Analyticity and subnormality of operator-valued functions

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Introduction

$\mathcal{H}$, $\mathcal{K}$ - complex Hilbert spaces, $\mathbf{B}(\mathcal{H})$ - bounded linear operators on $\mathcal{H}$

$S \in \mathbf{B}(\mathcal{H})$ is *subnormal* if
• there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ (isometric embedding) and
• a normal operator $N \in \mathbf{B}(\mathcal{K})$ such that $Sh = Nh$ for all $h \in \mathcal{H}$.

Let $\varphi\colon \Omega \to \mathbf{B}(\mathcal{H})$ (Ω - any set), then φ is *jointly subnormal* if
• there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and
• a function $\Phi\colon \Omega \to \mathbf{B}(\mathcal{K})$ such that $\Phi(\omega)$, $\omega \in \Omega$, are commuting normal operators and
• $\varphi(\omega) = \Phi(\omega)|_{\mathcal{H}}$ for all $\omega \in \Omega$.
Φ is called a *normal extension of* φ.

Φ is called a *minimal* normal extension if additionally:
$\mathcal{K}$ is a unique closed linear subspace of $\mathcal{K}$ such that
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$$\varOmega \ni z \mapsto (z - S)^{-1} \in \mathbf{B}(\mathcal{H})$$

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Some history

Itô (1958): commutative families of subnormals

Lubin (1970's):

• discrete semigroup of subnormals with no normal extension

• neither sum nor product of commuting subnormals need to be subnormal

S, T – commuting subnormals, such that $aS + bT$ is subnormal ($a, b \in \mathbb{C}$). Is the function $(z, w) \mapsto zS + wT$ jointly subnormal?

Striking example by Catepillán and Szymanski (2004): V_1, V_2 isometries with orthogonal ranges. Then

$$\begin{aligned}\|(aV_1 + bV_2)f\|^2 &= |a|^2\|V_1 f\|^2 + |b|^2\|V_2 f\|^2 \\ &= (|a|^2 + |b|^2)\|f\|^2\end{aligned}$$

So $aV_1 + bV_2$ is a multiple of an isometry, hence subnormal. However, V_1 and V_2 can never commute.

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Theorem

Let $\varphi\colon \Omega \to \mathbf{B}(\mathcal{H})$ *be a function defined on a nonempty set* Ω*. Then the following conditions are equivalent*:

(A) φ *is jointly subnormal,*

(B) *for every integer* $n \geqslant 1$*, for all n-sequences* $\{h_\alpha\}_{\alpha\in\mathbb{N}^n} \subseteq \mathcal{H}$ *with finite number of nonzero entries and for all n-tuples* $(\omega_1, \ldots, \omega_n) \in \Omega^n$

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Mapping Θ in action

Corollary

$\varphi\colon \Omega \to \mathbf{B}(\mathcal{H})$ is such that $\varphi|_{\Omega_0}$ is jointly subnormal ($\varnothing \neq \Omega_0 \subseteq \Omega$) and $\varphi(\Omega) \subseteq \overline{\operatorname{alg}\varphi(\Omega_0)}^{\mathrm{SOT}}$. Then

- *φ is jointly subnormal,*
- *Φ is minimal $\Rightarrow \Phi|_{\Omega_0}$ is minimal.*

Proof. $\Phi = \Theta \circ \varphi$, where $\Theta : \overline{\operatorname{alg}\varphi(\Omega_0)}^{\mathrm{SOT}} \to \mathbf{B}(\mathcal{K})$.

Θ is a tool for deriving many nice properties of Φ from those of φ, e.g. φ continuous $\Rightarrow \Phi$ continuous,
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Analytic operator-valued functions

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This is a generalized theorem of Globevnik & Vidav, 1973; they considered only $\mathcal{X} = \mathbb{C}$.

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$U \in \mathbf{B}(\mathcal{H})$ unitary, $T \in \mathbf{B}(\mathcal{H})$, satisfy

$$T^*U = T^*T \quad \text{and} \quad TU \neq UT. \tag{1}$$

Remark. *T satisfies (1) $\iff$ $T = UP$ with some orthogonal projection P such that $PU \neq UP$.*

Let $\varphi\colon \mathbb{C} \to \mathbf{B}(\mathcal{H})$ be defined by $\quad \varphi(z) = U + zT, \quad z \in \mathbb{C}$.

Then

- *$E \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |1 + z| = 1\}$ is a set of uniqueness in $\mathbb{C}$ and*
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$$T^*U = T^*T \quad \text{and} \quad TU \neq UT. \tag{1}$$

Remark. *T satisfies (1) $\iff$ $T = UP$ with some orthogonal projection P such that $PU \neq UP$.*

Let $\varphi\colon \mathbb{C} \to \mathbf{B}(\mathcal{H})$ be defined by $\quad \varphi(z) = U + zT, \quad z \in \mathbb{C}$.

Then

- *$E \overset{\text{def}}{=} \{z \in \mathbb{C}\colon |1 + z| = 1\}$ is a set of uniqueness in $\mathbb{C}$ and*
- *$\varphi(E)$ consists of unitary operators,* **but**
- *$\varphi(\mathbb{C})$ is not commutative and $\varphi(\mathbb{C})$ is not a family of normal operators.*

If moreover U and T are such that $T(\mathcal{H}) \not\subseteq T^*(\mathcal{H})$ and $T^*(\mathcal{H}) \not\subseteq T(\mathcal{H})$, then

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Proof. $\varphi(E)$ is composed of unitaries because

$$(U+zT)(U+zT)^* = UU^* + zTU^* + \bar{z}UT^* + |z|^2 TT^* = \ldots$$

now, $T^*U = T^*T$ implies $TU^* = TT^*$ ($\leftarrow$ nice exercise):

$$\ldots = I + (2\,\mathfrak{Re}\, z + |z|^2)TT^* = I + (\underbrace{|z+1|^2 - 1}_{=0 \text{ on } E})TT^* = I$$

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One more look on the G-V theorem

Theorem

$\mathcal{X}$ is a normed space, $\Omega \subseteq \mathcal{X}$ is open and connected,
$\varphi\colon \Omega \to \mathbf{B}(\mathcal{H})$ is analytic, $\varnothing \neq \Omega_0 \subseteq \Omega$ is open,
$\varphi(\Omega_0)$ consists of normal operators.
Then $\varphi(\Omega)$ consists of commuting normal operators.

Question. May we assume that $\varphi(\Omega_0)$ consists of **subnormal** operators?
Answer is yes/no/I don't know.
The Catepillán–Szymanski example shows that the function

$$\mathbb{C} \ni z \mapsto V_1 + zV_2 \in \mathbf{B}(\mathcal{H})$$

solves the problem in the negative. (Recall: V_1, V_2 – isometries with orthogonal ranges.)

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Example

There exists analytic $\varphi\colon \mathbb{C} \to \mathbf{B}(\mathcal{H})$ *such that*

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Construction.
Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $\mathcal{H}_1 \neq \{0\}$ *and* $\mathcal{H}_2 \neq \{0\}$,
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Why?

Proof of 1° follows from an auxiliary fact:
If V is an isometry, then

$$S = \begin{bmatrix} V & X \\ 0 & 0 \end{bmatrix} \in \mathbf{B}(\mathcal{H}), \quad (\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2)$$

is subnormal $\iff$ $\|S\| \leqslant 1 \iff \|X\| \leqslant 1$ and $V(\mathcal{H}_1) \perp X(\mathcal{H}_2)$

The proof of subnormality is based on
• $\{\|S^n h\|^2\}_{n=0}^\infty$ is constant for $n \geqslant 1$
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$$\|S^n h\|^2 = \int_0^\infty t^n \mathrm{d}\mu_h(t), \quad n \geqslant 0,$$

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If $S = \begin{bmatrix} V & X \\ 0 & 0 \end{bmatrix} \in \mathbf{B}(\mathcal{H})$ is subnormal, then

$$N \stackrel{\text{def}}{=} \begin{bmatrix} V & X & Q & P & 0 & 0 \\ 0 & 0 & 0 & 0 & |X|^2 & |X|D \\ 0 & 0 & 0 & 0 & D^*|X| & D^*D \\ 0 & 0 & 0 & V^* & 0 & 0 \\ 0 & 0 & 0 & X^* & 0 & 0 \\ 0 & 0 & 0 & Q^* & 0 & 0 \end{bmatrix}$$

defined on $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ is a normal extension of S, where
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Question

$\Omega \subseteq \mathbb{C}$ nonempty connected open,
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$\varphi(\Omega)$ – commuting family.
Does φ have to be jointly subnormal?

Answer is yes/no/I don't know

Thank you!

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