\mathcal{K}(z,w)$, $\lambda>0$, is a positive definite kernel and

(ii) it transforms according with $m_\lambda(\varphi,z)=\left(\det_{\mathbb{C}} D\varphi(z)^{2+\lambda}D\varphi(z)^{\dagger}\right)^{-1}$ in place of $m_2(\varphi,z)$.

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(ii) it transforms according with $m_\lambda(\varphi,z) = \left(\det_{\mathbb{C}} D\varphi(z)^{2+\lambda} D\varphi(z)^{\dagger}\right)^{-1}$ in place of $m_2(\varphi,z)$.

rewrite the transformation rule

Or equivalently,

$$\begin{aligned}\mathcal{K}(\varphi(z),\varphi(w)) &= D\varphi(z)^{\sharp^{-1}}\mathcal{K}(z,w)\overline{D\varphi(z)}^{-1} \\ &= D\varphi(z)^{\sharp^{-1}}\mathcal{K}(z,w)\left(D\varphi(w)^{\sharp^{-1}}\right)^{*} \\ &= m_0(\varphi,z)\mathcal{K}(z,w)m_0(\varphi,w)^{*},\end{aligned}$$

where $m_0(\varphi,z) = D\varphi(z)^{\sharp^{-1}}$ and multiplying both sides by K^2, we have

$$K(\varphi(z),\varphi(w))^2\mathcal{K}(\varphi(z),\varphi(w)) = m_2(\varphi,z)K(z,w)^2\mathcal{K}(z,w)m_2(\varphi,w)^{*},$$

where $m_2(\varphi,z) = \left(\det_{\mathbb{C}} D\varphi(w)^2 D\varphi(z)^{\sharp}\right)^{-1}$ is a multiplier. Of course, we now have that

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Thank you!