Schur coupling and other equivalence relations for operators on a Hilbert space.

Dan Timotin

IMAR

OTOA – Bangalore 2014

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Definition (Bart–Gohberg–Kaashoek 1983)

T, S *matricially coupled* $\Leftrightarrow$ there exist A_{ij}, B_{ij} such that

$$\begin{pmatrix} T & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & S \end{pmatrix}$$

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Use: transfer of properties, formulas, etc from T to S.
Application (BGK): $S = I - K$, K integral operator; T finite dimensional matrix.

Equivalence after extension

$$\begin{pmatrix} T & 0 \\ 0 & I_Y \end{pmatrix} = \overbrace{\begin{pmatrix} A_{12} & T \\ A_{22} & A_{21} \end{pmatrix}}^{\text{invertible}} \begin{pmatrix} S & 0 \\ 0 & I_X \end{pmatrix} \overbrace{\begin{pmatrix} -A_{21} & I_Y \\ B_{11}T & B_{12} \end{pmatrix}}^{\text{invertible}}$$

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Bart-Tsekanovski 1991

T, S matricially coupled $\Leftrightarrow T, S$ equivalent after extension.

Schur

$$D \text{ invertible } \Rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}$$

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BGKR:

T, S Schur coupled $\Rightarrow$ T, S equivalent after extension.

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T, S equivalent after one-sided extension $\Rightarrow$ T, S Schur coupled.

So:

$$\text{e.o.e} \Rightarrow \text{S.c.} \Rightarrow \text{e.e.}$$

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The simple case

If T, S have closed range, then all relations are equivalent to the kernel conditions:

$$\dim\ker T = \dim\ker S, \quad \dim\ker T^* = \dim\ker S^*.$$

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1. T is not in the norm closure of invertible operators.
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- Otherwise, we use an approximation procedure.

But it is not a concrete description. For instance, it does not answer the simple question: When are two compact operators equivalent?

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Köthe (1936)...

Illustration: compact positive operators

A, B compact positive operators

Eigenvalues:

$$A : \alpha_1 \geqslant \alpha_2 \geqslant \cdots \to 0;$$
$$B : \beta_1 \geqslant \beta_2 \geqslant \cdots \to 0.$$

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$$A : \alpha_1 \geqslant \alpha_2 \geqslant \cdots \to 0;$$
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- $A \sim B$ if and only if there exists $\delta > 0$ such that

$$\delta \leqslant \frac{\alpha_j}{\beta_j} \leqslant \frac{1}{\delta}, \quad \forall j \geqslant 1.$$

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- e.o.e. $\Leftrightarrow$ S.c. $\Leftrightarrow$ e.e. $\Leftrightarrow$

there exists $\delta > 0$ and $k \geqslant 0$ such that

$$\text{either } \delta \leqslant \frac{\alpha_{j+k}}{\beta_j} \leqslant \frac{1}{\delta} \text{ or } \delta \leqslant \frac{\alpha_j}{\beta_{j+k}} \leqslant \frac{1}{\delta} \quad \forall j \geqslant 1,.$$

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If $A = UBV$, with U, V unitary, then

$$\dim E_A([a,b]) = \dim E_B([a,b])$$

for any $0 < a \leqslant b < \infty$.

Strong equivalence

Condition $\mathfrak{S}$ for $A, B \geqslant 0$

There exists $0 < \delta < 1$ such that for any $0 < a \leqslant b < \infty$

$$\dim E_A([a, b]) \leqslant \dim E_B([\delta a, \frac{1}{\delta} b]),$$
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Theorem (Fillmore-Williams)

TFAE:

1. $T \sim S$,
2. - Kernel conditions
   ($\dim\ker T = \dim\ker S$, $\quad \dim\ker T^* = \dim\ker S^*$).
   - $|T|, |S|$ satisfy condition $\mathfrak{S}$.

The weaker form

Condition $\tilde{\mathfrak{S}}$ for $A, B \geqslant 0$

There exists $0 < \delta < 1$ and $\rho > 0$ such that for any $0 < a \leqslant b < \rho$

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Theorem (DT)

TFAE:

1. T, S equivalent after extension.
2. - Kernel conditions,
   - $|T|, |S|$ satisfy condition $\widetilde{\mathfrak{S}}$.
3. T, S equivalent after one-sided extension.

A taste of the details

Main implication: ❷ $\Rightarrow$ ❸

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Particular case

Suppose $A, B \geqslant 0$ are compact, $\dim\ker A = \dim\ker B = 0$, and eigenvalues are simple:

$$A : \alpha_1 > \alpha_2 > \cdots \to 0;$$
$$B : \beta_1 > \beta_2 > \cdots \to 0.$$

If A, B satisfy condition $\widetilde{\mathfrak{S}}$, then they are equivalent after one sided finite extension.

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$$A = U^* BU.$$
- If
$$\delta \leqslant \frac{\alpha_i}{\beta_i} \leqslant \frac{1}{\delta},$$
then $E : y_i \mapsto \frac{\alpha_i}{\beta_i} x_i$ is invertible, and
$$A = EBU.$$

Reminder

Condition $\tilde{\mathfrak{S}}$ for $A, B \geqslant 0$

There exists $0 < \delta < 1$ and $\rho > 0$ such that for any $0 < a \leqslant b < \rho$

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In particular, for $a = b = \alpha_i < \rho$:

There exists β_j with

$$\delta \leqslant \frac{\alpha_i}{\beta_j} \leqslant \frac{1}{\delta}$$

Starting the proof

We want to prove:

Suppose $A, B \geqslant 0$ are compact, $\dim\ker A = \dim\ker B = 0$, and eigenvalues are simple:

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If A, B satisfy condition $\widetilde{\mathfrak{S}}$, then they are equivalent after one sided finite extension.

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1. Define $\mathfrak{a} \subset \mathbb{N}$ by $\mathfrak{a} = \{j \in \mathbb{N} : \alpha_j \leqslant \rho\}$.

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- For any $j_1, \dots, j_N \in \mathfrak{a}$, the union $\bigcup_\ell p(j_\ell)$ contains at least N elements.

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Hall's Marriage Lemma: There exists a one–to–one map $\phi : \mathfrak{a} \to \mathbb{N}$ such that $\phi(j) \in p(j)$ for all j.

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Equivalently,

$$\delta \leqslant \frac{\alpha_j}{\beta_{\phi(j)}} \leqslant \frac{1}{\delta}$$

Proof (continued)

2. Similarly, if $\mathfrak{b} := \{j \in \mathbb{N} : \beta_j \leqslant \rho\}$, there exists a one-to-one map $\psi : \mathfrak{b} \to \mathbb{N}$, such that for all j

$$\delta \leqslant \frac{\alpha_{\psi(j)}}{\beta_j} \leqslant \frac{1}{\delta}$$

Proof (continued)

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3. Define $\Phi : \mathcal{P}(\mathbb{N})$ by

$$\Phi(E) = \mathbb{N} \setminus \psi\Big[\big(\mathbb{N} \setminus \phi(E \cap \mathfrak{a})\big) \cap \mathfrak{b}\Big]$$

Proof (continued)

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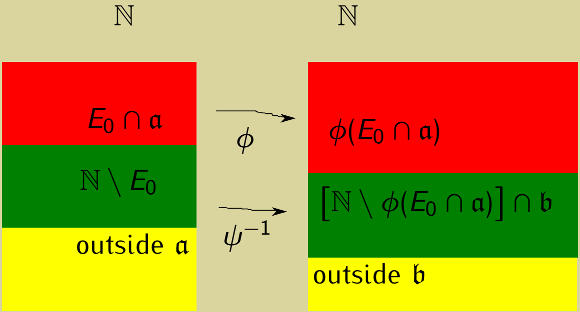
The Cantor–Bernstein argument: there exists a fixed point of this map!

Proof (continued)

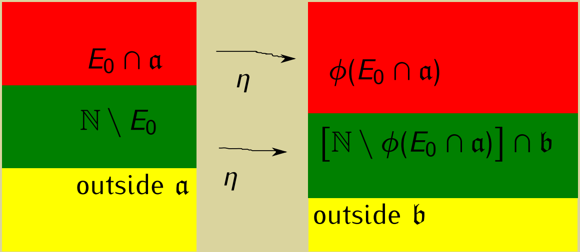
$$\mathbb{N} \setminus E_0 = \psi\Big[\big(\mathbb{N} \setminus \phi(E_0 \cap \mathfrak{a})\big) \cap \mathfrak{b}\Big]$$



Proof (continued)

4. We may define a bijection $\eta : ■ \cup ■ \to ■ \cup ■$:

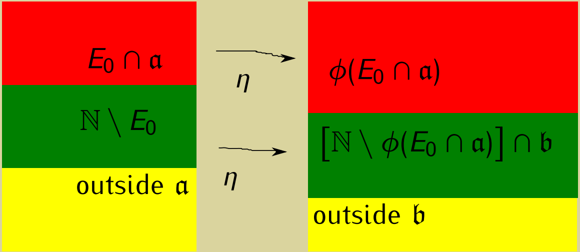
$$\eta = \begin{cases} \phi & : E_0 \cap \mathfrak{a} \to \phi(E_0 \cap \mathfrak{a}) \\ \psi^{-1} & : \mathbb{N} \setminus E_0 \to \big(\mathbb{N} \setminus \phi(E_0 \cap \mathfrak{a})\big) \cap \mathfrak{b} \end{cases}$$



Then

$$\delta \leqslant \frac{\alpha_j}{\beta_{\eta(j)}} \leqslant \frac{1}{\delta} \text{ for } j \in ■ \cup ■$$

Proof (continued)



Then

$$\mathfrak{a}' := \text{left } \blacksquare \subset \{j \in \mathbb{N} : \alpha_j \geqslant \rho\}$$
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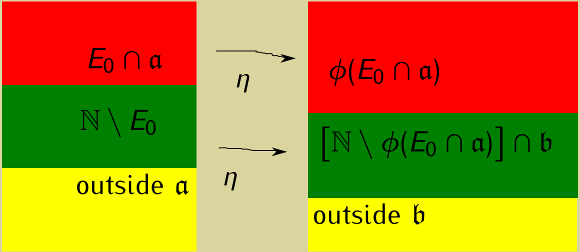

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So $\mathfrak{a}', \mathfrak{b}'$ are finite, and for any $i \in \mathfrak{a}'$, $j \in \mathfrak{b}'$,

$$\frac{\rho}{\max\{||A||, ||B||\}} \leqslant \frac{\alpha_i}{\beta_j} \leqslant \frac{\max\{||A||, ||B||\}}{\rho}$$

Proof (continued)

Case 1. If $\#\mathfrak{a}' = \#\mathfrak{b}'$, we may extend η to a bijection $\tilde{\eta} : \mathbb{N} \to \mathbb{N}$, such that for any $j \in \mathbb{N}$,

$$\delta' \leqslant \frac{\alpha_j}{\beta_{\tilde{\eta}(j)}} \leqslant \frac{1}{\delta'}$$

where

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$$\delta' = \min\{\delta, \frac{\rho}{\max\{||A||, ||B||\}}\}.$$

$$E : x_i \mapsto y_{\tilde{\eta}(i)} \text{ is unitary}$$

$$F : y_i \mapsto \frac{\alpha_i}{\beta_i} x_{\tilde{\eta}^{-1}(i)} \text{ is invertible}$$

and $A = FBE$, so $A \sim B$.

Proof (end)

Case 2. If $\#\mathfrak{a}' > \#\mathfrak{b}'$, we add to Y a finite dimensional space Y', $\dim Y' = \ell := \#\mathfrak{a}' - \#\mathfrak{b}'$, and consider $A, B' := B \oplus I_{Y'}$. If β'_j are the eigenvalues of B', then again we obtain a bijection $\tilde{\eta} : \mathbb{N} \to \mathbb{N} \cup \{1, \dots, \ell\}$, such that for any $j \in \mathbb{N}$,

$$\delta' \leqslant \frac{\alpha_j}{\beta_{\tilde{\eta}(j)}} \leqslant \frac{1}{\delta'}$$

Proof (end)

Case 2. If $\#\mathfrak{a}' > \#\mathfrak{b}'$, we add to Y a finite dimensional space Y', $\dim Y' = \ell := \#\mathfrak{a}' - \#\mathfrak{b}'$, and consider $A, B' := B \oplus I_{Y'}$. If β_j' are the eigenvalues of B', then again we obtain a bijection $\tilde{\eta} : \mathbb{N} \to \mathbb{N} \cup \{1, \dots, \ell\}$, such that for any $j \in \mathbb{N}$,

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Case 3. Similarly, if $\#\mathfrak{a}' < \#\mathfrak{b}'$, then

$A \oplus I_\ell \sim B$ for some $\ell > 0$.

General compact operators

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Corollary (for compact operators)

T, S are equivalent after extension if and only if

1. the kernel conditions are satisfied;
2. their respective singular values are comparable after a shift: for some $k \geqslant 0$
   - either
   $$\delta' < \frac{\alpha_{i+k}}{\beta_i} < \frac{1}{\delta'}$$
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Corollary

If T, S are equivalent after extension and T belongs to some ideal of compact operators, then S belongs to the same ideal.

THANK YOU!