The Cauchy dual subnormality problem

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joint work with A. Anand, S. Chavan and J. Stochel
A solution to the Cauchy dual subnormality problem for 2-isometries
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$$T' := T(T^*T)^{-1}.$$

2. The operator T' is again left-invertible and has the property that $(T')' = T$.
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- Given $T \in \boldsymbol{B}(\mathcal{H})$, set

$$\mathcal{S}^n(T)f := \sum_{0 \leq p \leq n} (-1)^p \binom{n}{p} \|T^p f\|^2, \quad f \in \mathcal{H}.$$

- An operator $T \in \boldsymbol{B}(\mathcal{H})$ is said to be a *n-isometry* if

$$\mathcal{S}^n(T)f = 0 \text{ for } f \in \mathcal{H},$$

- *k-hyperexpansive* if

$$\mathcal{S}^n(T)f \leq 0 \text{ for } n = 1, \ldots, k \text{ and } f \in \mathcal{H},$$

- *completely hyperexpansive* if

$$\mathcal{S}^n(T)f \leq 0 \text{ for } n \geq 1 \text{ and } f \in \mathcal{H}.$$

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Set $\triangle_T = T^*T - I$ for $T \in \boldsymbol{B}(\mathcal{H})$.

Recall that an operator $T \in \boldsymbol{B}(\mathcal{H})$ is said to be

- a *Brownian isometry* if T is a 2-isometry such that

$$\triangle_T \triangle_{T^*} \triangle_T = 0,$$

- a $\triangle_T$*-regular* operator if $\triangle_T \geqslant 0$ and

$$\triangle_T T = \triangle_T^{1/2} T \triangle_T^{1/2},$$

- a *quasi-Brownian isometry* if T is a $\triangle_T$-regular 2-isometry.

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Theorem (Agler-Stankus'95, Majdak-Mbekhta-Suciu '16, Anand-Chavan-J-Stochel)

If $T \in \boldsymbol{B}(\mathcal{H})$, then the following conditions are equivalent:

(i) *T is a quasi-Brownian isometry* (*resp., Brownian isometry*),

(ii) *T has the block matrix form*

$$T = \begin{bmatrix} V & E \\ 0 & U \end{bmatrix}$$

with respect to an orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ (one of the summands may be absent), where $V \in \boldsymbol{B}(\mathcal{H}_1)$, $E \in \boldsymbol{B}(\mathcal{H}_2, \mathcal{H}_1)$ and $U \in \boldsymbol{B}(\mathcal{H}_2)$ are such that

$$V^*V = I,\ V^*E = 0,\ U^*U = I \text{ and } UE^*E = E^*EU$$

$$(\textit{resp., } V^*V = I,\ V^*E = 0,\ U^*U = I = UU^*,\ UE^*E = E^*EU).$$

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An operator $S \in \boldsymbol{B}(\mathcal{H})$ is

- *subnormal* if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator $N \in \boldsymbol{B}(\mathcal{K})$ such that $N\mathcal{H} \subseteq \mathcal{H}$ and $S = N|\mathcal{H}$,
- *hyponormal* if $\|S^* f\| \leq \|Sf\|, \quad f \in \mathcal{H}.$

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- All positive integer powers of the Cauchy dual of a 2-hyperexpansive operator turn out to be hyponormal (Chavan 2013).
- If T is a completely hyperexpansive weighted shift, then T' is a subnormal contraction and the reverse implication is not true (A. Athavale, 1996).
- **Cauchy dual subnormality problem.** Is the Cauchy dual of a 2-isometry (or more general, a completely hyperexpansive operator) a subnormal contraction?

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The kernel condition

- We say that T satisfies the *kernel condition*, if

$$T^*T(\ker T^*) \subseteq \ker T^*.$$

Recall that if $\mathcal{M}$ is a nonzero Hilbert space and $\{W_n\}_{n=0}^{\infty} \subseteq \boldsymbol{B}(\mathcal{M})$, then the operator $W \in \boldsymbol{B}(\ell^2_{\mathcal{M}})$ defined by

$$W(h_0, h_1, \ldots) = (0, W_0 h_0, W_1 h_1, \ldots), \quad (h_0, h_1, \ldots) \in \ell^2_{\mathcal{M}},$$

is said to be an *operator valued unilateral weighted shift* with weights $\{W_n\}_{n=0}^{\infty}$. Putting $\mathcal{M} = \mathbb{C}$, we arrive at the well-known notion of a unilateral weighted shift in ℓ^2.

Characterization of 2-isometric operators satisfying the kernel condition.

Theorem

If T is a 2-isometry in $\boldsymbol{B}(\mathcal{H})$, then the following are equivalent:

(i) $T^*T(\ker T^*) \subseteq \ker T^*$,

(ii) $T^*T(\ker T^*) = \ker T^*$,

(iii) $T(\ker T^*) \perp T^n(\ker T^*)$ *for every integer* $n \geqslant 2$,

(iv) *the spaces* $\{T^n(\ker T^*)\}_{n=0}^{\infty}$ *are mutually orthogonal,*

(v) $T \cong U \oplus W$, *where* U *is a unitary operator and* W *is an operator valued unilateral weighted shift with invertible weights,*

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(vi) *$T \cong U \oplus W$, where U is a unitary operator and W is an operator valued unilateral weighted shift in $\ell^2_{\mathcal{M}}$ with weights*

$$W_n = \int_{[1,\infty)} \xi_n(x)\, E(dx), \quad n \geqslant 0,$$

where

$$\xi_n(x) = \sqrt{\frac{1+(n+1)(x^2-1)}{1+n(x^2-1)}}, \quad x \in [1,\infty),\ n = 0, 1, \ldots,$$

and E is a compactly supported $\boldsymbol{B}(\mathcal{M})$-valued Borel spectral measure on the interval $[1,\infty)$.

Moreover, if T is as in (vi)*, then T is a* 2*-isometry in $\boldsymbol{B}(\mathcal{H})$ and $\dim \ker T^* = \dim \mathcal{M}$*

Theorem (Lambert '76)

An operator $S \in \boldsymbol{B}(\mathcal{H})$ is subnormal if and only if for every $f \in \mathcal{H}$, the sequence $\{\|S^n f\|^2\}_{n=0}^{\infty}$ is a Stieltjes moment sequence, i.e., there exists a positive Borel measure μ_f on $[0, \infty)$ such that

$$\|S^n f\|^2 = \int_{[0,\infty)} t^n d\mu_f(t), \quad n = 0, 1, 2, \ldots .$$

Lemma

Let $a, b \in \mathbb{R}$ be such that $a + bn \neq 0$ for every $n \in \mathbb{Z}_+$ and let $\gamma_{a,b} = \{\gamma_{a,b}(n)\}_{n=0}^{\infty}$ be a sequence given by

$$\gamma_{a,b}(n) = \frac{1}{a + bn}, \quad n \in \mathbb{Z}_+.$$

Then $\gamma_{a,b}$ is a Hamburger moment sequence if and only if $a > 0$ and $b \geqslant 0$. If this is the case, then $\gamma_{a,b}$ is a Hausdorff moment sequence and its unique representing measure $\mu_{a,b}$ is given by

$$\mu_{a,b}(\varDelta) = \begin{cases} \frac{1}{b} \int_{\varDelta} t^{\frac{a}{b}-1} dt & \text{if } a > 0 \text{ and } b > 0, \\ \frac{1}{a}\delta_1(\varDelta) & \text{if } a > 0 \text{ and } b = 0, \end{cases} \qquad \varDelta \in \mathfrak{B}([0,1]).$$

Lemma

Let $(X, \mathcal{A}, \mu)$ *be a measure space and* $\{\gamma_n\}_{n=0}^{\infty}$ *be a sequence of* $\mathcal{A}$*-measurable real valued functions on* X*. Assume that*

$$\{\gamma_n(x)\}_{n=0}^{\infty} \text{ is a Hamburger moment sequence}$$

(*resp., Stieltjes, Hausdorff moment sequence*) *for* μ*-almost every* $x \in X$ *and* $\int_X |\gamma_n| d\mu < \infty$ *for all* $n \in \mathbb{Z}_+$*. Then*

$$\Big\{ \int_X \gamma_n d\mu \Big\}_{n=0}^{\infty} \text{ is a Hamburger moment sequence}$$

(*resp., Stieltjes, Hausdorff moment sequence*).

Automatic subnormality

Theorem

Let T be a 2-isometry in $\boldsymbol{B}(\mathcal{H})$ such that $T^*T(\ker T^*) \subseteq \ker T^*$. Then T' is a subnormal contraction such that

$$T'^{*n}T'^{n} = (n(T^*T - I) + I)^{-1} = (T^{*n}T^{n})^{-1} \text{ for all integers } n \geqslant 0.$$

Theorem

Suppose $T \in \boldsymbol{B}(\mathcal{H})$ is a quasi-Brownian isometry. Then T' is a subnormal contraction such that

$$T'^{*n}T'^{n} = (I + T^*T)^{-1}(I + (T^*T)^{1-2n}), \quad n \in \mathbb{Z}_+.$$

The proof is based on the formula

$$T'^{*n}T'^{n} = r_n(T^*T), \quad n \in \mathbb{Z}_+.$$

where

$$r_n\colon [1,\infty) \ni x \to \frac{1+x^{1-2n}}{1+x} = \frac{1}{1+x} + \frac{x}{1+x}(x^{-2})^n \in (0,\infty).$$

A weighted shifts on a directed trees

- Let $\mathscr{T} = (V, E)$ be a directed tree.
- Let $\ell^2(V)$ be the space of all square summable function on V with a scalar products

$$\langle f, g \rangle = \sum_{u \in V} f(u)\overline{g(u)}, \quad f, g \in \ell^2(V).$$

- For $u \in V$, let us define $e_u \in \ell^2(V)$ by

$$e_u(v) = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases}$$

- $\{e_u\}_{u \in V}$ is an orthonormal basis in $\ell^2(V)$.

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A weighted shifts on a directed trees

- For a family $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$ let us define an operator $S_{\boldsymbol{\lambda}}$ in $\ell^2(V)$ by

$$\mathcal{D}(S_{\boldsymbol{\lambda}}) = \{f \in \ell^2(V) \colon \varLambda_{\mathscr{T}} f \in \ell^2(V)\},$$
$$S_{\boldsymbol{\lambda}} f = \varLambda_{\mathscr{T}} f, \quad f \in \mathcal{D}(S_{\boldsymbol{\lambda}}),$$

  where $\varLambda_{\mathscr{T}}$ is define on functions $f \colon V \to \mathbb{C}$ by

$$(\varLambda_{\mathscr{T}} f)(v) = \begin{cases} \lambda_v \cdot f\big(\mathrm{par}(v)\big) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \mathrm{root}\,. \end{cases}$$

- An operator $S_{\boldsymbol{\lambda}}$ is called a *weighted shift on a directed tree* $\mathscr{T}$ with weights $\{\lambda_v\}_{v \in V^\circ}$.

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Lemma

Let $S_{\boldsymbol\lambda}$ be a weighted shift on $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. Then

(i) e_u *is in* $\mathcal{D}(S_{\boldsymbol\lambda})$ *if and only if* $\sum_{v\in\mathrm{Chi}(u)} |\lambda_v|^2 < \infty$; *if* $e_u \in \mathscr{D}(S_{\boldsymbol\lambda})$, *then* $S_{\boldsymbol\lambda} e_u = \sum_{v\in\mathrm{Chi}(u)} \lambda_v e_v$ *and* $\|S_{\boldsymbol\lambda} e_u\|^2 = \sum_{v\in\mathrm{Chi}(u)} |\lambda_v|^2$,

(ii) $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ *if and only if* $\sup_{u\in V} \sum_{v\in\mathrm{Chi}(u)} |\lambda_v|^2 < \infty$; *if this is the case, then* $\|S_{\boldsymbol\lambda}\|^2 = \sup_{u\in V} \|S_{\boldsymbol\lambda} e_u\|^2 = \sup_{u\in V} \sum_{v\in\mathrm{Chi}(u)} |\lambda_v|^2$.

Moreover, if $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$, *then*

(iii) $S_{\boldsymbol\lambda}^* e_u = \bar\lambda_u e_{\mathrm{par}(u)}$ *if* $u \in V^\circ$ *and* $S_\lambda^* e_u = 0$ *otherwise,*

(iv) $|S_{\boldsymbol\lambda}| e_u = \|S_{\boldsymbol\lambda} e_u\| e_u$ *for all* $u \in V$,

(v) $\triangle_{S_{\boldsymbol\lambda}}(e_u) = (\|S_{\boldsymbol\lambda} e_u\|^2 - 1) e_u$ *for every* $u \in V$,

(vi) $$\triangle_{S_{\boldsymbol\lambda}^*}(e_u) = \begin{cases} \left(\sum_{v\in\mathrm{Chi}(\mathrm{par}(u))} \lambda_v \bar\lambda_u e_v\right) - e_u & \text{if } u \in V^\circ, \\ -e_u & \text{if } u = \omega. \end{cases}$$

Given a weighted shift $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$, we set

$$\{\boldsymbol{\lambda} \neq 0\} = \{v \in V^\circ \colon \lambda_v \neq 0\} \quad \text{and} \quad V_{\boldsymbol{\lambda}}^+ = \{u \in V \colon \|S_{\boldsymbol{\lambda}} e_u\| > 0\}.$$

Proposition

Let $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\{\lambda_v\}_{v \in V^\circ}$. Assume that $S_{\boldsymbol{\lambda}}$ is left-invertible. Then $V_{\boldsymbol{\lambda}}^+ = V$ and the Cauchy dual $S'_{\boldsymbol{\lambda}}$ of $S_{\boldsymbol{\lambda}}$ is a weighted shift on $\mathscr{T}$ with weights $\big\{\lambda_v \|S_{\boldsymbol{\lambda}} e_{\mathrm{par}(v)}\|^{-2}\big\}_{v \in V^\circ}$.

Lemma

A weighted shift $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ on $\mathscr{T}$ is a 2-isometry if and only if either of the following two equivalent conditions holds:

$$1 - 2\|S_{\boldsymbol{\lambda}} e_u\|^2 + \sum_{v \in \mathrm{Chi}(u)} |\lambda_v|^2 \|S_{\boldsymbol{\lambda}} e_v\|^2 = 0, \quad u \in V,$$

$$\sum_{v \in \mathrm{Chi}(u)} |\lambda_v|^2 (2 - \|S_{\boldsymbol{\lambda}} e_v\|^2) = 1, \quad u \in V.$$

If $S_{\boldsymbol{\lambda}}$ is a 2-isometry, then $\|S_{\boldsymbol{\lambda}} e_u\| \geqslant 1$ for all $u \in V$, $V_{\boldsymbol{\lambda}}^+ = V$ and $\mathscr{T}$ is leafless.

Weighted shifts - the kernel condition

Proposition

Let $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V^\circ}$. If $\mathcal{T}$ is leafless and $S_{\boldsymbol\lambda}$ has nonzero weights, then the following conditions are equivalent:

(i) $S_{\boldsymbol\lambda}^* S_{\boldsymbol\lambda}(\ker S_{\boldsymbol\lambda}^*) \subseteq \ker S_{\boldsymbol\lambda}^*$,

(ii) *there exists a family $\{\alpha_v\}_{v \in V} \subseteq \mathbb{R}_+$ such that*

$$\|S_{\boldsymbol\lambda} e_u\| = \alpha_{\mathrm{par}(u)}, \quad u \in V^\circ. \tag{1}$$

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The definition

For $x \in [1, \infty)$, we denote by $S_{[x]}$ the unilateral weighted shift in ℓ^2 with weights $\{\xi_n(x)\}_{n=0}^{\infty}$, where

$$\xi_n(x) = \sqrt{\frac{1 + (n+1)(x^2 - 1)}{1 + n(x^2 - 1)}}, \quad x \in [1, \infty), \ n = 0, 1, \ldots.$$

Proposition

If $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ is a weighted shift on a rooted directed tree $\mathscr{T}$, then the following conditions are equivalent:

(i) *$S_{\boldsymbol{\lambda}}$ is a 2-isometry satisfying the condition (1) for some $\{\alpha_v\}_{v \in V} \subseteq \mathbb{R}_+$,*

(ii) *$\|S_{\boldsymbol{\lambda}} e_\omega\| \geqslant 1$ and $\|S_{\boldsymbol{\lambda}} e_v\| = \xi_n(\|S_{\boldsymbol{\lambda}} e_\omega\|)$ for all $v \in \mathrm{Chi}^{\langle n \rangle}(\omega)$ and $n \in \mathbb{Z}_+$.*

Proposition

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(i) $S_{\boldsymbol{\lambda}}$ *is a* 2*-isometry satisfying the condition* (1) *for some* $\{\alpha_v\}_{v \in V} \subseteq \mathbb{R}_+$,

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Theorem

Let $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ be a 2-isometric weighted shift on a rooted directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V^\circ}$ which satisfies the condition (1) *for some $\{\alpha_v\}_{v\in V} \subseteq \mathbb{R}_+$. Then*

$$S_{\boldsymbol\lambda} \cong S_{[x]} \oplus \bigoplus_{k=1}^{\infty} \big(S_{[\xi_k(x)]}\big)^{\oplus j_k},$$

where $x = \|S_{\boldsymbol\lambda} e_\omega\|$ and

$$j_k = \sum_{u \in \mathrm{Chi}^{\langle k-1\rangle}(\omega)} (\deg u - 1), \quad k \in \mathbb{N}.$$

Moreover, if the weights of $S_{\boldsymbol\lambda}$ are nonzero, then $j_k \leqslant \aleph_0$ for all $k \in \mathbb{N}$.

The perturbed kernel condition

Theorem

Let $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ be a 2-isometric weighted shift on a rooted directed tree $\mathscr{T} = (V, E)$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V^\circ}$. Assume that there exist $k \in \mathbb{N}$ and a family $\{\alpha_v\}_{v \in \mathrm{Des}(\mathrm{Chi}^{\langle k \rangle}(\omega))} \subseteq \mathbb{R}_+$ such that

$$\|S_{\boldsymbol\lambda} e_u\| = \alpha_{\mathrm{par}(u)}, \quad u \in \mathrm{Des}(\mathrm{Chi}^{\langle k+1 \rangle}(\omega)),$$

and $\lambda_v \neq 0$ for all $v \in \bigsqcup_{i=1}^{k} \mathrm{Chi}^{\langle i \rangle}(\omega)$. Then the following conditions are equivalent:

The perturbed kernel condition

Theorem

(i) *the Cauchy dual S'_λ of S_λ is subnormal,*

(ii) *there exists a family $\{\alpha_v\}_{v \in \bigsqcup_{i=0}^{k-1} \mathrm{Chi}^{\langle i \rangle}(\omega)} \subseteq \mathbb{R}_+$ such that*

$$\|S_\lambda e_u\| = \alpha_{\mathrm{par}(u)}, \quad u \in \bigsqcup_{i=1}^{k} \mathrm{Chi}^{\langle i \rangle}(\omega),$$

(iii) $S_\lambda^* S_\lambda(\ker S_\lambda^*) \subseteq \ker S_\lambda^*$.

The perturbed kernel condition

Theorem

(i) *the Cauchy dual $S'_{\boldsymbol\lambda}$ of $S_{\boldsymbol\lambda}$ is subnormal,*

(ii) *there exists a family $\{\alpha_v\}_{v \in \bigsqcup_{i=0}^{k-1} \mathrm{Chi}^{\langle i \rangle}(\omega)} \subseteq \mathbb{R}_+$ such that*

$$\|S_{\boldsymbol\lambda} e_u\| = \alpha_{\mathrm{par}(u)}, \quad u \in \bigsqcup_{i=1}^{k} \mathrm{Chi}^{\langle i \rangle}(\omega),$$

(iii) $S^*_{\boldsymbol\lambda} S_{\boldsymbol\lambda}(\ker S^*_{\boldsymbol\lambda}) \subseteq \ker S^*_{\boldsymbol\lambda}$.

The perturbed kernel condition

Theorem

(i) *the Cauchy dual $S'_{\boldsymbol\lambda}$ of $S_{\boldsymbol\lambda}$ is subnormal,*

(ii) *there exists a family $\{\alpha_v\}_{v \in \bigsqcup_{i=0}^{k-1} \mathsf{Chi}^{\langle i \rangle}(\omega)} \subseteq \mathbb{R}_+$ such that*

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(iii) $S_{\boldsymbol\lambda}^* S_{\boldsymbol\lambda}(\ker S_{\boldsymbol\lambda}^*) \subseteq \ker S_{\boldsymbol\lambda}^*$.

An example

Example

(1) Let $y_1, y_2 \in \mathbb{R}$ be such that $1 < y_1, y_2 < \sqrt{2}$ and $y_1 \neq y_2$.

(2) Then there exist positive real numbers x_1 and x_2 such that

$$\sum_{i=1}^{2} x_i^2(2 - y_i^2) = 1 \qquad \left(\text{e.g., } x_i = \frac{1}{\sqrt{2(2 - y_i^2)}} \text{ for } i = 1, 2\right).$$

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An example

Example

(3) Let $S_{\boldsymbol{\lambda}}$ be the weighted shift on $\mathscr{T}_{2,0}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V_{2,0}^{\circ}}$ defined by

$$\lambda_{i,j} = \begin{cases} x_i & \text{if } j = 1, \\ \xi_{j-2}(y_i) & \text{if } j \geqslant 2, \end{cases} \qquad i = 1, 2.$$

(4) The Cauchy dual $S'_{\boldsymbol{\lambda}}$ of $S_{\boldsymbol{\lambda}}$ is not subnormal.

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