Orthogonality in C^*-algebras

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Presence of algebraic orthogonality in C^*-algebras.
Let A be a C^*-algebra. Then for each $a \in A_{sa}$, there exists a unique pair $a^+, a^- \in A^+$ such that

- $a = a^+ - a^-$
- $a^+a^- = 0$
- $\|a\| = \max\{\|a^+\|, \|a^-\|\}$.

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(T) $|a + b| \leq |a| + |b|$ for all $a, b \in A_{sa}$.

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Theorem. Let A be a C^*-algebra. If A is commutative, then the following condition holds:

(T) $|a + b| \leq |a| + |b|$ for all $a, b \in A_{sa}$.

Conversely, if (T) holds, then A_{sa} is isometrically order isomorphic to the self-adjoint part of a commutative C^*-algebra.

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Then $x * y \geq x$ and $x * y \geq y$.
Next, let $z \in A_{sa}$ such that $x \leq z$ and $y \leq z$. Then

$$\begin{aligned} 2(z - x * y) &= 2z - x - y - |x - y| \\ &= (z - x) + (z - y) - |(z - x) - (z - y)| \\ &\geq (z - x) + (z - y) - |z - x| - |z - y| = 0. \end{aligned}$$

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Thus $x * y = x \vee y$.
Now, by Kakutani's representation theorem for AM-spaces, A_{sa} is order isomorphic to $C_{\mathbb{R}}(K)$ for some compact and Hausdorff space K.

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Definition. Let (V, V^+) be a real ordered vector space such that V^+ is proper and generating and let $\| \ \|$ be a norm on V such that V^+ is $\| \ \|$-closed. For a fixed real number p, $1 \leq p < \infty$, consider the following conditions on V:

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- (O.p.1) For $u, v, w \in V$ with $u \leq v \leq w$, we have $\|v\| \leq (\|u\|^p + \|w\|^p)^{1/p}$.

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- (O.p.2) For $v \in V$ and $\epsilon > 0$, there are $u_1, u_2 \in V^+$ such that $v = u_1 - u_2$ and $(\|u_1\|^p + \|u_2\|^p)^{1/p} \leq \|v\| + \epsilon$.

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- (O.p.2) For $v \in V$ and $\epsilon > 0$, there are $u_1, u_2 \in V^+$ such that $v = u_1 - u_2$ and $(\|u_1\|^p + \|u_2\|^p)^{1/p} \leq \|v\| + \epsilon$.
- (OS.p.2) For $v \in V$, there are $u_1, u_2 \in V^+$ such that $v = u_1 - u_2$ and $(\|u_1\|^p + \|u_2\|^p)^{1/p} = \|v\|$.

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Theorem. Let (V, V^+) be a real ordered vector space such that V^+ is proper and generating and let $\| \cdot \|$ be a norm on V such that V^+ is $\| \cdot \|$-closed. For a fixed $p, 1 \leq p \leq \infty$, we have

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Theorem. Let (V, V^+) be a real ordered vector space such that V^+ is proper and generating and let $\|\cdot\|$ be a norm on V such that V^+ is $\|\cdot\|$-closed. For a fixed $p, 1 \leq p \leq \infty$, we have

- $\|\cdot\|$ satisfies the condition $(O.p.1)$ on V if and only if $\|\cdot\|'$ satisfies the condition $(OS.p'.2)$ on V'.

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Definition (Order smooth p-normed space).
Let (V, V^+) be a real ordered vector space such that V^+ is proper and generating and let $\|\|$ be a norm on V such that V^+ is $\|\|$-closed. For a fixed p, $1 \leq p \leq \infty$, we say that V is an order smooth p-normed space, if $\|\|$ satisfies conditions $O.p.1$ and $O.p.2$ on V.

Geometric orthogonality

Definition. Let $1 \leq p \leq \infty$ and let V be an order smooth p-normed space. For $u, v \in V$ we say that u is p-orthogonal to v, ($u \perp_p v$), if

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Definition. Let $1 \leq p \leq \infty$ and let V be an order smooth p-normed space. For $u, v \in V$ we say that u is p-orthogonal to v, ($u \perp_p v$), if

$$\|u + kv\|^p = \|u\|^p + \|kv\|^p, \quad 1 \leq p < \infty$$

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for all $k \in \mathbb{R}$.

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Further, we say that $\perp_p$ is additive in V, if $u \perp_p v$ and $u \perp_p w$ implies $u \perp_p (v + w)$. Note that in this case $u^{\perp_p} = \{v \in V : u \perp_p v\}$ is a subspace of V.

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A subset S of V is called p-orthogonal if $0 \notin S$ and $u \perp_p v$ for every pair $u, v \in S$ with $u \neq v$. If, in addition, $\|v\| = 1$ for all $v \in S$, we say that S is a p-orthonormal set in V. We say that S is total if the linear span of S is dense in V.

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Theorem. Let $1 \leq p \leq \infty$ and let V be a (norm) complete order smooth p-normed space. If $\perp_p$ is additive in V^+ and U is a total p-orthonormal set in V^+, then V is isometrically order isomorphic to $\ell_p(U)$. For $p = \infty$, we replace $\ell_p(U)$ by $c_0(U)$.

Geometric orthogonality

Definition. Let V be an order smooth p-normed space, $1 \leq p \leq \infty$. For $v \in V \setminus \{0\}$ we say that $f \in V'$ supports v if $\|f\| = 1$ and $\|v\| = f(v)$. The set of all supports of v will be denoted by $Supp(v)$. For $u \in V^+ \setminus \{0\}$, we write, $Supp_+(u)$ for $Supp(u) \cap V'^+$.

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By Hahn-Banach theorem, $Supp(v) \neq \emptyset$, if $v \in V \setminus \{0\}$. Moreover, it is weak*-compact and convex too.

Proposition. Let V be an order smooth p-normed space, $1 \leq p \leq \infty$. For $u \in V^+ \setminus \{0\}$, $Supp_+(u) \neq \emptyset$.

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Proposition. Let V be an order smooth p-normed space, $1 \leq p \leq \infty$. For $u \in V^+ \setminus \{0\}$, $Supp_+(u) \neq \emptyset$.
The extreme points in $Supp_+(v)$ will be denoted by $Peak_+(v)$.

Geometric ∞-orthogonality

Theorem. Let V be an order smooth ∞-normed space. Suppose that $u_1, u_2 \in V^+ \setminus \{0\}$ and let W be the linear span of u_1, u_2. Then the following statements are equivalent:

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3. For $f_i \in Supp_+(u_i)$, $i = 1, 2$, we have $g_1 \perp_1 g_2$ with $g_i(u_j) = 0$ if $i \neq j$, where $g_i = f_i|_W$, $i = 1, 2$.

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Proof. Without any loss of generality, we may assume that $\|u_i\| = 1$, $i = 1, 2$. First, let $\|u_1 + u_2\| = 1$. Let $\lambda > 0$.

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Case.1: $\lambda \leq 1$.
In this case,

$$\begin{aligned} \|u_1 + \lambda u_2\| &= \|\lambda(u_1 + u_2) + (1 - \lambda)u_1\| \\ &\leq \lambda\|u_1 + u_2\| + (1 - \lambda)\|u_1\| = 1. \end{aligned}$$

Geometric ∞-orthogonality

Case.2: $\lambda > 1$.
In this case,

$$\begin{aligned}\|u_1 + \lambda u_2\| &= \|(u_1 + u_2) + (\lambda - 1)u_2\| \\ &\leq \|u_1 + u_2\| + (\lambda - 1)\|u_2\| = \lambda.\end{aligned}$$

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Thus, in either case, $\|u_1 + \lambda u_2\| \leq \max(1, \lambda)$. Further, as $\lambda > 0$ we also have

$$\max(1, \lambda) = \max(\|u_1\|, \|\lambda u_2\|) \leq \|u_1 + \lambda u_2\|.$$

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Thus for $\lambda \geq 0$, we get $\|u_1 + \lambda u_2\| = \max(\|u_1\|, \|\lambda u_2\|)$.

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Again, for $\lambda > 0$, $-\lambda u_2 \leq u_1 - \lambda u_2 \leq u_1$ so that

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Next, let $f_i \in Supp_+(u_i)$, $i = 1, 2$. Then $f_i(u_i) = 1 = \|f_i\|$, $i = 1, 2$. As $\|u_1 + u_2\| = 1$ we get

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Now it follows that

$$\|u_1 - \lambda u_2\| \geq |f_1(u_1 - \lambda u_2)| = 1$$

and

$$\|u_1 - \lambda u_2\| \geq |f_2(u_1 - \lambda u_2)| = \lambda$$

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$$\|u_1 - \lambda u_2\| \geq \max(1, \lambda) = \max(\|u_1\|, \|\lambda u_2\|).$$

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Therefore, $u_1 \perp_\infty u_2$.
Next, assume that $u_1 \perp_\infty u_2$. Let $f_i \in Supp_+(u_i)$, $i = 1, 2$ and put $g_i = f_i|_W$, $i = 1, 2$. Then as above, we have $g_i(u_j) = 0$ if $i \neq j$.

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For $\alpha_1, \alpha_2 \in \mathbb{R}$, we have

$$\begin{aligned}\|\alpha_1 g_1 + \alpha_2 g_2\| &= \sup\{|(\alpha_1 g_1 + \alpha_2 g_2)(\lambda_1 u_1 + \lambda_2 u_2)| : \\ &\qquad\qquad \|\lambda_1 u_1 + \lambda_2 u_2\| \leq 1\} \\ &= \sup\{|\alpha_1 \lambda_1 + \alpha_2 \lambda_2| : \max(|\lambda_1|, |\lambda_2|) \leq 1\} \\ &= |\alpha_1| + |\alpha_2|.\end{aligned}$$

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Finally assume that for $f_i \in Supp_+(u_i)$, $i = 1, 2$, we have $g_1 \perp_1 g_2$ with $g_i(u_j) = 0$ if $i \neq j$ where $g_i = f_i|_W$, $i = 1, 2$. It is easy to note that $\|g_i\| = 1$, $i = 1, 2$. Thus

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$$\begin{aligned}\|u_1 + u_2\| &= \sup\{|(\alpha_1 g_1 + \alpha_2 g_2)(u_1 + u_2)| : \\ &\qquad\qquad \|\alpha_1 g_1 + \alpha_2 g_2\| \leq 1\} \\ &= \sup\{|\alpha_1 + \alpha_2| : |\alpha_1| + |\alpha_2| \leq 1\} = 1.\end{aligned}$$

This completes the proof.

Geometric ∞-orthogonality

Remark. Let $u, v \in V^+ \setminus \{0\}$ such that $u \perp_\infty v$. Then for $f \in Supp_+(v), f(u) = 0$.

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Further, $Support_+(u) \subset Crust_+(v)$ and consequently, $Peak_+(u) \subset Sink_+(v)$.

Orthogonality in C^*-algebras

Theorem. Let A be a C^*-algebra and let $a, b \in A^+ \setminus \{0\}$. Let B be the C^*-subalgebra of A generated by a and b. Then $ab = 0$ if and only if $P(B) = Sink_+^B(a) \cup Sink_+^B(b)$.
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Proof. Let $ab = 0$. Then $P(B)$ is multiplicative on B. Thus

$$0 = f(ab) = f(a)f(b)$$

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Conversely, assume that $P(B) = Sink_+^B(a) \cup Sink_+^B(b)$. If $f \in P(B)$, then by the Cauchy-Schwarz' inequality,

$$0 \leq |f(ab)|^2 \leq f(a^2)f(a^2) \leq \|a\|\|b\| f(a)f(b).$$

Thus by assumption, $f(ab) = 0$ for all $f \in P(B)$ so that $ab = 0$.

Orthogonality in C^*-algebras

Definition. Let A be any C^*-algebra and let M be any self-adjoint subspace of A such that M_{sa} is an order smooth ∞-normed sub space of A_{sa} satifying $(OS.\infty.2)$. (For example, M is a C^*- subalgebra of A.) For $a, b \in M^+$, we say that a is orthogonal to b with respect to M ($a \perp^M b$) if $P(M) \subset Sink_+(a) \cup Sink_+(b)$.

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Question. Can we have a “space-free” geometric characterization of algebraic Orthogonality?

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Theorem. Let a qnd b be any two positive elements of a C^*-algebra A. Then $ab = 0$ if and only if $ab = ba$ and the following property holds:

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Thus $c \perp_\infty b$ or equivalently,

$$\big\| \|c\|^{-1} c + \|b\|^{-1} b \big\| = 1.$$

Orthogonality in C^*-algebras

Since $0 \leq c \leq b$ we have

$$1 = \big\| \|c\|^{-1}c + \|b\|^{-1}b \big\| \geq \big\| \|c\|^{-1}c + \|b\|^{-1}c \big\| = 1 + \|b\|^{-1}\|c\| > 1$$

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Conversely assume that $ab = ba$ and that $0 \leq c \leq a$ and $c \leq b$ implies $c = 0$.
Since $a, b \in A^+$, we have $ab \in A^+$. Also, then $ab \leq \|a\|b$ and $ab \leq \|b\|a$.

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Conversely assume that $ab = ba$ and that $0 \le c \le a$ and $c \le b$ implies $c = 0$.
Since $a, b \in A^+$, we have $ab \in A^+$. Also, then $ab \le \|a\| b$ and $ab \le \|b\| a$.
Thus $0 \le k^{-1} ab \le a$ and $k^{-1} ab \le b$ where $k = \max\{\|a\|, \|b\|\}$.
Now, by assumption, $ab = 0$.

Orthogonality in C^*-algebras

Conjecture. Let a qnd b be any two positive elements of a C^*-algebra A. Then $ab = 0$ if and only if $0 \leq c \leq a$ and $0 \leq d \leq b$ implies $c \perp_\infty d$.

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Proof. Let $0 \leq c \leq a$ and $0 \leq d \leq b$. Then $0 \leq bcb \leq bab = 0$ so that $cb = 0$. Thus $0 \leq cdc \leq cbc = 0$ so that $cd = 0$.

Orthogonality in C^*-algebras

Theorem. Let a qnd b be any two positive elements in M_n^+. Then $ab = 0$ if and only if $0 \leq c \leq a$ and $0 \leq d \leq b$ implies $c \perp_\infty d$.

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Proof. Assume that $0 \leq c \leq a$ and $0 \leq d \leq b$ implies $c \perp_\infty d$. Consider the spectral decompositions $a = \sum_{i=1}^k \lambda_i p_i$ and $b = \sum_{j=1}^l \mu_j q_j$ where $\lambda_i \neq 0$ and $\mu_j \neq 0$ for $1 \leq i \leq k$ and $1 \leq j \leq l$.

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Then $\lambda_i p_i \leq a$ and $\mu_j q_j \leq b$. Thus by assumption, $p_i \perp_\infty q_j$.

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Then $\lambda_i p_i \leq a$ and $\mu_j q_j \leq b$. Thus by assumption, $p_i \perp_\infty q_j$. It follows that $\|p_i + q_j\| = 1$ so that $p_i + q_j \leq 1$. Since p_i and q_j are projections, we get that $p_i q_j = 0$. Hence $ab = 0$.

Thank you!